ISOGENY FROM SU(2) TO SO(3)

This writeup constructs a 2-to-1 epimorphism $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$, quickly demonstrating methods by example without full discussion. In general, a group that doubly covers an orthogonal group is called a *spin group*. See Paul Garrett's writeup

`http://www-users.math.umn.edu/~garrett/m/v/sporadic_isogenies.pdf`

for many more examples.

1. Unitary group and its Lie algebra

The unitary group $\mathrm{U}(2) \subset \mathrm{GL}_2(\mathbb{C})$ is defined by the condition (in which "*" denotes the hermitan operator, i.e., transpose-conjugate)

$$g^* g = 1.$$

Its Lie algebra $\mathfrak{u}(2) \subset \mathrm{M}_2(\mathbb{C})$ is defined by the condition

$$e^{\mathbb{R}x} \subset \mathrm{U}(2).$$

The Lie algebra condition that for any $x \in \mathfrak{u}(2)$,

$$1 = (e^{tx})^* e^{tx} = e^{tx^*} e^{tx}, \quad t \in \mathbb{R}$$

differentiates at $t = 0$ to $0 = x^* + x$; and conversely if $x^* = -x$ then

$$(e^{tx})^* e^{tx} = e^{tx^*} e^{tx} = e^{-tx} e^{tx} = 1, \quad t \in \mathbb{R},$$

and so $x \in \mathfrak{u}(2)$. Thus the Lie algebra consists of the skew hermitian matrices, i.e., the Lie algebra $\mathfrak{u}(2) \subset \mathrm{M}_2(\mathbb{C})$ is defined by the condition

$$x^* + x = 0.$$

Here $\mathrm{U}(2)$ and $\mathfrak{u}(2)$ are a real Lie group and Lie algebra notwithstanding their elements having complex entries. Their shared real dimension is 4.

The Lie algebra $\mathfrak{su}(2)$ of the special unitary group $\mathrm{SU}(2)$ carries the additional condition that the trace vanishes,

$$\mathfrak{su}(2) = \{\, x \in \mathrm{M}_2(\mathbb{C}) : x^* = -x,\ \mathrm{tr}\, x = 0 \,\}.$$

This reduces its dimension to 3, also the manifold dimension of $\mathrm{SU}(2)$. Here the argument is that the condition $\det e^{tx} = 1$ is $e^{t\,\mathrm{tr}\,x} = 1$, which differentiates at $t = 0$ to $\mathrm{tr}\, x = 0$; and conversely if $\mathrm{tr}\, x = 0$ then $\det e^{tx} = e^{t\,\mathrm{tr}\,x} = e^0 = 1$.

The $\mathfrak{su}(2)$ conditions $x^* = -x$ and $\mathrm{tr}\, x = 0$ are preserved under addition, real scaling, and the Lie bracket. For example,

$$(rx)^* = \overline{r}\, x^* = -r\, x \quad \text{for real } r,$$

and

$$(xy - yx)^* = y^* x^* - x^* y^* = (-y)(-x) - (-x)(-y) = yx - xy = -(xy - yx).$$

2. Inner Product, Invariance

A real symmetric bilinear inner product on $\mathfrak{su}(2)$ is

$$\langle\cdot,\cdot\rangle : \mathfrak{su}(2)\times\mathfrak{su}(2)\longrightarrow\mathbb{R},\qquad \langle x,y\rangle = -\tfrac{1}{2}\mathrm{Re}(\mathrm{tr}\,(xy)).$$

The group $\mathrm{SU}(2)$ acts linearly on the algebra $\mathfrak{su}(2)$ by conjugation,

$$g\cdot x = gxg^{-1},$$

and this action preserves the inner product,

$$\langle g\cdot x, g\cdot y\rangle = \langle x,y\rangle.$$

Indeed, to see that $g\cdot x$ again lies in $\mathfrak{su}(2)$ for all $g\in\mathrm{SU}(2)$ and $x\in\mathfrak{su}(2)$, note that because g and $e^{\mathbb{R}x}$ and g^{-1} lie in $\mathrm{SU}(2)$, also

$$e^{\mathbb{R}gxg^{-1}} = ge^{\mathbb{R}x}g^{-1}\quad\text{lies in } \mathrm{SU}(2),$$

and to see that the action preserves the inner product, compute that

$$\langle g\cdot x, g\cdot y\rangle = -\tfrac{1}{2}\mathrm{Re}(\mathrm{tr}\,(gxg^{-1}gyg^{-1})) = -\tfrac{1}{2}\mathrm{Re}(\mathrm{tr}\,(xy)) = \langle x,y\rangle.$$

Note that the results in this section rely only on general Lie group and Lie algebra properties, not on any particulars of the specific Lie group $\mathrm{SU}(2)$ and Lie algebra $\mathfrak{su}(2)$.

3. Orthonormal Basis

The defining conditions $x^* = -x$ and $\mathrm{tr}\,x = 0$ say that $\mathfrak{su}(2)$ consists of the matrices

$$x = \begin{bmatrix} i\eta_1 & \xi_2 + i\eta_2 \\ -\xi_2 + i\eta_2 & -i\eta_1 \end{bmatrix},\quad \eta_1,\xi_2,\eta_2\in\mathbb{R}.$$

The $\mathfrak{su}(2)$-basis

$$x_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\qquad x_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\qquad x_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

is orthonormal under the $\mathfrak{su}(2)$ inner product. For example,

$$\langle x_1,x_1\rangle = -\tfrac{1}{2}\mathrm{Re}(\mathrm{tr}\,(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix})) = 1$$
$$\langle x_2,x_3\rangle = -\tfrac{1}{2}\mathrm{Re}(\mathrm{tr}\,(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix})) = 0.$$

4. Homomorphism

For each $g\in\mathrm{SU}(2)$ there exist nine real constants c_{ij} for $i,j=1,2,3$ such that

$$g\cdot x_j = \sum_{i=1}^{3} c_{ij}x_i,\quad j=1,2,3.$$

The corresponding 3×3 real matrix of the action of g is

$$A_g = [c_{ij}]_{i,j=1,2,3}.$$

Thus

$$A_g \text{ has } j\text{th column } \sum_{i=1}^{3} c_{ij}e_i,\quad j=1,2,3.$$

We show that A_g is orthogonal. Because the $\mathbb{R}^3$ standard basis (e_1, e_2, e_3) and the $\mathfrak{su}(2)$ basis (x_1, x_2, x_3) are both orthonormal, for any two column indices $j, k \in \{1, 2, 3\}$ the inner product of the jth and kth columns of A_g is

$$\langle\sum_{i=1}^{3} c_{ij}e_i, \sum_{i=1}^{3} c_{ik}e_i\rangle_{\mathbb{R}^3} = \sum_{i=1}^{3} c_{ij}c_{ik} = \langle\sum_{i=1}^{3} c_{ij}x_i, \sum_{i=1}^{3} c_{ik}x_i\rangle_{\mathfrak{su}(2)}.$$

That is, now denoting the columns c_j and c_k,

$$\langle c_j, c_k\rangle = \langle g \cdot x_j, g \cdot x_k\rangle_{\mathfrak{su}(2)} = \langle x_j, x_k\rangle_{\mathfrak{su}(2)} = \delta_{jk} \text{ (Kronecker delta).}$$

Thus A_g is orthogonal as claimed. This argument uses only the existence of an orthonormal basis of $\mathfrak{su}(2)$, with no reference to the values of the c_{ij}.

The map $g \mapsto A_g$ is innately a homomorphism, because the action property $(gg')\cdot x = g\cdot(g'\cdot x)$ for $g, g' \in \mathrm{SU}(2)$ and $x \in \mathfrak{su}(2)$ combines with the fact that matrix multiplication is compatible with linear map composition to give $A_{gg'} = A_g A_{g'}$.

5. Homomorphism Formula

Specifically, the action of any group element

$$g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -\beta_1 + i\beta_2 & \alpha_1 - i\alpha_2 \end{bmatrix} \in \mathrm{SU}(2)$$

on the basis elements is a matter of direct computation, albeit a bit tedious,

$$\begin{aligned} g \cdot x_1 &= (\alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2)x_1 + 2(\alpha_2\beta_1 + \alpha_1\beta_2)x_2 + 2(\alpha_2\beta_2 - \alpha_1\beta_1)x_3 \\ g \cdot x_2 &= 2(\alpha_2\beta_1 - \alpha_1\beta_2)x_1 + (\alpha_1^2 - \alpha_2^2 + \beta_1^2 - \beta_2^2)x_2 + 2(\alpha_1\alpha_2 + \beta_1\beta_2)x_3 \\ g \cdot x_3 &= 2(\alpha_1\beta_1 + \alpha_2\beta_2)x_1 + 2(-\alpha_1\alpha_2 + \beta_1\beta_2)x_2 + (\alpha_1^2 - \alpha_2^2 - \beta_1^2 + \beta_2^2)x_3. \end{aligned}$$

Thus the matrix of the action of g is

$$A_g = \begin{bmatrix} \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 & 2(\alpha_2\beta_1 - \alpha_1\beta_2) & 2(\alpha_1\beta_1 + \alpha_2\beta_2) \\ 2(\alpha_2\beta_1 + \alpha_1\beta_2) & \alpha_1^2 - \alpha_2^2 + \beta_1^2 - \beta_2^2 & 2(-\alpha_1\alpha_2 + \beta_1\beta_2) \\ 2(\alpha_2\beta_2 - \alpha_1\beta_1) & 2(\alpha_1\alpha_2 + \beta_1\beta_2) & \alpha_1^2 - \alpha_2^2 - \beta_1^2 + \beta_2^2 \end{bmatrix} \in \mathrm{O}(3).$$

6. Isogeny

To determine the kernel of the map $g \mapsto A_g$, note that the diagonal conditions

$$\alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 = \alpha_1^2 - \alpha_2^2 + \beta_1^2 - \beta_2^2 = \alpha_1^2 - \alpha_2^2 - \beta_1^2 + \beta_2^2 = 1$$

are $\alpha_1^2 - 1 = \alpha_2^2 = \beta_1^2 = \beta_2^2$. Now the $(1, 2)$-entry condition $\alpha_2\beta_1 - \alpha_1\beta_2 = 0$ is $\pm\beta_2^2 = \pm\sqrt{\beta_2^2 + 1}\beta_2$; if $\beta_2 \neq 0$ then canceling β_2 and then squaring both sides gives the impossible condition $\beta_2^2 = \beta_2^2 + 1$, so $\beta_2 = 0$. This forces $g = \pm 1_2$. Conversely, if $g = \pm 1$ then $A_g = 1$. In sum, the map has kernel is $\pm 1_2$.

Because the manifold dimension 3 of $\mathrm{O}(3)$ matches that of the connected group $\mathrm{SU}(2)$, the map $g \mapsto A_g$ surjects to the connected component $\mathrm{SO}(3)$ of its codomain.

7. Generalization

Again, for many more examples see the writeup by Paul Garrett mentioned at the beginning of this writeup. Here we note that if a finite-dimensional real Lie group G has Lie algebra $\mathfrak{g}$, and if $\mathfrak{g}$ carries a symmetric bilinear inner product and has an orthogonal anisotropic basis $\{x_i\}$, so that after permuting and scaling the basis elements,

$$\langle x_i, x_{i'}\rangle = \delta_{ii'}\lambda_i, \quad \text{with } \lambda_1 = \cdots = \lambda_p = 1 \text{ and } \lambda_{p-1} = \cdots = \lambda_n = -1,$$

then we may carry out the same process as above: Let $g \cdot x_j = \sum_{i=1}^n c_{ij} x_i$ for each j, and let $A_g = [c_{ij}] = [c_1 \cdots c_n]$ where the c_j are column vectors. For any column indices j and k (all sums to follow run from 1 to n),

$$\begin{aligned}
c_j^{\mathsf{T}} \begin{bmatrix} 1_p & 0 \\ 0 & -1_{n-p} \end{bmatrix} c_k &= \begin{bmatrix} c_{1j} & c_{2j} & \dots & c_{nj} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} c_{1k} \\ \vdots \\ c_{nk} \end{bmatrix} \\
&= \begin{bmatrix} c_{1j}\lambda_1 & c_{2j}\lambda_2 & \dots & c_{nj}\lambda_n \end{bmatrix} \begin{bmatrix} c_{1k} \\ \vdots \\ c_{nk} \end{bmatrix} \\
&= \sum_i c_{ij} c_{ik} \lambda_i \\
&= \sum_{i,i'} c_{ij} c_{i'k} \delta_{ii'} \lambda_i \\
&= \sum_{i,i'} c_{ij} c_{i'k} \langle x_i, x_{i'}\rangle \\
&= \langle \sum_i c_{ij} x_i, \sum_{i'} c_{i'k} x_{i'} \rangle \\
&= \langle g \cdot x_j, g \cdot x_k \rangle \\
&= \langle x_j, x_k \rangle \\
&= \delta_{jk} \lambda_j.
\end{aligned}$$

This says that

$$A_g^{\mathsf{T}} \begin{bmatrix} 1_p & 0 \\ 0 & -1_{n-p} \end{bmatrix} A_g = \begin{bmatrix} 1_p & 0 \\ 0 & -1_{n-p} \end{bmatrix}$$

Thus the map $g \mapsto A_g$ is a homomorphism from G to $\mathrm{O}(p, n-p)$. For $\mathrm{SU}(2)$ we had $n = p = 3$. The reader is invited to check that similarly there is a 2-to-1 isogeny from $\mathrm{SU}(1,1)$ to $\mathrm{SO}(2,1)$.