MAXWELL’S EQUATIONS

1. The Environment

Sources:

• $\rho$ is the distribution of charge in $\mathbb{R}^3$ (units: C/m$^3$).
• $\mathbf{J}$ is the current density field in $\mathbb{R}^3$ (units: A/m$^2$).

Induced fields:

• $\mathbf{E}$ is the electric field (units: N/C).
• $\mathbf{B}$ is the magnetic field (units: Ns/Cm).

Constants:

• $\varepsilon_0 = 8.85 \times 10^{-12}$ C$^2$/Nm$^2$ is the permittivity of free space.
• $\mu_0 = 4\pi \times 10^{-7}$ N/A$^2$ is the permeability of free space.

Here C=Coulomb, N=Newton, A=Ampere=C/s. The product of the two constants is the reciprocal square of the speed of light,

$$\varepsilon_0 \mu_0 = \frac{1}{c^2}, \quad c = 3 \times 10^8 \text{ m/s}.$$

2. Maxwell’s Equations Classically

2.1. Static electric charge induces an electric field. A particle at the point $p_0 \in \mathbb{R}^3$ with charge $q$ Coulomb induces a static electric field

$$\mathbf{E} : \mathbb{R}^3 - \{p_0\} \rightarrow \mathbb{R}^3.$$

The field is described by an inverse square law. For any point $p \in \mathbb{R}^3 - \{p_0\}$, let $r = p - p_0$, and let $\hat{r} = r/|r|$ be the corresponding unit vector. Then

$$\mathbf{E}(p) = \frac{q}{4\pi \varepsilon_0} \cdot \frac{\hat{r}}{|r|^2}.$$

Let $S$ be a spherical membrane of radius $a$ centered at $p_0$. Since the outward normal vector $\hat{n}$ is again $\hat{r}$, the flux of $\mathbf{E}$ through $S$ is

$$\iint_S \mathbf{E} \cdot \hat{n} = \frac{q}{4\pi \varepsilon_0 a^2} \iint_S 1 = \frac{q}{\varepsilon_0}.$$

This flux should be the integral of the divergence of $\mathbf{E}$ over the solid ball $B$ with boundary $S$,

$$\iiint_B \nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0}, \quad \nabla = [D_1 \ D_2 \ D_3].$$

However, it is easy to compute that (letting $(x, y, z)$ be the coordinates of $r = p - p_0$ where $p \neq p_0$)

$$\mathbf{E}(p) = \frac{q}{4\pi \varepsilon_0} \cdot \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} = (\nabla \cdot \mathbf{E})(p) = 0.$$
That is, $\nabla \cdot \mathbf{E} = 0$ on $\mathbb{R}^3 - \{p_0\}$. Thus the only way to make the divergence integral give the right answer is to conclude that $\nabla \cdot \mathbf{E}$ is $q/\varepsilon_0$ times the Dirac delta distribution at $p_0$,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad \rho = q\delta_{p_0}.$$ 

This argument extends to finitely many point charges to give

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad \rho = \sum_i q_i\delta_{p_i}.$$ 

View a continuous charge distribution $\rho$ as a limit of linear combinations $\rho_n$ of point mass distributions, so that the resulting electric field $\mathbf{E}$ is the corresponding limit of fields $\mathbf{E}_n$. Then presumably

$$\nabla \cdot \mathbf{E} = \nabla \cdot \lim_{n} \mathbf{E}_n = \lim_{n} (\nabla \cdot \mathbf{E}_n) = \lim_{n} \frac{\rho_n}{\varepsilon_0} = \frac{\rho}{\varepsilon_0}.$$ 

In sum, the first Maxwell equation is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

2.2. **Static magnetic charge would induce a magnetic field.** By similar reasoning, the magnetic field arising from a magnetic charge distribution according to an inverse square law satisfies an analogous equation. But apparently there is no magnetic charge, and so the second Maxwell equation is

$$\nabla \cdot \mathbf{B} = 0.$$

2.3. **A changing magnetic field induces an electric field.** Pass a magnet through a wire ring. Faraday’s Law is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$ 

2.4. **Current induces a magnetic field.** Run current in a wire that passes straight through a ring of compasses. Ampère’s Law is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$ 

However, something is missing here, because the left side is exact while the right side need not be. That is, in the resulting relation $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J}$, the left side is zero, but the right side is guaranteed to be zero only if the current is steady.

2.5. **A changing electric field induces a magnetic field.** Suppose that current is flowing, not necessarily steadily, through a sphere $S$, the boundary of a solid ball $B$. The conservation of charge implies an equality between the flux of current through $S$ and the (negative) rate of change of the charge distribution through $B$,

$$\int_S \mathbf{J} \cdot \hat{n} = -\frac{\partial}{\partial t} \int_B \rho.$$ 

Apply the Divergence Theorem to the left side and pass the time derivative through the space integral on the right side to get

$$\int_B \nabla \cdot \mathbf{J} = \int_B \left( -\frac{\partial \rho}{\partial t} \right).$$
Now let $B$ shrink to a point. This gives an equality of distributions,
\[ \nabla \cdot J = -\frac{\partial \rho}{\partial t}. \]
From the first Maxwell equation, $\rho = \varepsilon_0 \nabla \cdot E$, it follows that
\[ \nabla \cdot \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right) = 0. \]
And since closed forms are exact on $\mathbb{R}^3$, there is a field $B$ such that
\[ \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \nabla \times B. \]
The field $B$ is indeed the magnetic field. This is Maxwell’s correction to Ampère’s Law, the fourth Maxwell equation.

2.6. **Summary.** Gathering the results together and recalling that $\varepsilon_0 \mu_0 = 1/c^2$, we have
\[
\begin{align*}
\nabla \cdot E &= \frac{\rho}{\varepsilon_0}, \\
\nabla \times E + \frac{\partial B}{\partial t} &= 0, \\
\nabla \cdot B &= 0, \\
\nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} &= \mu_0 J.
\end{align*}
\]

3. **The Wave Equation**

For any vector field $F$, the Laplacian can be expressed in terms of the divergence, the gradient, and the curl,
\[ \nabla^2 F = \nabla(\nabla \cdot F) - \nabla \times \nabla \times F. \]
To see this, note that the Laplacian of $F$ has first entry
\[ D_{11}F_1 + D_{22}F_1 + D_{33}F_1, \]
while the gradient of the divergence of $F$ has first entry
\[ D_{11}F_1 + D_{12}F_2 + D_{13}F_3. \]
Subtracting the first of these from the second gives
\[ (D_{12}F_2 - D_{22}F_1) - (D_{33}F_1 - D_{13}F_3), \]
which is also the first entry of
\[ \nabla \times \nabla \times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ D_1 & D_2 & D_3 \\ D_2F_3 - D_3F_2 & D_3F_1 - D_1F_3 & D_1F_2 - D_2F_1 \end{vmatrix}. \]
Maxwell’s equations in a vacuum are
\[
\begin{align*}
\nabla \cdot E &= 0, \\
\nabla \times E &= -\frac{\partial B}{\partial t}, \\
\nabla \cdot B &= 0, \\
\nabla \times B &= \frac{1}{c^2} \frac{\partial E}{\partial t}.
\end{align*}
\]
Combine these with (1) to compute that
\[ \nabla^2 E = \nabla(\nabla \cdot E) - \nabla \times \nabla \times E = \nabla(0) + \nabla \times \frac{\partial B}{\partial t}. \]
The first term is 0, and we may exchange the time derivative and the space derivatives in the second term to get, once more using Maxwell’s equations,

\[ \nabla^2 E = \frac{\partial}{\partial t} (\nabla \times B) = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}. \]

Thus an electric field in a vacuum satisfies the wave equation,

\[ \nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}. \]

Similarly, a magnetic field in a vacuum also satisfies the wave equation,

\[ \nabla^2 B = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}. \]

4. **Maxwell’s Equations in the Language of Differential Forms**

Now normalize the speed of light to \( c = 1 \). Recall the definitions

\[ \vec{ds} \wedge dt = (dx \wedge dt, dy \wedge dt, dz \wedge dt), \]
\[ \vec{dn} = (dy \wedge dz, dz \wedge dx, dx \wedge dy). \]

For any field \( F \) introduce a pair of 2-forms,

\[ \lambda_F = F \cdot \vec{ds} \wedge dt, \]
\[ \omega_F = F \cdot \vec{dn}. \]

Then, recalling also that \( E, B, \) and \( J \) respectively denote the electric, magnetic, and current density fields, the four classical Maxwell equations rewrite as two equations in the language of differential forms,

\[ d(\omega_B + \lambda_E) = 0, \]
\[ d(\omega_E - \lambda_B) = \rho dV - \omega_J \wedge dt. \]

As before, the asymmetry is due to the absence of magnetic charge. However, we now see the two electric driving sources—charge distribution and current density—on the right side of one equation, and it is clear how the corresponding magnetic driving sources would fit into the other equation if they existed.

5. **The Wave Equation Again**

Gathering objects of various dimensions together allows a simultaneous derivation of the wave equation for the electric field \( \mathbb{E} \) and the magnetic field \( \mathbb{B} \). Introduce the Hodge-star operator, taking certain forms in \((x, y, z, t)\) to forms of complementary dimension and negating in one direction,

\[ *: \left\{ \begin{array}{c} \lambda_E, \ \omega_E \wedge dt \quad \mapsto^+ \quad \omega_E, \ \mathbb{F} \cdot \vec{ds} \\ f \ dV, \ f \quad \mapsto^+ \quad f \ dt, \ f \ dV \wedge dt \end{array} \right\}. \]
Also, the $d$ operator (including a time derivative and the corresponding $dt$ wedged in) acts as
\[
d : \begin{cases}
\begin{align*}
\lambda_F, \quad \omega_F & \quad \mapsto \omega \nabla \times F \wedge dt, \quad (\nabla \cdot F) dV + \omega dt \wedge dt, \\
\omega \wedge dt, \quad F \cdot ds & \quad \mapsto (\nabla \cdot F) dV \wedge dt, \quad \omega \nabla \times F - \lambda F, \\
\int_\mathcal{D} f dV, \quad f dt, \quad f & \quad \mapsto - f_t dV \wedge dt, \quad \lambda \nabla f, \quad \nabla f \cdot ds + f dt.
\end{align*}
\end{cases}
\]

Thus
\[
\begin{eqnarray*}
* \overset{d}{\star} * & \overset{d}{\star} \lambda_F & \overset{d}{\star} \omega_F \wedge dt \overset{\star}{\mapsto} (\nabla \times F) \cdot ds \\
& \overset{d}{\star} d & \omega \nabla \times \nabla \times F - \lambda \nabla \times F, \\
& \overset{\star}{\overset{d}{\star}} d & \omega \nabla \times \nabla \times F - \omega \nabla \times F, \\
& \overset{\star}{\overset{d}{\star}} \overset{\star}{d} & \omega \nabla \times \nabla \times F - \omega \nabla \times F - \lambda \nabla \times \nabla \times F.
\end{eqnarray*}
\]

and similarly,
\[
\begin{eqnarray*}
* \overset{d}{\star} d & \lambda_F & \overset{d}{\star} \omega_F \wedge dt \overset{\star}{\mapsto} (\nabla \times F) \cdot ds \\
& \overset{d}{\star} d & \omega \nabla \times \nabla \times F + \lambda \nabla (\nabla \cdot F - F_i) \\
& \overset{\star}{\overset{d}{\star}} d & \omega \nabla \times \nabla \times F + \lambda (\nabla \cdot F - F_i), \\
& \overset{\star}{\overset{d}{\star}} \overset{\star}{d} & \omega \nabla \times \nabla \times F - \omega \nabla \times \nabla \times F + \lambda \nabla \times F, \\
& \overset{\star}{\overset{d}{\star}} \overset{\star}{d} & \omega \nabla \times \nabla \times F + \lambda \nabla \times F.
\end{eqnarray*}
\]

In sum, if we introduce the operator
\[
h = * \overset{d}{\star} d + \overset{d}{\star} \overset{d}{\star} d,
\]
then gathering terms from the above calculations shows that altogether $h$ acts on each component of any $\lambda$-form and each component of any $\omega$-form by the space Laplacian minus the time Laplacian,
\[
h : \lambda_F, \quad \omega_F \overset{\star}{\mapsto} \lambda (\nabla \cdot F - F_i), \quad \omega (\nabla \cdot F - F_i).
\]

Thus $h$ annihilates certain 2-forms exactly when their coefficient vector fields satisfy the wave equation componentwise. (And incidentally, one also can check that
\[
h : f \overset{\star}{\mapsto} (\nabla^2 f - f_{tt}) \quad \text{for all} \quad f \in \{1, dt, dV, dV \wedge dt\},
\]

so that $h$ annihilates certain 0-forms, 1-forms, 3-forms, and 4-forms exactly when their coefficient functions satisfy the wave equation as well.)

Now we can derive the wave equation simultaneously for the electric field and the magnetic field. The $h$ operator gives
\[
h(\omega_B + \lambda_E) = \omega (\nabla^2 \omega_B - \omega_{tt}) + \lambda (\nabla^2 \omega_B - \omega_{tt}).
\]

But the first Maxwell equation in the language of differential forms gives
\[
* \overset{d}{\star} d (\omega_B + \lambda_E) = * \overset{d}{\star} 0 = 0,
\]
and in a vacuum, the second one gives
\[
d \overset{\star}{\overset{d}{\star}} (\omega_B + \lambda_E) = d \overset{\star}{\overset{d}{\star}} (\omega_E - \lambda_E) = d \overset{\star}{\overset{d}{\star}} 0 = 0.
\]

Thus
\[
0 = h(\omega_B + \lambda_E) = \omega (\nabla^2 \omega_B - \omega_{tt}) + \lambda (\nabla^2 \omega_B - \omega_{tt}).
\]

That is,
\[
\nabla^2 B = E_{tt}, \quad \nabla^2 E = E_{tt},
\]
as desired.
6. Maxwell’s Equations in the Language of Relativistic Tensors

Continue to work with the units renormalized so that the speed of light is $c = 1$. Write the electric field in components as $E = (E_x, E_y, E_z)$, and similarly for the magnetic field $B$. Consider the skew symmetric matrix

$$F = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}.$$ 

Similarly define

$$G = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}.$$ 

View time $t$ as the zeroth of four variables $(t, x, y, z)$. Consider a four-vector of differential operators,

$$D = [D_0 \ \nabla] = [D_0 \ D_1 \ D_2 \ D_3].$$ 

Then

$$DF = \left[ \nabla \cdot E - \frac{\partial E}{\partial t} + \nabla \times B \right], \quad DG = \left[ \nabla \cdot B - \frac{\partial B}{\partial t} - \nabla \times E \right].$$ 

So Maxwell’s equations rewrite as

$$DF = [\rho \ J], \quad DG = 0.$$