Chapter 6

6.1.1. (a) Let \( I = [0, 1] \), let \( P = \{0, 1/2, 1\} \), let \( P' = \{0, 3/8, 5/8, 1\} \), and let \( P'' \) be the common refinement of \( P \) and \( P' \). What are the subintervals of \( P \), and what are their lengths? Same question for \( P' \). Same question for \( P'' \).

The subintervals of \( P \) are \([0, 1/2] \) and \([1/2, 1] \), both of length 1/2. The subintervals of \( P' \) are \([0, 3/8] \), \([3/8, 5/8] \), and \([5/8, 1] \), of lengths 3/8, 1/4, and 3/8.

The common refinement of \( P \) and \( P' \) is

\[ P'' = \{0, 3/8, 1/2, 5/8, 1\}. \]


(b) Let \( B = I \times I \), let \( Q = P \times \{0, 1/2, 1\} \), let \( Q' = P' \times \{0, 1/2, 1\} \), and let \( Q'' \) be the common refinement of \( Q \) and \( Q' \). What are the subboxes of \( Q \) and what are their areas? Same question for \( Q' \). Same question for \( Q'' \).

The subrectangles of \( Q \) are

\[ [0, 1/2] \times [0, 1/2], [1/2, 1] \times [0, 1/2], [0, 1/2] \times [1/2, 1], [1/2, 1] \times [1/2, 1], \]

all of area 1/4. The subrectangles of \( Q' \) are

\[ [0, 3/8] \times [0, 1/2], [3/8, 5/8] \times [0, 1/2], [5/8, 1] \times [0, 1/2], [0, 3/8] \times [1/2, 1], [3/8, 5/8] \times [1/2, 1], [5/8, 1] \times [1/2, 1], \]

of areas 3/16, 1/8, 3/16, 3/16, 1/8, 3/16. The common refinement of \( Q \) and \( Q' \) is

\[ Q'' = \{0, 3/8, 1/2, 5/8, 1\} \times \{0, 1/2, 1\}. \]

Its subrectangles are

\[ [0, 3/8] \times [0, 1/2], [3/8, 1/2] \times [0, 1/2], [1/2, 5/8] \times [0, 1/2], [5/8, 1] \times [0, 1/2], [0, 3/8] \times [1/2, 1], [3/8, 1/2] \times [1/2, 1], [1/2, 5/8] \times [1/2, 1], [5/8, 1] \times [1/2, 1], \]

of areas 3/16, 1/16, 1/16, 3/16, 3/16, 1/16, and 3/16.

6.1.2. Show that the lengths of the subintervals of any partition of \([a, b]\) sum to the length of \([a, b]\). Same for the areas of the subboxes of \([a, b] \times [c, d]\). Generalize to \( \mathbb{R}^n \).

For an interval, the partition is

\[ P = \{t_0, t_1, \ldots, t_k\}, \quad a = t_0 < t_1 < \ldots < t_k = b. \]

The subintervals and their lengths are

\[ J_j = [t_{j-1}, t_j], \quad j = 1, \ldots, k, \quad \text{length} (J_j) = t_j - t_{j-1}. \]
So the sum of the lengths of the subintervals is

$$\sum_{j=1}^{k} \text{length}(J_j) = \sum_{j=1}^{k} (t_j - t_{j-1})$$

$$= (t_1 - t_0) + (t_2 - t_1) + \cdots + (t_k - t_{k-1})$$

$$= t_k - t_0$$

$$= b - a.$$

(A sum that mostly-cancels in this fashion is called a \textit{telescoping} sum.)

For a 2-dimensional box \([a, b] \times [c, d]\), the partition is

$$P = \{t_{1,0}, t_{1,1}, \ldots, t_{1,k_1}\} \times \{t_{2,0}, t_{2,1}, \ldots, t_{2,k_2}\},$$

where

$$a = t_{1,0} < t_{1,1} < \cdots < t_{1,k_1} = b, \quad c = t_{2,0} < t_{2,1} < \cdots < t_{2,k_2} = d.$$

So the sum of the areas of the subboxes is

$$\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} (t_{1,j_1} - t_{1,j_1-1})(t_{2,j_2} - t_{2,j_2-1}).$$

This is a doubly-indexed sum of twofold products. Note that in each product, each factor depends on only \textit{one} index of summation. Therefore, one factor of each product passes through the sum over the other index, and the sum of the areas is

$$\sum_{j_1=1}^{k_1} (t_{1,j_1} - t_{1,j_1-1}) \sum_{j_2=1}^{k_2} (t_{2,j_2} - t_{2,j_2-1}).$$

This is \((b - a)(d - c)\) by two applications of the 1-dimensional result.

The discussion of the 2-dimensional case has made clear how the general argument will go. For an \(n\)-dimensional box \([a_1, b_1] \times \cdots \times [a_n, b_n]\), proceed either by induction or by ellipsis. First we give a solution by ellipsis. The partition is

$$P = \{t_{1,0}, t_{1,1}, \ldots, t_{1,k_1}\} \times \cdots \times \{t_{n,0}, t_{n,1}, \ldots, t_{n,k_n}\},$$

where

$$a_1 = t_{1,0} < t_{1,1} < \cdots < t_{1,k_1} = b_1, \quad \ldots, \quad a_n = t_{n,0} < t_{n,1} < \cdots < t_{n,k_n} = b_n.$$

So the sum of the volumes of the subboxes is

$$\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} (t_{1,j_1} - t_{1,j_1-1}) \cdots (t_{n,j_n} - t_{n,j_n-1}).$$
In this \( n \)-fold sum of \( n \)-fold products, each term of each product depends on only one index of summation, and therefore passes through all the other sums. This shows that the sum of the volumes of the subboxes is

\[
\sum_{j_1=1}^{k_1} (t_{1,j_1} - t_{1,j_1-1}) \cdots \sum_{j_n=1}^{k_n} (t_{n,j_n} - t_{n,j_n-1}).
\]

and this is \((b_1 - a_1) \cdots (b_n - a_n)\) by \( n \) applications of the 1-dimensional result.

Alternatively, for a solution by induction, note (or prove by induction) that an \( n \)-dimensional box is the cartesian product of an \((n - 1)\)-dimensional box and an interval,

\[ B_n = B_{n-1} \times I. \]

And similarly for its \( n \)-dimensional subboxes,

\[ J_n = J_{n-1} \times J_1. \]

Let \( \text{vol}_n \) denote \( n \)-dimensional volume and let \( \text{vol}_{n-1} \) denote \((n - 1)\)-dimensional volume. Then

\[
\sum_{J_n} \text{vol}_n(J_n) = \sum_{J_{n-1} \times J_1} \text{vol}_n(J_{n-1} \times J_1)
\]

\[
= \sum_{J_{n-1}} \sum_{J_1} \text{vol}_{n-1}(J_{n-1}) \text{length}(J_1)
\]

\[
= \sum_{J_{n-1}} \text{vol}_{n-1}(J_{n-1}) \sum_{J_1} \text{length}(J_1).
\]

By inductive hypothesis,

\[
\sum_{J_{n-1}} \text{vol}_{n-1}(J_{n-1}) = \text{vol}_{n-1}(B_{n-1}),
\]

and by the result in one dimension,

\[
\sum_{J_1} \text{length}(J_1) = \text{length}(I).
\]

So our calculation has shown that

\[
\sum_{J_n} \text{vol}_n(J_n) = \text{vol}_{n-1}(B_{n-1}) \text{length}(I) = \text{vol}_n(B_n).
\]

This is the desired result.

6.1.3. Let \( J = [0, 1] \). Compute \( m_J(f) \) and \( M_J(f) \) for each of the following functions \( f : J \to \mathbb{R} \).

(a) \( f(x) = x(1 - x) \).

One-variable calculus shows that \( m_J(f) = 0 \) and \( M_J(f) = 1/4 \).
(b) \( f(x) = \begin{cases} 
1 & \text{if } x \text{ is irrational} \\
1/m & \text{if } x = n/m \text{ in lowest terms, } n, m \in \mathbb{Z} \text{ and } m > 0,
\end{cases} \)

All values taken by \( f \) are positive. The values \( 1/m \) for \( m \in \mathbb{Z}^+ \) get arbitrarily close to 0 despite never reaching it. On the other hand, \( f \) takes a maximum value of 1. Thus, \( m_f(f) = 0, \quad M_f(f) = 1. \)

(c) \( f(x) = \begin{cases} 
(1 - x) \sin(1/x) & \text{if } x \neq 0 \\
0 & \text{if } x = 0.
\end{cases} \) (See figure 1.)

This function is trapped between \( 1 - x \) and \(-1 + x\). As \( x \) approaches 0, \( f(x) \) oscillates ever faster between \( 1 - x \) and \(-1 + x\), so it assumes values closer and closer to 1 (though always below 1), and it assumes values closer and closer to \(-1\) (though always above \(-1\)). Thus
\[
m_f(f) = -1, \quad M_f(f) = 1.
\]

6.1.4. (a) Let \( I, P, P' \) and \( P'' \) be as in exercise 6.1.1(a), and let \( f(x) = x^2 \) on \( I \). Compute the lower sums \( L(f, P), L(f, P'), L(f, P'') \) and the corresponding upper sums, and check that they conform to Lemma 6.1.6, Lemma 6.1.8, and Proposition 6.1.10. 

The sums are
\[
L(f, P) = 0^2 \cdot 1/2 + (1/2)^2 \cdot 1/2 = 1/8 = 64/512,
\]
\[
L(f, P') = 6^2 \cdot 3/8 + (3/8)^2 \cdot 1/4 + (5/8)^2 \cdot 3/8 = 93/512,
\]
\[
L(f, P'') = 0^2 \cdot 3/8 + (3/8)^2 \cdot 1/8 + (1/2)^2 \cdot 1/8 + (5/8)^2 \cdot 3/8 = 100/512,
\]
\[
U(f, P) = (1/2)^2 \cdot 1/2 + 1^2 \cdot 1/2 = 5/8 = 320/512,
\]
\[
U(f, P') = (3/8)^2 \cdot 3/8 + (5/8)^2 \cdot 1/4 + 1^2 \cdot 3/8 = 269/512,
\]
\[
U(f, P'') = (3/8)^2 \cdot 3/8 + (1/2)^2 \cdot 1/8 + (5/8)^2 \cdot 1/8 + 1^2 \cdot 3/8 = 260/512.
\]
And indeed,
\[
\max\{L(f, P), L(f, P')\} \leq L(f, P'') \leq U(f, P'') \leq \min\{U(f, P), U(f, P')\}.
\]

(b) Let \(B, Q, Q'\) and \(Q''\) be as in exercise 6.1.1(b), and define \(f : B \rightarrow \mathbb{R}\) by
\[
f(x, y) = \begin{cases} 
0 & \text{if } 0 \leq x < 1/2 \\
1 & \text{if } 1/2 \leq x \leq 1.
\end{cases}
\]
Compute \(L(f, Q), L(f, Q'), L(f, Q'')\) and the corresponding upper sums, and check that they conform to Lemma 6.1.6, Lemma 6.1.8, and Proposition 6.1.10.

The sums are
\[
L(f, Q) = 0 \cdot 1/4 + 1 \cdot 1/4 + 0 \cdot 1/4 + 1 \cdot 1/4 = 1/2,
\]
\[
L(f, Q') = 0 \cdot 3/16 + 0 \cdot 1/8 + 1 \cdot 1/4 + 0 \cdot 1/4 + 0 \cdot 1/4 + 1 \cdot 3/16 = 3/8,
\]
\[
L(f, Q'') = 0 \cdot 3/16 + 0 \cdot 1/16 + 1 \cdot 1/4 + 1 \cdot 3/16 + 0 \cdot 1/4 + 1 \cdot 3/16 + 0 \cdot 1/8 + 1 \cdot 3/16 = 3/8,
\]
\[
U(f, Q) = 1 \cdot 1/4 + 1 \cdot 1/4 + 1 \cdot 1/4 + 1 \cdot 1/4 = 1,
\]
\[
U(f, Q') = 0 \cdot 3/16 + 1 \cdot 1/8 + 1 \cdot 3/16 + 0 \cdot 3/16 + 1 \cdot 1/8 + 1 \cdot 3/16 = 5/8,
\]
\[
U(f, Q'') = 0 \cdot 3/16 + 1 \cdot 1/16 + 1 \cdot 3/16 + 1 \cdot 1/16 + 1 \cdot 3/16 + 0 \cdot 3/16 + 1 \cdot 1/16 + 1 \cdot 3/16 = 5/8.
\]

Again,
\[
\max\{L(f, Q), L(f, Q')\} \leq L(f, Q'') \leq U(f, Q'') \leq \min\{U(f, Q), U(f, Q')\}.
\]

6.1.5. Draw the cartesian product \(([a_1, b_1] \cup [c_1, d_1]) \times ([a_2, b_2] \cup [c_2, d_2]) \subset \mathbb{R}^2\) where \(a_1 < b_1 < c_1 < d_1\) and similarly for the other subscript.

The picture is four boxes arranged like panes of a window. (See figure 2.)

![Figure 2: Cartesian product](image)
6.2.1. Let $f : B \to \mathbb{R}$ be a bounded function. Explain how Lemma 6.2.2 shows that $L \int_B f \leq U \int_B f$.

Let $\mathcal{L}$ be the set of lower sums of $f$ over all partitions $P$ of $B$, and similarly for $U$. Then by Proposition 6.1.10,

$$\ell \leq u \quad \text{for all } \ell \in \mathcal{L} \text{ and } u \in \mathcal{U}.$$  

This is the required condition for the lemma. By the definitions of lower and upper integral, the lemma’s conclusion is precisely that $L \int_B f \leq U \int_B f$.

6.2.2. Let $U$ and $L$ be real numbers satisfying $U \geq L$. Show that $U = L$ if and only if $U - L < \varepsilon$ for all $\varepsilon > 0$.

If $U = L$ then $U - L = 0$, so certainly $U - L < \varepsilon$ for all $\varepsilon > 0$.

On the other hand, if $U - L < \varepsilon$ for all $\varepsilon > 0$, then

- The condition $U - L > 0$ is impossible because the given information that $U - L < \varepsilon$ for all $\varepsilon > 0$ says in particular that $U - L < U - L$, which is nonsense.

- So, since we are given that $U \geq L$, and we now know that $U > L$ is impossible, the only remaining possibility is $U = L$, as desired.

6.2.3. Let $f : B \to \mathbb{R}$ be the constant function $f(x) = k$ for all $x \in B$. Show that $f$ is integrable over $B$ and $\int_B f = k \cdot \text{vol}(B)$.

Let $P$ be any partition of $B$. For each subbox $J$ of $P$, we have

$$m_J(f) = k.$$  

It follows that the lower sum for $f$ and $P$ is

$$L(f, P) = \sum_J m_J(f) \text{vol}(J) = k \sum_J \text{vol}(J) = k \text{vol}(B).$$  

Since this is independent of the partition $P$, we have shown that all lower sums have the same value, $k \text{vol}(B)$. The lower integral is the least upper bound of the lower sums, so since they all have the same value, it is

$$L \int_B f = k \text{vol}(B).$$  

A virtually identical argument shows that also the upper integral is

$$U \int_B f = k \text{vol}(B).$$  

Since the lower and upper integrals agree, the integral exists, and it is their shared values,

$$\int_B f = k \text{vol}(B).$$
6.2.4. Fill in the details in the argument that the function \( f : [0, 1] \rightarrow \mathbb{R} \) with \( f(x) = 0 \) for irrational \( x \) and \( f(x) = 1 \) for rational \( x \) is not integrable over \([0, 1]\).

Every positive-length interval \( J \) of real numbers contains both rational and irrational numbers. (We take this fact for granted here.) Consequently,

\[
m_J(f) = 0 \quad \text{and} \quad M_J(f) = 1.
\]

Consequently, every lower sum for this function is

\[
L(f, P) = \sum_J 0 \cdot \text{length}(J) = 0,
\]

and therefore the lower integral is

\[
L \int_{[0,1]} f = 0.
\]

Also, every upper sum for this function is

\[
L(f, P) = \sum_J 1 \cdot \text{length}(J) = \sum_J \text{length}(J) = 1,
\]

and therefore the upper integral is

\[
U \int_{[0,1]} f = 1.
\]

Since the lower and upper integrals are unequal, the integral does not exist.

6.2.5. Let \( B = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2 \). Define a function \( f : B \rightarrow \mathbb{R} \) by

\[
f(x, y) = \begin{cases} 
0 & \text{if } 0 \leq x < 1/2, \\
1 & \text{if } 1/2 \leq x \leq 1.
\end{cases}
\]

Show that \( f \) is integrable and \( \int_B f = 1/2 \).

Let \( P = \{0, 1/2, 1\} \times \{0, 1\} \). By a small calculation, \( L(f, P) = 1/2 \).

Consequently,

\[
L \int_B f \geq 1/2.
\]

On the other hand, let \( \varepsilon \) be a small positive number, and let

\[
P_\varepsilon = \{0, (1 - \varepsilon)/2, 1/2, 1\} \times \{0, 1\}.
\]

By another calculation, \( U(f, P) = 1/2 + \varepsilon/2 \), so \( U \int_B f \leq 1/2 + \varepsilon/2 < 1/2 + \varepsilon \).

Since \( \varepsilon > 0 \) was arbitrary, \( U \int_B f \leq 1/2 \). Putting all of this together gives

\[
1/2 \leq L \int_B f \leq U \int_B f \leq 1/2.
\]
It follows that $L \int_B f = U \int_B f = \frac{1}{2}$, showing that the integral exists and is $1/2$.

6.2.6. This exercise shows that integration is linear. Let $f : B \rightarrow \mathbb{R}$ and $g : B \rightarrow \mathbb{R}$ be integrable.

(a) Let $P$ be a partition of $B$ and let $J$ be some subbox of $P$. Show that

$$m_J(f) + m_J(g) \leq m_J(f + g) \leq M_J(f + g) \leq M_J(f) + M_J(g).$$

Show that consequently,

$$L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P).$$

This is left as an exercise.

(b) Part (a) of this exercise obtained comparisons between lower and upper sums, analogously to the first paragraph of the proof of Proposition 6.2.4. Argue analogously to the rest of the proof to show $\int_B (f+g)$ exists and equals $\int_B f + \int_B g$.

Let $\varepsilon > 0$ be given. There exist partitions $P'$ and $P''$ of $B$ such that

$$U(f, P') - L(f, P') < \varepsilon/2, \quad U(g, P'') - L(g, P'') < \varepsilon/2.$$

Let $P$ be the common refinement of $P'$ and $P''$. Then

$$U(f, P) - L(f, P) < \varepsilon/2, \quad U(g, P) - L(g, P) < \varepsilon/2,$$

so that

$$(U(f, P) + U(g, P)) - (L(f, P) + L(g, P)) < \varepsilon.$$

By the estimate from part (a), it follows that

$$U(f + g, P) - L(f + g, P) < \varepsilon.$$

Thus $\int_B (f + g)$ exists by the Integrability Criterion.

Similarly, given any $\varepsilon > 0$ we can find a partition $P$ of $B$ such that

$$\int_B f + \int_B g - \varepsilon \leq L(f, P) + L(g, P) \leq U(f, P) + U(g, P) \leq \int_B f + \int_B g + \varepsilon,$$

and so consequently

$$\int_B f + \int_B g - \varepsilon \leq L(f + g, P) \leq U(f + g, P) \leq \int_B f + \int_B g + \varepsilon.$$

Since $\varepsilon$ can be arbitrarily small,

$$\int_B f + \int_B g \leq L \int_B (f + g) \leq U \int_B (f + g) \leq \int_B f + \int_B g.$$

The quantities at either end are equal, so the inequalities all must be inequalities. Hence $\int_B (f + g)$ exists and equals $\int_B f + \int_B g$. 8
(c) Let \( c \geq 0 \) be any constant. Let \( P \) be any partition of \( B \). Show that for any subbox \( J \) of \( P \),

\[
m_J(cf) = cm_J(f) \quad \text{and} \quad M_J(cf) = cM_J(f).
\]

If \( c = 0 \) then all the quantities are 0, so we may assume that \( c > 0 \).

Since \( m_J(f) \leq f(x) \) for all \( x \in J \), it follows that \( cm_J(f) \leq cf(x) = (cf)(x) \) for all \( x \in J \), so that \( cm_J(f) \leq m_J(cf) \). Since \( m_J(cf) \leq (cf)(x) = cf(x) \) for all \( x \in J \), it follows that \( m_J(cf)/c \leq f(x) \) for all \( x \in J \), so that \( m_J(cf)/c \leq m_J(f) \) and therefore \( m_J(cf) \leq cm_J(f) \). The two inequalities show that \( m_J(cf) = cm_J(f) \). The argument for \( M_J \) is virtually identical.

Explain why consequently

\[
L(cf, P) = cL(f, P) \quad \text{and} \quad U(cf, P) = cU(f, P).
\]

This follows immediately,

\[
L(cf, P) = \sum_J m_J(cf) \text{vol}(J) = \sum_J cm_J(f) \text{vol}(J) = c \sum_J m_J(f) \text{vol}(J)
= cL(f, P),
\]

and similarly for upper sums.

Explain why consequently

\[
\int_B(cf) = c\int_B f \quad \text{and} \quad \int_B(cf) = c\int_B f.
\]

Similarly to the argument at the beginning of this part of the problem,

\[
\inf\{cL(f, P)\} = c\inf\{L(f, P)\}.
\]

It follows that

\[
L \int_B cf = \inf\{L(cf, P)\} = \inf\{cL(f, P)\} = c \inf\{L(f, P)\} = cL \int_B f.
\]

And similarly for the upper integral.

Explain why consequently \( \int_B cf \) exists and

\[
\int_B cf = c \int_B f.
\]

Since \( L \int_B f = U \int_B f = \int_B f \) we have in fact established that

\[
L \int_B cf = U \int_B cf = c \int_B f,
\]
which is exactly what we need to show.

(d) Let \( P \) be any partition of \( B \). Show that for any subbox \( J \) of \( P \),
\[
m_J(-f) = -M_J(f) \quad \text{and} \quad M_J(-f) = -m_J(f).
\]
Since \( m_J(-f) \leq -f(x) \) for all \( x \in J \), it follows that \( -m_J(-f) \geq f(x) \) for all \( x \in J \), so that \( m_J(-f) \leq M_J(f) \), and so \( M_J(-f) \leq -M_J(f) \). Since \( M_J(f) \geq f(x) \) for all \( x \in J \), it follows that \( -M_J(f) \leq f(x) \) for all \( x \in J \), so that \( -M_J(f) \leq m_J(-f) \). The two inequalities show that \( m_J(-f) = -M_J(f) \).

Explain why consequently
\[
L(-f, P) = -U(f, P) \quad \text{and} \quad U(-f, P) = -L(f, P),
\]
and so
\[
L \int_B (-f) = -U \int_B f \quad \text{and} \quad U \int_B (-f) = -L \int_B f,
\]
and so \( \int_B (-f) \) exists and
\[
\int_B (-f) = - \int_B f.
\]
The argument is very similar to the argument in part (c).

Explain why the work so far here in part (d) combines with part (c) to show that for any \( c \in \mathbb{R} \), \( \int_B cf \) exists and
\[
\int_B cf = c \int_B f.
\]
If \( c \geq 0 \) then we have the result from part (c). If \( c < 0 \) then \( c = -\tilde{c} \) where \( \tilde{c} > 0 \), and so by the various results that we have established, skipping some pedantic basic algebra steps,
\[
\int_B cf = \int_B (-\tilde{c}f)
= -\int_B \tilde{c}f \quad \text{by (d), since} \int_B \tilde{c}f \text{ exists by (c)}
= -\tilde{c} \int_B f \quad \text{by (c)}
= c \int_B f.
\]

6.2.7. This exercise shows that integration preserves order. Let \( f : B \to \mathbb{R} \) and \( g : B \to \mathbb{R} \) both be integrable, and suppose that \( f \leq g \), meaning that \( f(x) \leq g(x) \) for all \( x \in B \). Show that \( \int_B f \leq \int_B g \).

By exercise 6.2.6, the integral \( \int_B (g - f) \) exists and is equal to \( \int_B g - \int_B f \). So it suffices to prove that \( \int_B (g - f) \geq 0 \). Simply the notation by replacing the symbol-string “\( g - f \)” by “\( g \)”. Now we only have to prove that if \( g \geq 0 \) then
also \( \int_B g \geq 0 \). Take any partition \( P \) of \( B \). For each subbox, we have \( m_J(g) \geq 0 \), so that \( L(f, P) \geq 0 \). That condition that each lower sum \( L(f, P) \) is at least 0 means that 0 is a lower bound of the set of lower sums, and so the lower integral, being the \textit{greatest} lower bound, is at least 0. Since \( g \) is integrable, its integral is equal to its lower integral, giving the desired result.

6.3.2. \textit{Let} \( f : \mathbb{R} \to \mathbb{R} \) \textit{be the cubing function} \( f(x) = x^3 \). \textit{Give a direct proof that} \( f \) \textit{is} \( \varepsilon \)-\( \delta \) \textit{continuous on} \( \mathbb{R} \).

Note that for any \( x, \hat{x} \in \mathbb{R} \),

\[
|x^3 - \hat{x}^3| = |(x - \hat{x})(x^2 + x\hat{x} + \hat{x}^2)|
\]

\[
= |x - \hat{x}| |x^2 + x\hat{x} + \hat{x}^2|
\]

\[
\leq |x - \hat{x}|(|x^2| + |x\hat{x}| + |\hat{x}^2|)
\]

Now take \( |x - \hat{x}| < 1 \). We can compute
\[
|x^2 + x| |x| + |x|^2 = |x - \hat{x} + x| |x| + |x|^2
\]

\[
\leq (|x - \hat{x}| + |x|)^2 + (|x - \hat{x}| + |x|)|x| + |x|^2
\]

\[
< (1 + |x|)^2 + (1 + |x|)|x| + |x|^2
\]

\[
= 1 + 2|x| + |x|^2 + |x| + |x|^2 + |x|^2 = 1 + 3|x| + 3|x|^2,
\]

or we can note that \( |\hat{x}| < |x| + 1 \) and therefore, again,
\[
|x^2 + x| |x| + |x|^2 < (1 + |x|)^2 + (1 + |x|)|x| + |x|^2 = 1 + 3|x| + 3|x|^2.
\]

Now, pick any \( x \in \mathbb{R} \) and let \( \varepsilon > 0 \) be given. Define
\[
\delta = \min\{1, \varepsilon/(1 + 3|x| + 3|x|^2)\}.
\]

Then for any \( \hat{x} \in \mathbb{R} \) such that \( |\hat{x} - x| < \delta \), we have
\[
|x^3 - \hat{x}^3| \leq |x - \hat{x}|(|\hat{x}^2| + |\hat{x}| |x| + |x|^2)
\]

\[
< |x - \hat{x}| (1 + 3|x| + 3|x|^2)
\]

\[
< \frac{\varepsilon}{1 + 3|x| + 3|x|^2} (1 + 3|x| + 3|x|^2)
\]

\[
= \varepsilon.
\]

This is the desired result.

6.3.3. \textit{Is the cubing function of the previous exercise uniformly continuous on} \( \mathbb{R} \)?

The cubing function is not uniformly continuous on \( \mathbb{R} \). Let any \( \delta > 0 \) be given. The claim is that this \( \delta \) fails to satisfy the definition of uniform continuity for \( \varepsilon = 1 \). To see this, set
\[
x = 1/\sqrt{3}, \quad \hat{x} = 1/\sqrt{3} + \delta/3.
\]
Then certainly $|\tilde{x} - x| < \delta$, and also

$$|\tilde{x}^3 - x^3| = |(1/\sqrt{\delta}+\delta/3)^3 - (1/\sqrt{\delta})^3| = |1/\delta^{3/2} + 1 + \delta^{3/2}/3 + \delta^{3}/(7 - 1 \delta^{3/2})| > 1 = \varepsilon.$$  

So uniform continuity fails at $x$, as claimed.

On $[0, 500]$?

Yes: The cubing function is continuous on the set $[0, 500]$, and the set is compact, so the continuity is uniform.

6.3.4. (a) Show: If $I \subset \mathbb{R}$ is an interval (possibly all of $\mathbb{R}$), $f : I \rightarrow \mathbb{R}$ is differentiable, and there exists a positive constant $R$ such that $|f'(x)| \leq R$ for all $x \in I$ then $f$ is uniformly continuous on $I$.

This is an application of the Mean Value Theorem. Given $\varepsilon > 0$, let $\delta = \varepsilon/R$. Then for all $x, \tilde{x} \in I$,

$$f(\tilde{x}) - f(x) = f'(c)(\tilde{x} - x) \quad \text{for some } c \text{ between } x \text{ and } \tilde{x}.$$  

It follows that

$$|f(\tilde{x}) - f(x)| = |f'(c)||\tilde{x} - x| \leq R|\tilde{x} - x|.$$  

So in particular, if $|\tilde{x} - x| < \delta$ then (recalling that $\delta = \varepsilon/R$)

$$|f(\tilde{x}) - f(x)| < R\delta = \varepsilon,$$  

as desired.

(b) Prove that sine and cosine are uniformly continuous on $\mathbb{R}$.

This is immediate from (a) since $|\sin'| = |\cos| \leq 1$ and similarly for $\cos'$.

6.3.5. Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be the square root function $f(x) = \sqrt{x}$. Take for granted that $f$ is $\varepsilon$-$\delta$ continuous on $[0, +\infty)$.

(a) What does part (a) of the previous problem say about the uniform continuity of $f$?

Nothing. The derivative $-1/(2\sqrt{x})$ of the square root function is not defined at 0, and it is unbounded near 0, so the hypotheses are not met. However, the fact that a particular diagnostic tool fails to show that $f$ is uniformly continuous does not preclude the possibility that it is.

(b) Is $f$ uniformly continuous on $[0, +\infty)$?

Yes. The idea of the proof is that given $\varepsilon > 0$, whatever $\delta$ works at $x = 0$ should work all along the graph, because the graph is steepest at the origin. To quantify this statement about the graph, the claim is that:

For any $x, \tilde{x}$ such that $0 \leq x \leq \tilde{x}$, $\sqrt{\tilde{x}} - \sqrt{x} \leq \sqrt{\tilde{x} - x}.$  

One way to prove the claim is to note first that in general, if $0 \leq a \leq b$ then certainly

$$\sqrt{b} - a \leq \sqrt{b + a}.$$  

Multiply both sides of the inequality by $\sqrt{b - a}$ to get

$$b - a \leq \sqrt{b^2 - a^2}.$$  

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In particular this holds for \( b = \sqrt{x} \) and \( a = \sqrt{\tilde{x}} \), proving the claim. A variant proof of the claim is to observe that if \( 0 \leq x \leq \tilde{x} \) then
\[
0 \leq (\sqrt{\tilde{x}} - \sqrt{x})^2 = \tilde{x} - 2\sqrt{\tilde{x}}x + x \leq \tilde{x} - 2x + x = \tilde{x} - x,
\]
and then the claim follows by taking square roots.
A little thought shows that the box that works at \( x = 0 \) has \( \varepsilon = \sqrt{\delta} \), i.e.,
\[
\delta = \varepsilon^2.
\]
This is the \( \delta \) that should work everywhere. Now we can write the proof:
Given \( \varepsilon > 0 \), let \( \delta = \varepsilon^2 \). Take any \( x, \tilde{x} \in [0, \infty) \), and assume without loss of generality that \( x \leq \tilde{x} \). Then
\[
|\tilde{x} - x| < \delta \implies |\sqrt{\tilde{x}} - \sqrt{x}| \leq \sqrt{\tilde{x}} - x < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon.
\]
Another solution is to argue that
- the square root function is pointwise continuous,
- so it is uniformly continuous on \([0, 1]\) since \([0, 1]\) is compact,
- and it is uniformly continuous on \([1, \infty)\) since it is differentiable with bounded derivative there,
- and the two uniform continuities concatenate to make the square root function uniformly continuous on \([0, \infty)\).

Invoking the continuity of the square root function for the first bullet is fine, and the second and third bullets are supported. But the fourth bullet requires showing that two uniform continuities concatenate to give a single uniform continuity, and the general concatenation argument is not quite as trivial as a person might think: taking \( \delta = \min\{\delta_1, \delta_2\} \) is not guaranteed to work (though it will work for the square root function). For example, consider the function
\[
f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} -1 & \text{if } x \leq -1/2, \\ 2x & \text{if } -1/2 < x < 1/2, \\ 1 & \text{if } x \geq 1/2. \end{cases}
\]
Let \( \varepsilon = 1.5 \). On \((-\infty, 0]\) we may take \( \delta_1 = 1.1 \) in response, and on \([0, \infty)\) we may take \( \delta_2 = 1.1 \) in response, but \( \delta = 1.1 \) does not work on \((-\infty, \infty)\).

Generalizing the problem, for any \( \alpha \) such that \( 0 \leq \alpha \leq 1 \), the function \( f_\alpha(x) = x^\alpha \) on \([0, +\infty)\) is uniformly continuous. In particular, the square root function is the case \( \alpha = 1/2 \). To solve the generalized problem, use calculus. For any fixed positive \( h \), introduce the function
\[
g_\alpha,h(x) = f_\alpha(x + h) - f_\alpha(x), \quad x \geq 0.
\]
Compute that for any $x > 0$, because $\alpha - 1 \leq 0$ we have
$$g'_{\alpha,h}(x) = \alpha((x + h)^{\alpha - 1} - x^{\alpha - 1}) \leq 0.$$ Thus $g_{\alpha,h}$ is decreasing, which is to say that for any $x \geq 0$, $f_{\alpha}(x + h) - f_{\alpha}(x) = g_{\alpha,h}(x) \leq g_{\alpha,h}(0) = f_{\alpha}(h)$.

In particular, given nonnegative $x$ and $\tilde{x}$ with $\tilde{x} > x$, let $h = \tilde{x} - x$ to get
$$f_{\alpha}(\tilde{x}) - f_{\alpha}(x) \leq f_{\alpha}(\tilde{x} - x) = (\tilde{x} - x)^\alpha.$$ Now we can solve the problem as before. Let $\varepsilon > 0$ be given, and set $\delta = \varepsilon^{1/\alpha}$.

Let $0 \leq x < \tilde{x}$ with $\tilde{x} - x < \delta$. Then
$$f_{\alpha}(\tilde{x}) - f_{\alpha}(x) \leq (\tilde{x} - x)^\alpha < \delta^\alpha = \varepsilon.$$ 6.3.6. Let $J$ be a box in $\mathbb{R}^n$ with sides of length less than $\delta/n$. Show that any points $x$ and $\tilde{x}$ in $J$ satisfy $|\tilde{x} - x| < \delta$.

Let $x = (x_1, \ldots, x_n)$ and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$. Then
$$\tilde{x} - x = (\tilde{x}_1 - x_1, \ldots, \tilde{x}_n - x_n) \text{ where } |\tilde{x}_1 - x_1| < \delta/n, \ldots, |\tilde{x}_n - x_n| < \delta/n.$$ And so by the Size Bounds,
$$|\tilde{x} - x| \leq \sum_{i=1}^n |\tilde{x}_i - x_i| < \sum_{i=1}^n \delta/n = \delta.$$ 6.3.7. For $\int_B f$ to exist, it is sufficient that $f : B \rightarrow \mathbb{R}$ be continuous, but it is not necessary. What preceding exercise provides an example of this?

Exercise 6.2.5.

Here is another example. Let $B = [0, 1]$ and let $f : B \rightarrow \mathbb{R}$ be monotonic increasing, meaning that if $x_1 < x_2$ in $B$ then $f(x_1) \leq f(x_2)$. Show that such a function is bounded, though it need not be continuous. Use the Integrability Criterion to show that $\int_B f$ exists.

The function is bounded because its outputs lie in a compact interval,
$$f(x) \in [f(0), f(1)] \text{ for all } x \in [0, 1].$$

A discontinuous such function is
$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2, \\ 1 & \text{if } 1/2 < x \leq 1. \end{cases}$$

To use the Integrability Criterion to show that $\int_B f$ exists, let $\varepsilon > 0$ be given. For some positive integer $n$ we have
$$(f(1) - f(0))/n < \varepsilon.$$
Partition the interval \([0, 1]\) into \(n\) subintervals of equal length \(1/n\). The lower and upper sums for the partition are

\[
L(f, P) = 1/n \left( f(0) + f(1/n) + f(2/n) + \cdots + f((n-1)/n) \right),
\]

\[
U(f, P) = 1/n \left( f(1/n) + f(2/n) + f(3/n) \cdots + f(1) \right).
\]

Their difference telescopes, and thus by our choice of \(n\),

\[
U(f, P) - L(f, P) = (f(1) - f(0))/n < \varepsilon.
\]

Consequently \(\int_{[0,1]} f\) exists by the Integrability Criterion.

6.4.1. (a) Show that for three points \(a, b, c \in \mathbb{R}\) in any order, and any integrable function \(f : [\min\{a, b, c\}, \max\{a, b, c\}] \to \mathbb{R}\), \(\int_c^a f = \int_b^c f + \int_c^b f\).

For example, suppose that \(b \leq c \leq a\). Then we know that

\[
\int_b^a f = \int_c^c f + \int_c^a f,
\]

and so by algebra,

\[
-\int_c^a f = \int_b^c f - \int_b^a f.
\]

Consequently,

\[
\int_c^c f = -\int_c^c f \quad \text{by definition}
\]

\[
= \int_b^c f - \int_b^a f \quad \text{by the previous display}
\]

\[
= \int_b^c f + \int_a^b f \quad \text{by definition}
\]

\[
= \int_a^b f + \int_c^c f \quad \text{obviously.}
\]

(b) Show that if \(f : [\min\{a, b\}, \max\{a, b\}] \to \mathbb{R}\) takes the constant value \(k\) then \(\int_a^b f = k(b - a)\), regardless of which of \(a\) and \(b\) is larger.

If \(a \leq b\) then we already have the result from exercise 6.2.3. If \(a > b\) then compute

\[
\int_a^b f = -\int_b^a f \quad \text{by definition}
\]

\[
= -(k(a - b)) \quad \text{by the result that we already have}
\]

\[
= k(b - a) \quad \text{by algebra.}
\]

6.4.3. Show that if \(F_1, F_2 : [a, b] \to \mathbb{R}\) are differentiable and \(F_1' = F_2'\), then \(F_1 = F_2 + C\) for some constant \(C\).
This is a consequence of the Mean Value Theorem. Let $G = F_1 - F_2$. Then $G' = F'_1 - F'_2 = 0$. For any $x \in [a, b]$, 

$$G(x) - G(a) = (x - a)G'(c) \text{ for some } c \in [a, x] = 0 \text{ since } G' = 0.$$  

This shows that $G(x) = G(a)$ for all $x \in [a, b]$. That is, $G$ is some constant $C$. Since $G = F_1 - F_2$, we are done.

6.4.5. Let $f : [0, 1] \to \mathbb{R}$ be continuous and suppose that for all $x \in [0, 1]$, $\int_0^x f = \int_x^1 f$. What is $f$?

We are given that $\int_0^x f = \int_x^1 f = - \int_x^1 f$ for all $x$. Differentiate to obtain $f(x) = -f(x)$ for all $x$. Thus $f$ is identically zero.

6.4.6. Find all differentiable functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that for all $x \in \mathbb{R}_{\geq 0}$, $(f(x))^2 = \int_0^x f$.

Differentiate the given relation to get

$$2f(x)f'(x) = f(x).$$

It follows that at every $x$ such that $f(x) \neq 0$ we have $f'(x) = 1/2$.

Next note that $(f(0))^2 = \int_0^0 f = 0$, so $f(0) = 0$.

One solution is $f(x) = 0$ for all $x \geq 0$.

For any other solution, $f(x) \neq 0$ for some $x$. Take any such $x$ and consider the associated set

$$S_x = \{ \hat{x} \in \mathbb{R}_{\geq 0} : f(\hat{x}) = 0 \text{ and } \hat{x} < x \}.$$  

Then $S_x$ is nonempty (since it contains 0), and $S_x$ is bounded above by $x$. The completeness of the real number system now says that sup$(S_x)$ exists. Call it $x_0$. If $x_0$ is an isolated point of $S_x$ then it belongs to $S_x$ and so $f(x_0) = 0$. On the other hand, if $x_0$ is a limit point of $S_x$ then by the continuity of $f$ and the definition of $x_0$, also $f(x_0) = 0$. That is, $f(x_0) = 0$ in all cases.

Recall that we have fixed some $x$ such that $f(x) \neq 0$, while on the other hand $f(x_0) = 0$. So $x_0 \neq x$. Since $x$ and $x_0$ are upper bounds of $S_x$ and since $x_0$ is the least upper bound, it follows that $x_0 < x$. For any $c$ between $x_0$ and $x$, necessarily $f(c) \neq 0$, and so $f'(c) = 1/2$. Now compute that by the Mean Value Theorem,

$$f(x) = f(x) - f(x_0) = f'(c)(x - x_0) \text{ for some } c \in (x_0, x) = \frac{1}{2}(x - x_0) \text{ as just explained.}$$

This calculation shows that $f(x)$ is positive.

Next we show that for every $x' > x$, also $f(x')$ is positive. The alternative is that for some $x' > x$, $f(x') = 0$. In this case define another set

$$T_x = \{ \hat{x} \in \mathbb{R}_{\geq 0} : f(\hat{x}) = 0 \text{ and } \hat{x} > x \}. $$
Then $T_x$ is nonempty (since it contains $x'$), and $T_x$ is bounded below by $x$. Similarly to the previous argument, the infimum (greatest lower bound) $x_1$ of $T_x$ satisfies $f(x_1) = 0$ and $x_1 \geq x$, and so $x_1 > x$. And then another Mean Value Theorem calculation gives

$$-f(x) = f(x_1) - f(x) = \frac{1}{2}(x_1 - x) > 0.$$ 

But since $f(x) > 0$ this is a contradiction. So it is impossible to have any $x' > x$ such that $f(x') = 0$. That is, $f(x') > 0$ for all $x' \geq x$, and so the reasoning of the previous paragraph shows that

$$f(x') = \frac{1}{2}(x - x_0) \quad \text{for all } x' > x.$$ 

All the quantization that has gone on shows that in this expression, $x_0$ is the largest $x$-value where $f$ is 0, and that $f$ is identically 0 for all values up to $x_0$. Thus

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq x_0, \\ \frac{1}{2}(x - x_0) & \text{if } x > x_0. \end{cases}$$

But this function fails to be differentiable at $x_0$ unless $x_0 = 0$. Thus finally, either $f$ is identically 0, or $f$ is the function $f(x) = x/2$.

**6.4.7.** Define $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(u) = e^{(u+1)/u}/u$ and $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(x) = \int_1^x f$. Show that $F$ behaves somewhat like a logarithm in that $F(1/x) = -F(x)$ for all $x \in \mathbb{R}^+$. Interpret this property of $F$ as a statement about area under the graph of $f$.

We have $f(u) = e^{(u+1)/u}/u$. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be $\phi(u) = 1/u$. Then

$$((f \circ \phi) \cdot \phi')(u) = \frac{e^{1/u+u}}{1/u} \cdot -\frac{1}{u^2} = -\frac{e^{u+1/u}}{u} = -f(u).$$

That is, $(f \circ \phi) \cdot \phi' = -f$. Consequently,

$$F(1/x) = \int_1^{1/x} f = \int_{1/x}^1 (-f) = \int_{1/x}^1 f \circ \phi \cdot \phi' = \int_{1/x}^1 f = -\int_1^x f = -F(x).$$

This says that $f$ has the following property: For any $x > 1$, the area under the graph of $f$ between 1 and $x$ equals the area under the graph from 1/x to 1.

**6.5.4.** Let $S$ be the set of rational numbers in $[0, 1]$. Show that $S$ does not have a volume (i.e., a length) under Definition 6.5.1.

The unit interval $[0, 1]$ is a box containing $S$, so according to the definition, the volume is

$$\int_{[0,1]} \chi_S,$$

if this integral exists. But by exercise 6.2.4, it doesn’t.
6.5.5. Prove the Volume Zero Criterion.

By Definition 6.5.1, a set $S$ contained in the box $B$ has volume zero if and only if $\int_B \chi_S$ exists and equals 0. Since $\chi_S(x) \geq 0$ for all $x \in B$, each lower sum $L(\chi_S, P)$ is nonnegative. Thus the lower integral $L \int_B \chi_S$ is nonnegative. And we have established that the upper integral $U \int_B \chi_S$ is at least the lower integral. So we have the inequalities

$$0 \leq L \int_B \chi_S \leq U \int_B \chi_S,$$

and therefore $\int_B \chi_S$ exists and equals 0 if and only if $U \int_B \chi_S \leq 0$. Since the upper integral is the greatest lower bound of the upper sums, it suffices to show that no positive number is a lower bound of the upper sums. So we need to show that given any $\varepsilon > 0$, some upper sum is less than $\varepsilon$. But the upper sum is the sum of the areas of the type I subboxes, and so the criterion is that $\int_B \chi_S$ exists and equals 0 if and only if

$$\sum_{J \text{ type I}} \text{vol}(J) < \varepsilon.$$

6.5.9. Use Theorem 6.5.4, the discussion immediately after its proof, Proposition 6.5.3, and any other results necessary to explain why for each set $K$ and function $f : K \to \mathbb{R}$ below, the integral $\int_K f$ exists.

(a) $K = \{(x, y) : 2 \leq y \leq 3, 0 \leq x \leq 1 + \ln y/y\}$, $f(x, y) = e^{xy}$.

Put the shaded set $K$ inside the box $B = [0, 1 + \ln 3/3] \times [2, 3]$. Extend $f$ from $K$ to $B$ by defining $f = 0$ on $B - K$. Then $f$ is discontinuous only on the boundary curve between the shaded and unshaded regions, and this curve is the graph of the function

$$\varphi : [2, 3] \to \mathbb{R}, \quad \varphi(y) = 1 + \ln y/y.$$

Thus the boundary curve has volume zero by Proposition 6.5.3. Consequently, $\int_B f$ exists by Theorem 6.5.4. This integral is $\int_K f$, as explained near the end of the section.

(b) $K = \{(x, y) : 1 \leq x \leq 4, 1 \leq y \leq \sqrt{x}\}$, $f(x, y) = e^{x/y^2}/y^5$.

This is very similar to (a). This time it involves the graph of the function

$$\varphi : [1, 4] \to \mathbb{R}, \quad \varphi(x) = \sqrt{x}.$$

(c) $K$ is the region between the curves $y = 2x^2$ and $x = 4y^2$, $f(x, y) = 1$.

This is again very similar to (a), but it involves two graphs rather than one.

(d) $K = \{(x, y) : 1 \leq x^2 + y^2 \leq 2\}$, $f(x, y) = x^2$.

This is again very similar to (a), but it involves four graphs. The functions
in question are
\[ \varphi_1 : [-\sqrt{2}, \sqrt{2}] \rightarrow \mathbb{R}, \quad \varphi_1(x) = \sqrt{2 - x^2}, \]
\[ \varphi_2 : [-\sqrt{2}, \sqrt{2}] \rightarrow \mathbb{R}, \quad \varphi_2(x) = -\sqrt{2 - x^2}, \]
\[ \varphi_3 : [-1, 1] \rightarrow \mathbb{R}, \quad \varphi_3(x) = \sqrt{1 - x^2}, \]
\[ \varphi_4 : [-1, 1] \rightarrow \mathbb{R}, \quad \varphi_4(x) = -\sqrt{1 - x^2}. \]

(e) \( K = \) the pyramid with vertices \((0, 0, 0), (3, 0, 0), (0, 3, 0), (0, 0, 3/2)\), and \( f(x, y, z) = x \).

Put the pyramid in a box \( B \). Extend \( f \) from \( K \) to \( B \) by defining \( f = 0 \) outside the pyramid. Then \( f \) is discontinuous only on the tilted pyramid-roof. This roof is a subset of the graph of the function
\[ \varphi : [0, 3] \times [0, 3] \rightarrow \mathbb{R}, \quad \varphi(x, y) = (3 - x - y)/2. \]

Hence the roof has area zero by Proposition 6.5.3 and exercise 6.5.6.

(f) \( K = \{ x \in \mathbb{R}^n : |x| \leq 1 \} \) (the solid unit ball in \( \mathbb{R}^n \)), \( f(x_1, \ldots, x_n) = x_1 \cdots x_n \).

Extend \( f \) from the ball \( K \) to the box \([-1, 1]^n\). Then \( f \) is discontinuous only on the boundary of the \( K \), i.e., on the unit sphere. The upper half of the unit sphere is a subset of the graph of the continuous function
\[ \varphi : [-1, 1]^{n-1} \rightarrow \mathbb{R} \]
given by
\[ \varphi(x_1, \ldots, x_{n-1}) = \begin{cases} \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2} & \text{if } x_1^2 + \cdots + x_{n-1}^2 \leq 1, \\ 0 & \text{if } x_1^2 + \cdots + x_{n-1}^2 > 1. \end{cases} \]

Similarly, the lower half of the unit sphere is a subset of the graph of \(-\varphi\). Consequently, \( \int_B f \) exists by Theorem 6.5.4, and this is \( \int_K f \) by definition.

6.6.1. Let \( S \) be the set of points \((x, y) \in \mathbb{R}^2 \) between the \( x \)-axis and the sine curve as \( x \) varies between 0 and \( 2\pi \). Since the sine curve has two arches between 0 and \( 2\pi \), and since the area of an arch of sine is 2,
\[ \int_S 1 = \frac{4}{2} = 2. \]

On the other hand,
\[ \int_{x=0}^{2\pi} \int_{y=0}^{\sin x} 1 = \int_{x=0}^{2\pi} \sin x = 0. \]

Why doesn’t this contradict Fubini’s Theorem?

The set \( S \) has equal parts of its area above and below the \( x \)-axis. The double integral simply measures area without being not sensitive to this, but
the iterated integral is sensitive to it since each of its one-dimensional piece-integrals takes orientation into account.

6.6.2. Exchange the order of integration in \( \int_{x=a}^{b} \int_{y=a}^{b} f(x, y) \).

The other iterated integral is \( \int_{y=a}^{b} \int_{x=a}^{b} \).

6.6.3. Exchange the order of integration in \( \int_{x=0}^{1} \int_{y=0}^{1} + \int_{y=x}^{1-x} \) f.

The result is \( \int_{x=0}^{1} \left( \int_{z=x}^{1} \int_{y=0}^{1-x} + \int_{z=x}^{1-x} \right) f \).

6.6.4. Exchange the inner order of integration in \( \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{x+y} f \). Sketch the region of integration.

The result is \( \int_{x=0}^{1} \left( \int_{z=x}^{1} \int_{y=0}^{1-x} + \int_{z=x}^{1-x} \right) f \).

6.6.5. Evaluate \( \int_K f \) from in parts (a), (b), (c), (f) of exercise 6.5.9.

(a) The integral is
\[
\int_{y=2}^{1} \int_{x=0}^{1+\ln y/y} e^{x/y} = \int_{y=2}^{1} \left( \frac{1}{y} e^{x/y} \right)_{x=0}^{1+\ln y/y} = \int_{y=2}^{1} \left( \frac{1}{y} (e^{y+\ln y} - 1) \right) = \int_{y=2}^{1} \left( e^y - 1/y \right) = (e^y - \ln y)_{y=2}^{3} = e^3 - e^2 - \ln(3/2).
\]

(b) The integral is
\[
\int_{x=1}^{2} \int_{y=1}^{\sqrt{y}} e^{x/y} / y^5 = \int_{y=1}^{2} \int_{x=1}^{y} e^{x/y} / y^5 = \int_{y=1}^{2} e^{x/y} / y^5 \int_{x=1}^{y} = \int_{y=1}^{2} \left( e^{y}/y^5 - e/y^3 \right) = (1/8)(e^4 - e) - e/2(1 - 1/4) = (e^4/8 - e/2).
\]

(c) The integral is
\[
\int_{x=0}^{1/16^{1/3}} \int_{y=2x^2}^{\sqrt{y} / 2} 1 = \int_{x=0}^{1/16^{1/3}} \left( \sqrt{y}/2 - 2x^2 \right) = \left( (1/3)x^{3/2} - (2/3)x^3 \right)_{x=0}^{1/16^{1/3}} = 1/(3 \cdot 4) - 2/(3 \cdot 16) = 1/12 - 1/24 = \frac{1}{24}.
\]

(f) The integral is
\[
\int_{x_1=0}^{1} \int_{x_2=0}^{1} \cdots \int_{x_n=0}^{1} x_1 x_2 \cdots x_n = \int_{x_1=0}^{1} \int_{x_2=0}^{1} \cdots \int_{x_n=0}^{1} x_n = (1/2)^n.
\]
6.6.6. Find the volume of the region $K$ bounded by the coordinate planes, $x+y = 1$, and $z = x^2 + y^2$. Sketch $K$.

Compute

$$\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{x^2+y^2} 1 = \int_{x=0}^{1} \int_{y=0}^{1-x} (x^2 + y^2)$$

$$= \int_{x=0}^{1} (x^2(1-x) + \frac{1}{3}(1-x)^3)$$

$$= \left( \frac{1}{3} x^3 - \frac{1}{4} x^4 - \frac{1}{12} (1-x)^4 \right) \bigg|_{x=0}^{1}$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}.$$

6.6.7. Evaluate $\int_{K} (1 + x + y + z)^{-3}$ where $K$ is the unit simplex.

Compute

$$\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (1 + x + y + z)^{-3}$$

$$= -\frac{1}{2} \int_{x=0}^{1} \int_{y=0}^{1-x} (1 + x + y + z)^{-2} \bigg|_{z=0}^{1-x-y}$$

$$= -\frac{1}{2} \int_{x=0}^{1} \int_{y=0}^{1-x} (1/4 - (1 + x + y)^{-2})$$

$$= -\frac{1}{2} \int_{x=0}^{1} ((1/4)(1-x) - (1 + x + y)^{-1}) \bigg|_{y=0}^{1-x}$$

$$= -\frac{1}{2} \int_{x=0}^{1} ((1/4)(1-x) + 1/2 - (1 + x)^{-1})$$

$$= -\frac{1}{2} \left( -\frac{1}{8}(1-x^2) \bigg|_{x=0}^{1} + \frac{1}{2} - \ln(1 + x) \bigg|_{x=0}^{1} \right)$$

$$= -\frac{1}{2} \left( \frac{1}{8} + \frac{1}{2} - \ln 2 \right) = -\frac{1}{2} \left( \frac{1}{8} + \frac{1}{2} - \ln 2 \right)$$

$$= \frac{\ln 2}{2} - \frac{5}{16}.$$

6.6.8. Find the volume of the region $K$ in the first octant bounded by $x = 0$, $z = 0$, $z = y$, and $x = 4 - y^2$. Sketch $K$.

Compute,

$$\int_{y=0}^{2} \int_{x=0}^{4-y^2} \int_{z=0}^{y} 1 = \int_{y=0}^{2} y \int_{x=0}^{4-y^2} 1 = \int_{y=0}^{2} y(4-y^2)$$

$$= \int_{y=0}^{2} (4y - y^3) = (2y^2 - (1/4)y^4) \bigg|_{y=0}^{2} = 8 - 4 = \frac{4}{16}.$$
6.6.11. Let $K$ and $L$ be compact subsets of $\mathbb{R}^n$ with boundaries of volume zero. Suppose that for each $x_1 \in \mathbb{R}$, the cross sectional sets

$$K_{x_1} = \{(x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in K\}$$

$$L_{x_1} = \{(x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in L\}$$

have equal $(n-1)$-dimensional volumes. Show that $K$ and $L$ have the same volume. Illustrate for $n = 2$.

Compute

$$\text{vol}(K) = \int_{K} 1 = \int_{x_1} \int_{K_{x_1}} 1 = \int_{x_1} \text{vol}(K_{x_1}) = \int_{x_1} \text{vol}(L_{x_1}) = \int_{x_1} \int_{L_{x_1}} 1 = \int_{L} 1 = \text{vol}(L).$$

6.6.13. Let $n \in \mathbb{Z}^+$ and $r \in \mathbb{R}_{\geq 0}$. The $n$-dimensional simplex of side $r$ is

$$S_n(r) = \{(x_1, \ldots, x_n) : 0 \leq x_1, \ldots, 0 \leq x_n, x_1 + \cdots + x_n \leq r\}.$$

(a) Show that for $n > 1$, $S_n(r) = \bigcup_{x_n \in [0, r]} S_{n-1}(r-x_n) \times \{x_n\}$. That is, $S_n(r)$ is a disjoint union of cross-sectional $(n-1)$-dimensional simplices of side $r-x_n$ at height $x_n$ as $x_n$ varies from 0 to $r$. Make sketches for $n = 2$ and $n = 3$.

For all $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and any fixed $x_n \in [0, r]$, we have the equivalences

$$(x_1, \ldots, x_n) \in S_n(r) \iff \left\{\begin{array}{l}
0 \leq x_1, 0 \leq x_2, \ldots, 0 \leq x_{n-1}, \\
x_1 + x_2 + \cdots + x_{n-1} \leq r - x_n
\end{array}\right\}$$

$$\iff (x_1, \ldots, x_{n-1}) \in S_{n-1}(r-x_n).$$

That is, the cross-section of $S_n(r)$ at $x_n$ is $S_{n-1}(r-x_n)$. The result follows.

(b) Prove that $\text{vol}(S_1(r)) = r$.

Compute $\text{vol}(S_1(r)) = \int_{x_1=0}^{r} 1 = r$.

Use Fubini’s Theorem to prove that

$$\text{vol}(S_n(r)) = \int_{x_n=0}^{r} \text{vol}(S_{n-1}(r-x_n)) \quad \text{for } n > 1,$$

and show by induction that $\text{vol}(S_n(r)) = r^n/n!$.

Compute

$$\text{vol}(S_n(r)) = \int_{S_n(r)} 1 = \int_{x_n=0}^{r} \int_{S_{n-1}(r-x_n)} 1 \quad \text{by Fubini’s Theorem}$$

$$= \int_{x_n=0}^{r} \text{vol}(S_{n-1}(r-x_n))$$

$$= \int_{x_n=0}^{r} \frac{(r-x_n)^{n-1}}{(n-1)!} \quad \text{by induction hypothesis}$$

$$= \left[-\frac{(r-x_n)^n}{n!}\right]_{x_n=0}^{r} = \frac{r^n}{n!}.$$
(c) Use Fubini’s Theorem to show that $\int_{S_n(r)} x_n = \int_{x_n=0}^{r} x_n \frac{(r-x_n)^{n-1}}{(n-1)!}$.

Work this integral by parts to get $\int_{S_n(r)} x_n = r^{n+1}/(n+1)!$.

This is similar to (b). Compute

$$\int_{S_n(r)} x_n = \int_{x_n=0}^{r} x_n \int_{S_{n-1}(r-x_n)} 1 \quad \text{by Fubini’s Theorem}$$

$$= \int_{x_n=0}^{r} x_n \frac{(r-x_n)^{n-1}}{(n-1)!} \quad \text{as in (c)}.$$

Now let $u = x_n$ and $dv = (r-x_n)^{n-1}/(n-1)!$, so that $du = 1$ and $v = -(r-x_n)^{n}/n!$ and continue,

$$\int_{S_n(r)} x_n = -x_n \frac{(r-x_n)^n}{n!} \bigg|_{x_n=0}^{r} + \int_{x_n=0}^{r} \frac{(r-x_n)^n}{n!}$$

$$= - \frac{(r-x)^{n+1}}{(n+1)!} \bigg|_{x_n=0}^{r} = \frac{r^{n+1}}{(n+1)!}.$$

(d) The centroid of $S_n(r)$ is $(\overline{x}_1, \ldots, \overline{x}_n)$, where $\overline{x}_j = \int_{S_n(r)} x_j / \text{vol}(S_n(r))$ for each $j$. What are these coordinates explicitly?

By the previous calculations,

$$\overline{x}_n = \frac{r^{n+1}/(n+1)!}{r^n/n!} = \frac{r}{r+1}.$$

By symmetry, $\overline{x}_j = r/(n+1)$ for all $j$. When $r = 1$ and $n = 3$, this gives $\overline{x} = \overline{y} = \overline{z} = 1/4$, as in the text.
6.7.1. Evaluate $\int_S x^2 + y^2$ where $S$ is the region bounded by $x^2 + y^2 = 2z$ and $z = 2$. Sketch $S$.

Using cylindrical coordinates, compute

$$\int_S x^2 + y^2 = \int_{\phi(K)} x^2 + y^2 = \int_K r^2 \cdot r = \int_{\theta=0}^{2\pi} \int_{r=0}^{r^3} \int_{z=r^2/2}^1 \frac{1}{4}$$

$$= 2\pi \int_{r=0}^{r^3} r^3 \left(2 - \frac{r^2}{2}\right) = 2\pi \left(r^4/2 - r^6/12\right) = 2\pi \left(8 - \frac{64}{12}\right)$$

$$= \frac{16\pi}{3}.$$

6.7.2. Find the volume of the region $S$ between $x^2 + y^2 = 4z$ and $x^2 + y^2 + z^2 = 5$. Sketch $S$.

The surfaces intersect at $z$ such that $4z = x^2 + y^2 = 5 - z^2$, i.e., $z = 1$, and $x^2 + y^2 = 4$, i.e., $r^2 = 4$. So, using cylindrical coordinates, compute

$$\int_S 1 = \int_{\theta=0}^{2\pi} \int_{r=0}^{r^5} \int_{z=r^4/4}^{r^3} \frac{1}{4}$$

$$= 2\pi \left(-\frac{1}{3} (5 - r^2)^{3/2} - \frac{r^4}{16}\right)_{r=0}^{r=4} = 2\pi \left(\frac{1}{3} (5^{3/2} - 1) - 1\right)$$

$$= \frac{2\pi}{3} (5\sqrt{5} - 4).$$

6.7.3. Find the volume of the region between the graphs of $z = x^2 + y^2$ and $z = (x^2 + y^2 + 1)/2$.

The graphs intersect where $x^2 + y^2 = (x^2 + y^2 + 1)/2$, i.e., $x^2 + y^2 = 1$, i.e., $r = 1$. So, using cylindrical coordinates, compute

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{(r^2+1)/2} 1 = 2\pi \int_{r=0}^{1} r(r^2+1)/2 - r^3 = 2\pi \int_{r=0}^{1} (r - r^3)/2$$

$$= 2\pi(r^2/4 - r^4/8)_{r=0}^{r=1} = 2\pi(1/4 - 1/8) = \frac{\pi}{4}.$$

6.7.5. Let $\phi$ be the spherical coordinate mapping. Describe $\phi(K)$ where

$$K = \{(\rho, \theta, \varphi) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi/2, 0 \leq \rho \leq \cos \varphi\}.$$

The range of $\theta$- and $\varphi$-values shows that $\phi(K)$ sits in the upper half of $(x, y, z)$-space, where $z \geq 0$, and that $\phi(K)$ is symmetric about the $z$-axis. The curve $\rho = \cos \varphi$ in parameter space also has the equation

$$\rho^2 = \rho \cos \varphi,$$

so long as we rule out $\rho = 0$ unless $\varphi = \pi/2$. The spherical coordinate map $\phi$ takes the curve to points $(x, y, z)$ such that

$$x^2 + y^2 + z^2 = z.$$
That is,\[ x^2 + y^2 + z^2 - z + (1/2)^2 = (1/2)^2, \]
or\[ x^2 + y^2 + (z - 1/2)^2 = (1/2)^2. \]
This is a sphere of radius 1/2 centered at (0, 0, 1/2), i.e., a sphere centered on the positive z-axis, tangent to the (x, y)-plane, of diameter 1.

Same question for\[ K = \{(\rho, \theta, \varphi) : 0 \leq \theta \leq 2\pi, \ 0 \leq \varphi \leq \pi, \ 0 \leq \rho \leq \sin \varphi\}. \]
The analysis here is similar. This time \( \phi(K) \) is again symmetric about the z-axis but is not restricted to the upper half space. The curve \( \rho = \sin \varphi \) in parameter space also has the equation\[ \rho^2 = \rho \cos \theta \sin \varphi, \]
so long as we rule out \( \rho = 0 \) unless \( \varphi \in \{0, \pi\} \). Since \( \phi(K) \) is symmetric about the z-axis, fix \( \theta \) at 0 and note that the spherical coordinate map \( \phi \) takes the curve to points \((x, 0, z)\) such that\[ x^2 + z^2 = x. \]
That is,\[ x^2 - x + (1/2)^2 + z^2 = (1/2)^2, \]
or\[ (x - 1/2)^2 + z^2 = (1/2)^2. \]
This is a circle of radius 1/2 in the \((x, z)\)-plane, centered at \((1/2, 0, 0)\). So its rotation about the z-axis is a sort of degenerate torus of inner radius 0 and outer radius 1.

6.7.6. Evaluate \( \int_S xyz \) where \( S \) is the first octant of \( B_3(1) \).
Compute, using spherical coordinates,
\[
\int_S xyz = \int_{\phi(K)} x y z = \int_K \rho \cos \theta \sin \varphi \cdot \rho \sin \theta \sin \varphi \cdot \rho \cos \varphi \cdot \rho^2 \sin \varphi \\
= \int_{\theta=0}^{2\pi} \cos \theta \sin \theta \int_{\rho=0}^{1} \rho^5 \int_{\varphi=0}^{\pi/2} \sin^3 \varphi \cos \varphi \\
= \frac{1}{2} \sin^2 \theta \bigg|_{\theta=0}^{\pi/2} \cdot \frac{\rho^6}{6} \bigg|_{\rho=0}^{1} \cdot \frac{1}{4} \sin^4 \varphi \bigg|_{\varphi=0}^{\pi/2} \\
= \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{48}. 
\]

6.7.7. Find the mass of a solid figure filling the spherical shell
\[ S = B_3(b) - B_3(a) \]
with density \( \delta(x, y, z) = x^2 + y^2 + z^2 \).

Use spherical coordinates and Fubini’s Theorem. In terms of spherical coordinates, the density is \( \rho^2 \), so the mass is

\[
M = 2\pi \cdot \frac{\rho^5 - \rho^5}{5} = \frac{4\pi (b^5 - a^5)}{5}.
\]

6.7.8. A solid sphere of radius \( r \) has density \( \delta(x, y, z) = e^{-(x^2+y^2+z^2)^{3/2}} \). Find its mass, \( \int_{B_3(\delta)} \).

Again, use spherical coordinates. In spherical coordinates the density is \( e^{-\rho^3} \), and so the mass is

\[
M = 4\pi \cdot \left( -\frac{\rho}{3} \right) = \frac{4\pi}{3} (1 - e^{-r^3}).
\]

6.7.9. Find the centroid of the region \( S = B_3(a) \cap \{ x^2 + y^2 \leq z^2 \} \cap \{ z \geq 0 \} \). Sketch \( S \).

The region is an ice cream cone, so by symmetry, \( \overline{x} = \overline{y} = 0 \). Computing with spherical coordinates shows that the volume of the region is

\[
V = \int_{\rho=0}^{a} \rho \int_{\theta=0}^{\pi} \int_{\varphi=0}^{\pi/4} \sin \varphi = \frac{a^3}{3} \cdot 2\pi \cdot \left( 1 - \frac{\sqrt{2}}{2} \right) = \frac{\pi a^3}{3} (2 - \sqrt{2}).
\]

Similarly, since \( z \) in spherical coordinates is \( \rho \cos \varphi \), integrating \( z \) over the region gives

\[
\int_{S} z = \int_{\rho=0}^{a} \rho \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/4} \sin \varphi \cos \varphi = \frac{a^4}{4} \cdot \frac{1}{2} \sin^2 \varphi \bigg|_{\varphi=0}^{\pi/4} = \frac{\pi a^4}{8}.
\]

It follows that

\[
\overline{z} = \frac{\pi a^4/8}{\pi a^3(2 - \sqrt{2})/3} = \frac{3(2 + \sqrt{2})a}{16}.
\]

6.7.10. (a) Prove Pappus’s Theorem: Let \( K \) be a compact set in the \((x, z)\)-plane lying to the right of the \( z \)-axis and with boundary of area zero. Let \( S \) be the solid obtained by rotating \( K \) about the \( z \)-axis in \( \mathbb{R}^3 \). Then

\[
\text{vol}(S) = 2\pi \overline{x} \cdot \text{area}(K),
\]

where as always, \( \overline{x} = \int_{K} x/\text{area}(K) \).
Use triples \((x, \theta, z)\) rather than the usual \((r, \theta, z)\) to denote cylindrical coordinates. Let
\[ R = \{(x, \theta, z) : (x, z) \in K, \ 0 \leq \theta \leq 2\pi\}. \]

Use cylindrical coordinates to parametrize \(S\),
\[ \phi : R \rightarrow S, \ \phi(x, \theta, z) = (x \cos \theta, x \sin \theta, z). \]

Then by the Change of Variable Theorem and then Fubini’s Theorem, the volume of \(S\) is
\[
\text{vol}(S) = \int_S 1 = \int_{\phi(R)} 1 = \int_R x = \int_{\theta=0}^{2\pi} \int_{(x,z) \in K} x = 2\pi \int_K x = 2\pi \text{area}(K).
\]

(b) What is the volume of the torus \(T_{a,b}\) of cross-sectional radius \(a\) and major radius \(b\)?

By the formula from part (a), the volume is
\[
\text{vol}(T_{a,b}) = 2\pi b \cdot \pi a^2 = 2\pi^2 a^2 b.
\]

6.7.11. Prove the change of scale principle: If the set \(K \subset \mathbb{R}^n\) has volume \(v\) then for any \(r \geq 0\), the set \(rK = \{rx : x \in K\}\) has volume \(r^nv\).

The map
\[ \phi : K \rightarrow rK, \ \phi(x) = rx \]

is linear, so it is its own derivative. Thus \(\phi'(x) = rI\) (where \(I\) is the \(n\)-by-\(n\) identity matrix) has determinant \(r^n\). By the Change of Variable Theorem,
\[
\text{vol}(rK) = \int_{rK} 1 = \int_{\phi(K)} 1 = \int_K r^n = r^n \int_K 1 = r^n \text{vol}(K) = r^nv.
\]

6.7.12. (Volume of the \(n\)-ball, first version.) Let \(n \in \mathbb{Z}^+\) and \(r \in \mathbb{R}_{\geq 0}\). The \(n\)-dimensional ball of radius \(r\) is
\[ B_n(r) = \{ x : x \in \mathbb{R}^n \mid |x| \leq r \}. \]

Let
\[ v_n = \text{vol}(B_n(1)). \]

(a) Explain how exercise 6.7.11 reduces computing the volume of \(B_n(r)\) to computing \(v_n\).

It is straightforward to show that \(B_n(r) = rB_n(1)\) since in general \(|rx| = r|x|\) for \(r \geq 0\) and \(x \in \mathbb{R}^n\):
\[ B_n(r) = \{ x : x \in \mathbb{R}^n, |x| \leq r \} = \{ rx : x \in \mathbb{R}^n : |x| \leq 1 \} = rB_n(1). \]

Consequently, the change of scale principle shows that
\[
\text{vol}(B_n(r)) = \int_{B_n(r)} 1 = \int_{rB_n(1)} 1 = r^n \int_{B_n(1)} 1 = r^n \text{vol}(B_n(1)) = r^nv_n.
\]
(b) Show that $v_1 = 2$ and $v_2 = \pi$.
Note that $v_1$ is the length of $[-1, 1]$, which is 2, and that $v_2$ is the area of the unit ball, which is the very definition of $\pi$.

(c) Let $D$ denote the unit disk $B_2(1)$. Explain why for $n > 2$,

$$B_n(1) = \bigcup_{(x_1, x_2) \in D} \{(x_1, x_2)\} \times B_{n-2}(\sqrt{1 - x_1^2 - x_2^2}).$$

That is, the unit $n$-ball is a union of cross-sectional $(n-2)$-dimensional balls of radius $\sqrt{1 - x_1^2 - x_2^2}$ as $(x_1, x_2)$ varies through the unit disk. Make a sketch for $n = 3$, the only value of $n$ for which we can see this.

For any $x = (x_1, \ldots, x_n)$ we have the equivalences

$$x \in B_n(1) \iff x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 \leq 1 \quad \iff x_3^2 + \cdots + x_n^2 \leq 1 - x_1^2 - x_2^2 \quad \iff (x_1, x_2) \in D \text{ and } (x_3, \ldots, x_n) \in B_{n-2}(\sqrt{1 - x_1^2 - x_2^2}).$$

(d) The problem assumes that $n > 2$ and gives a string of equalities. The first equality is

$$v_n = v_{n-2} \int_{(x_1, x_2) \in D} (1 - x_1^2 - x_2^2)^{n/2 - 1}.$$ 

To show this, note that by the definition of volume and by part (c),

$$v_n = \int_{B_n(1)} 1 = \int_{\{(x_1, x_2) \times B_{n-2}(\sqrt{1 - x_1^2 - x_2^2}): (x_1, x_2) \in D\}} 1$$ 

Next, by Fubini’s Theorem the last integral is

$$\int_{(x_1, x_2) \in D} \int_{B_{n-2}(\sqrt{1 - x_1^2 - x_2^2})} 1.$$

The inner integral is the volume of $B_{n-2}(\sqrt{1 - x_1^2 - x_2^2})$, and by the change of scale principle this is $(1 - x_1^2 - x_2^2)^{n/2 - 1} v_{n-2}$. So now we have

$$v_{n-2} \int_{(x_1, x_2) \in D} (1 - x_1^2 - x_2^2)^{n/2 - 1}.$$ 

Switching to polar coordinates, the integral becomes

$$v_{n-2} \int_{(r, \theta) \in [0,1] \times [0, 2\pi]} r (1 - r^2)^{n/2 - 1}.$$ 

By Fubini’s Theorem, this is

$$v_{n-2} \int_{\theta = 0}^{2\pi} \int_{r = 0}^{1} r (1 - r^2)^{n/2 - 1}.$$
And by a short calculation, this is
\[-v_{n-2}\pi \frac{(1-r^2)^{n/2}}{n} \bigg|_{r=0} = v_{n-2} \pi \frac{n}{n/2}.\]

(e) Show by induction the even case of the formula

\[v_n = \begin{cases} 
\frac{\pi^{n/2}}{(n/2)!} & \text{for } n \text{ even,} \\
\frac{\pi^{(n-1)/2} 2^n ((n-1)/2)!}{n!} & \text{for } n \text{ odd.}
\end{cases}\]

For \(n = 2\), the claimed value of \(v_n\) is
\[\frac{\pi^{2/2}}{(2/2)!} = \pi,
\]
which is indeed \(v_2\). Now the induction can proceed in steps of 2. For even \(n \geq 2\), if the formula holds for \(n\) then the right side for \(n + 2\) is
\[\frac{\pi^{(n+2)/2}}{((n + 2)/2)!} = \frac{\pi}{(n + 2)/2} \frac{\pi^{n/2}}{(n/2)!} = \frac{\pi}{(n + 2)/2} v_n,
\]
and this is \(v_{n+2}\) by part (d). The induction is complete.

6.7.13. This exercise computes the “improper” integral \(I = \int_{x=0}^{\infty} e^{-x^2} \, dx\), defined as the limit \(\lim_{R \to \infty} \int_{x=0}^{R} e^{-x^2} \, dx\). Let \(I(R) = \int_{x=0}^{R} e^{-x^2} \, dx\) for any \(R \geq 0\).

(a) Use Fubini’s Theorem to show that \(I(R)^2 = \int_{S(R)} e^{-x^2-y^2} \, dx \, dy\), where \(S(R)\) is the square \(S(R) = \{(x, y) : 0 \leq x \leq R, 0 \leq y \leq R\}\).

Note that the variable of integration \(x\) in the formula for \(I(R)\) is a dummy variable. So we can compute
\[I(R)^2 = I(R) \cdot I(R) = \int_{x=0}^{R} e^{-x^2} \int_{y=0}^{R} e^{-y^2} \, dx \, dy = \int_{x=0}^{R} \int_{y=0}^{R} e^{-x^2-y^2} \, dx \, dy.
\]
By Fubini’s Theorem, this last integral is \(\int_{S(R)} e^{-x^2-y^2} \, dx \, dy\).

(b) Let \(Q(R)\) be the quarter disk
\[Q(R) = \{(x, y) : 0 \leq x, 0 \leq y, x^2 + y^2 \leq R^2\},
\]
and similarly for \(Q(\sqrt{2} R)\). Explain why
\[\int_{Q(R)} e^{-x^2-y^2} \leq \int_{S(R)} e^{-x^2-y^2} \leq \int_{Q(\sqrt{2} R)} e^{-x^2-y^2},\]
The inequalities between the integrals follow from two conditions. First, we have a containment of sets,

$$Q(R) \subset S(R) \subset Q(\sqrt{2} R),$$

and second, the integrand $e^{-x^2-y^2}$ is positive.

(c) Change variables, evaluate $\int_{Q(R)} e^{-x^2-y^2}$ and $\int_{Q(\sqrt{2} R)} e^{-x^2-y^2}$. What are the limits of these two quantities as $R \to \infty$?

Compute, using polar coordinates and Fubini’s Theorem, that

$$\int_{Q(R)} e^{-x^2-y^2} = \int_{\theta=0}^{\pi/2} \int_{r=0}^{R} r e^{-r^2} = \frac{\pi}{2} \left( -\frac{1}{2} e^{-R^2} \right) = \frac{\pi}{4} \left( 1 - e^{-R^2} \right).$$

Substitute $\sqrt{2} R$ for $R$ to get that also

$$\int_{Q(\sqrt{2} R)} e^{-x^2-y^2} = \frac{\pi}{4} \left( 1 - e^{-2R^2} \right).$$

(d) What is $I$?

Both of the integrals computed in (c) go to $\pi/4$ as $R$ goes to infinity. Since the quantity $I(R)^2$ is trapped between them, it is squeezed to $\pi/4$ as well. Hence the desired integral $I$ is

$$\int_{x=0}^{\infty} e^{-x^2} = \frac{\sqrt{\pi}}{2}.$$

6.7.14. (Volume of the $n$-ball, improved version) Define the gamma function as an integral,

$$\Gamma(s) = \int_{x=0}^{\infty} x^{s-1} e^{-x} dx, \quad s > 0.$$

(a) Show that $\Gamma(1) = 1$.

Compute

$$\Gamma(1) = \int_{x=0}^{\infty} x^0 e^{-x} dx = \int_{x=0}^{\infty} e^{-x} dx = -e^{-x}\big|_{x=0}^{\infty} = -(0 - 1) = 1.$$

Show that $\Gamma(1/2) = \sqrt{\pi}$.

Start from

$$\Gamma(1/2) = \int_{x=0}^{\infty} x^{-1/2} e^{-x} dx.$$

Let $x = y^2$, so that $y = x^{1/2}$. Then $dy = (1/2)x^{-1/2} dx$, and so by the previous exercise,

$$\Gamma(1/2) = 2 \int_{y=0}^{\infty} e^{-y^2} dy = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Show that $\Gamma(s+1) = s\Gamma(s)$.

Start from

$$\Gamma(s+1) = \int_{x=0}^{\infty} x^s e^{-x} dx.$$
Let \( u = x^s \) and \( dv = e^{-x} dx \). Then \( du = sx^{s-1} \) and \( v = -e^{-x} \). Integration by parts gives
\[
\Gamma(s + 1) = -x^s e^{-x}\bigg|_{x=0}^{x=\infty} + s \int_0^\infty x^{s-1} e^{-x} dx.
\]
For large \( x \), the exponential decay of \( e^{-x} \) dominates the polynomial growth of \( x^s \), and since \( s > 0 \), \( x^s e^{-x} \) is zero at \( x = 0 \). So the boundary term of the previous display vanishes. The other term is \( \Gamma(s) \), giving the desired result.

(b) Use part (a) to show that \( \Gamma(n) = (n-1)! \) for \( n = 1, 2, 3, \ldots \).

This is immediate by induction on \( n \) since \( \Gamma(1) = 1 = 0! \) and then for \( n \geq 1 \), if we assume inductively that \( \Gamma(n) = (n-1)! \) then also
\[
\Gamma(n + 1) = n \Gamma(n) = n(n-1)! = n!
\]
and this completes the induction.

(c) Use exercise 6.7.12(b), exercise 6.7.12(d), and the extended definition of the factorial in part (b) of this exercise to obtain a uniform formula for the volume of the unit \( n \)-ball,
\[
v_n = \frac{\pi^{n/2}}{(n/2)!}, \quad n = 1, 2, 3, \ldots.
\]
Thus the \( n \)-ball of radius \( r \) has volume
\[
\text{vol}(B_n(r)) = \frac{\pi^{n/2}}{(n/2)!} r^n, \quad n = 1, 2, 3, \ldots.
\]

We already have the formula for \( v_n \) if \( n \) even. For \( n \) odd, the argument is essentially identical to exercise 6.7.12(e) but starting at the base case \( n = 1 \). For the base case, we need to show that
\[
\frac{\pi^{1/2}}{\Gamma(1/2 + 1)} = 2.
\]
So compute that indeed
\[
\frac{\pi^{1/2}}{\Gamma(3/2)} = \frac{\sqrt{\pi}}{1/2 \cdot \Gamma(1/2)} = \frac{\sqrt{\pi}}{1/2 \cdot \sqrt{\pi}} = 2.
\]
For the induction step, assume that
\[
v_{n-2} = \frac{\pi^{(n-2)/2}}{\Gamma((n-2)/2 + 1)} = \frac{\pi^{(n-2)/2}}{\Gamma(n/2)}.
\]
Then
\[
v_n = \frac{\pi}{n/2} \cdot v_{n-2} = \frac{\pi}{n/2} \cdot \frac{\pi^{(n-2)/2}}{\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.
\]
This completes the induction.
Chapter 8

8.2.1. (a) Let $\alpha : I \rightarrow \mathbb{R}^n$ be a regular curve that doesn’t pass through the origin, but has a point $\alpha(t_0)$ of nearest approach to the origin. Show that the position vector $\alpha(t_0)$ and the velocity vector $\alpha'(t_0)$ are orthogonal.

The scalar-valued function $f(t) = |\alpha(t)|^2 = (\alpha(t), \alpha(t))$ has a minimum at $t_0$. Its derivative is

$$f'(t) = 2(\alpha(t), \alpha'(t)),$$

so that especially since $f'(t_0) = 0$,

$$\langle \alpha(t_0), \alpha'(t_0) \rangle = 0.$$

The previous display says precisely that $\alpha(t_0) \perp \alpha'(t_0)$ as desired. Yes, geometrically it is clear that at the point of nearest approach to the origin, the velocity is orthogonal to the position.

(b) Find a regular curve $\alpha : I \rightarrow \mathbb{R}^n$ that does not pass through the origin and does not have a point of nearest approach to the origin.

Let $I = (0, 1)$, and let $n = 1$, and let $\alpha$ be the identity map.

Does an example exist with $I$ compact?

No. If $I$ is compact then as in part (a), the continuous function $f(t) = |\alpha(t)|$ assumes a minimum.

8.2.2. Let $\alpha$ be a regular parametrized curve with $\alpha''(t) = 0$ for all $t \in I$. What is the nature of $\alpha$?

The curve $\alpha$ is a line. Since $\alpha''$ vanishes componentwise, $\alpha$ takes the form

$$\alpha(t) = (a_1 t + b_1, a_2 t + b_2, \ldots, a_n t + b_n).$$

That is, letting $d = (a_1, \ldots, a_n)$ and $p = (b_1, \ldots, b_n),

$$\alpha(t) = td + p.$$ 

Furthermore, since $\alpha'(t) = d$ for all $t$, it follows that $d \neq 0$ since $\alpha$ is regular. In sum, the trace of $\alpha$ is the line through the point $p$ having direction $d$.

8.2.3. Let $\alpha : I \rightarrow \mathbb{R}^n$ be a parametrized curve and let $v \in \mathbb{R}^n$ be a fixed vector. Assume that $\langle \alpha'(t), v \rangle = 0$ for all $t \in I$ and that $\langle \alpha(t_0), v \rangle = 0$ for some $t_0 \in I$. Prove that $\langle \alpha(t), v \rangle = 0$ for all $t \in I$. What is the geometric idea?

For any $t \in I$ we have

$$\langle \alpha(t), v \rangle' = \langle \alpha'(t), v \rangle + \langle \alpha(t), v' \rangle = 0 + \langle \alpha(t), 0 \rangle = 0 + 0 = 0.$$

Thus $\langle \alpha(t), v \rangle$ is constant, and since $\langle \alpha(t_0), v \rangle = 0$ the constant is 0. The geometric idea is that the trace of $\alpha$ lies in the hyperplane orthogonal to $v$.

8.3.2. The parametrized curve $\alpha : [0, +\infty) \rightarrow \mathbb{R}^2$, $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$

(where $a > 0$ and $b < 0$ are real constants) is called a logarithmic spiral.
(a) Show that as \( t \to +\infty \), \( \alpha(t) \) spirals in toward the origin.

Compute that

\[
\lim_{t \to \infty} \alpha(t) = \lim_{t \to \infty} (ae^{bt}\cos t, ae^{bt}\sin t) = (0, 0).
\]

The approach to the origin winds around and around because \( \alpha(t) \) is a positive scalar multiple of \((\cos t, \sin t)\).

(b) Show that as \( t \to +\infty \), \( L(0, t) \) remains bounded. Thus the spiral has finite length.

Compute that since

\[
|\alpha'(t)|^2 = |abe^{bt}(\cos t, \sin t) + ae^{bt}(-\sin t, \cos t)|^2 = a^2e^{2bt}(b^2 + 1),
\]

it follows that

\[
L(0, t) = \int_{\tau=0}^{t} |\alpha'(|\tau)| d\tau = a\sqrt{b^2 + 1} \int_{\tau=0}^{t} e^{b\tau} d\tau
= \frac{a}{b}\sqrt{b^2 + 1}(e^{bt} - 1) = \frac{a}{|b|}\sqrt{b^2 + 1}(1 - e^{bt}),
\]

so that

\[
\lim_{t \to \infty} L(0, t) = \frac{a}{|b|}\sqrt{b^2 + 1}.
\]

8.3.3. Explicitly reparametrize each curve \( \alpha : I \to \mathbb{R}^n \) with a curve \( \gamma : I' \to \mathbb{R}^n \) parametrized by arc length.

(a) The ray \( \alpha : \mathbb{R} > 0 \to \mathbb{R}^n \) given by \( \alpha(t) = t^2v \) where \( v \) is some fixed nonzero vector.

Letting \( t_0 = 1 \) gives for all \( t > 0 \),

\[
\ell(t) = \int_{\tau=1}^{t} |\alpha'(|\tau)| d\tau = \int_{\tau=1}^{t} |2\tau v| d\tau = |v| \int_{\tau=1}^{t} 2\tau d\tau = |v|(t^2 - 1).
\]

Thus solving the equation \( s = |v|(t^2 - 1) \) for \( t \) gives \( \ell^{-1}(s) = \sqrt{1 + s/|v|} \). The parametrization by arc length is consequently

\[
\gamma : (-|v|, \infty) \to \mathbb{R}^n, \quad \gamma(s) = (1 + s/|v|)v.
\]

(b) The circle \( \alpha : \mathbb{R} \to \mathbb{R}^2 \) given by \( \alpha(t) = (\cos e^t, \sin e^t) \).

Letting \( t_0 = 0 \) gives for all \( t \in \mathbb{R} \),

\[
\ell(t) = \int_{\tau=0}^{t} |\alpha'(|\tau)| d\tau = \int_{\tau=0}^{t} e^\tau d\tau = e^t - 1.
\]

Thus solving the equation \( s = e^t - 1 \) for \( t \) gives \( \ell^{-1}(s) = \ln(1 + s) \). The parametrization by arc length is consequently

\[
\gamma : (-1, \infty) \to \mathbb{R}^2, \quad \gamma(s) = (\cos(1 + s), \sin(1 + s)).
\]

(c) The helix \( \alpha : [0, 2\pi] \to \mathbb{R}^3 \) given by \( \alpha(t) = (a \cos t, a \sin t, bt) \).
Letting $t_0 = 0$ gives for all $t \in \mathbb{R}$,
\[
\ell(t) = \int_{t_0}^{t} |\alpha'(\tau)| \, d\tau = \sqrt{a^2 + b^2} \int_{t_0}^{t} d\tau = \sqrt{a^2 + b^2} \cdot t.
\]
Thus solving the equation $s = \sqrt{a^2 + b^2} \cdot t$ for $t$ gives $\ell^{-1}(s) = s/\sqrt{a^2 + b^2}$. The parametrization by arc length is consequently
\[
\gamma : [0, 2\pi/\sqrt{a^2 + b^2}] \rightarrow \mathbb{R}^3
\]
where
\[
\gamma(s) = (a \cos(s/\sqrt{a^2 + b^2}), a \sin(s/\sqrt{a^2 + b^2}), b s/\sqrt{a^2 + b^2}).
\]

8.4.1. (a) Let $a$ and $b$ be positive. Find the curvature of the ellipse $\alpha(t) = (a \cos(t), b \sin(t))$ for $t \in \mathbb{R}$.

Compute,
\[
\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}} = \frac{ab \sin^2 t + ab \cos^2 t}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}.
\]
Assuming that $a > b$, at $t = 0$ the curvature is $\kappa = a/b^2 > 1/a$ (because $a^2/b^2 > 1$), and at $t = \pi/2$ the curvature is $\kappa = b/a^2 < 1/b$. These results agree with the geometry: at the point $(a,0)$ the ellipse is bending inside its tangent circle of radius $a$, and at the point $(0,b)$ the ellipse is bending outside its tangent circle of radius $b$.

(b) Let $a$ be positive and $b$ be negative. Find the curvature of the logarithmic spiral $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$ for $t \geq 0$.

In the formula
\[
\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}
\]
we have
\[
x'y'' = a^2 e^{2bt} (b \cos t - \sin t)((b^2 - 1) \sin t + 2b \cos t)
\]
and
\[
x''y' = a^2 e^{2bt} ((b^2 - 1) \cos t - 2b \sin t)(b \sin t + \cos t),
\]
so that (after a while)
\[
x''y' - x'y'' = (1 + b^2)a^2 e^{2bt}.
\]

Also,
\[
x'^2 + y'^2 = a^2 e^{2bt} (b^2 \cos^2 t + \sin^2 t + b^2 \sin^2 t + \cos^2 t) = a^2 e^{2bt}(1 + b^2)
\]
Thus the curvature is
\[
\kappa = \frac{(1 + b^2)a^2 e^{2bt}}{a^3 e^{3bt}(1 + b^2)^{3/2}} = \frac{1}{ae^{bt}(1 + b^2)^{1/2}} = \frac{e^{b t}}{a(1 + b^2)^{1/2}}.
\]
8.4.2. Let \( \gamma : I \rightarrow \mathbb{R}^2 \) be parametrized by arc length. Fix any unit vector \( v \in \mathbb{R}^2 \), and define a function 
\[ \theta : I \rightarrow \mathbb{R} \]
by the conditions
\[ \cos(\theta(s)) = \langle T(s), v \rangle, \quad \sin(\theta(s)) = -\langle N(s), v \rangle. \]
Thus \( \theta \) is the angle that the curve \( \gamma \) makes with the fixed direction \( v \). Show that \( \theta' = \kappa \). Thus our notion of curvature does indeed measure the rate at which \( \gamma \) is turning.

Differentiate the first condition to get
\[ -\sin(\theta(s)) \theta'(s) = \langle T'(s), v \rangle. \]

The second condition says that \( -\sin(\theta(s)) = \langle N(s), v \rangle \), and the Frenet equations say that \( T'(s) = \kappa(s)N(s) \). Thus
\[ \langle N(s), v \rangle \theta'(s) = \kappa(s)\langle N(s), v \rangle. \]
So long as \( \langle N(s), v \rangle \neq 0 \) we may cancel to get the result. In the exceptional case, proceed similarly but start by differentiating the second condition. This time the factor that we want to cancel will be \( \langle T(s), v \rangle \), and this is nonzero when \( \langle N(s), v \rangle = 0 \).

8.5.1. (a) Let \( a \) and \( b \) be positive. Compute the curvature \( \kappa \) and the torsion \( \tau \) of the helix \( \alpha(t) = (a \cos t, a \sin t, bt) \).

Routine calculations give
\[ \kappa = \frac{a}{a^2 + b^2} \quad \text{and} \quad \tau = \frac{b}{a^2 + b^2}. \]

(b) How do \( \kappa \) and \( \tau \) behave if \( a \) is held constant and \( b \to \infty \)?
Here \( \kappa \to 0 \) and \( \tau \to 0 \). These results are sensible since the helix is tending to a vertical line through \( (a,0,0) \).

(c) How do \( \kappa \) and \( \tau \) behave if \( a \) is held constant and \( b \to 0 \)?
Here \( \kappa \to 1/a \) and \( \tau \to 0 \). These results are sensible since the helix is tending to a circle of radius \( a \).

(d) How do \( \kappa \) and \( \tau \) behave if \( b \) is held constant and \( a \to \infty \)?
Here \( \kappa \to 0 \) and \( \tau \to 0 \). These results are perhaps sensible since the helix is acquiring an ever-larger radius but its pitch is being held constant.

(e) How do \( \kappa \) and \( \tau \) behave if \( b \) is held constant and \( a \to 0 \)?
Here \( \kappa \to 0 \) and \( \tau \to 1/b \). These results are not immediately intuitive to me. The helix is tending to a vertical line through \( (0,0,0) \), but somehow even in the limit it is twisting at a rate reciprocal to the pitch.
Chapter 9

9.1.1. Consider two vectors \( u = (x_u, y_u, z_u) \) and \( v = (x_v, y_v, z_v) \). Calculate that 
\[ |u|^2|v|^2 - (u \cdot v)^2 = |u \times v|^2. \]
Compute that
\[
|u|^2|v|^2 = (x_u^2 + y_u^2 + z_u^2)(x_v^2 + y_v^2 + z_v^2)
\]
\[
= x_u^2x_v^2 + x_u^2y_v^2 + x_u^2z_v^2 + y_u^2x_v^2 + y_u^2y_v^2 + y_u^2z_v^2 + z_u^2x_v^2 + z_u^2y_v^2 + z_u^2z_v^2,
\]
and that
\[
(u \cdot v)^2 = (x_u x_v + y_u y_v + z_u z_v)^2
\]
\[
= x_u^2 x_v^2 + x_u x_v y_u y_v + x_u x_v z_u z_v + y_u y_v x_u x_v + y_u y_v y_u y_v + y_u y_v z_u z_v + z_u z_v x_u x_v + z_u z_v y_u y_v + z_u z_v z_u z_v.
\]
Thus the difference is
\[
|u|^2|v|^2 - (u \cdot v)^2 = x_u^2 y_v^2 - 2x_u y_u y_v x_v + y_u^2 x_v^2
\]
\[
+ x_u^2 y_v^2 - 2x_u z_u z_v x_v + z_u^2 x_v^2
+ y_u^2 y_v^2 - 2y_u z_u z_v y_v + z_u^2 y_v^2
= (x_u y_v - y_u x_v)^2 + (x_u z_v - z_u x_v)^2 + (y_u z_v - z_u y_v)^2
= |u \times v|^2.
\]

9.1.3. Let \( f(x, y, z) = x^2 + yz \).
(a) Integrate \( f \) over the box \( B = [0, 1]^3 \).
This is an integral of the type studied in chapter 6, so Fubini’s Theorem applies immediately,
\[
\int_B f = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (x^2 + yz) = \int_{x=0}^{1} \int_{y=0}^{1} (x^2 z + \frac{1}{2} yz^2)_{z=0}^{1}
= \int_{x=0}^{1} \int_{y=0}^{1} (x^2 + \frac{1}{2} y) = \int_{x=0}^{1} (x^2 y + \frac{1}{4} y^2)_{y=0}^{1}
= \int_{x=0}^{1} (x^2 + \frac{1}{4}) = (\frac{1}{3} x^3 + \frac{1}{4} x)_{x=0}^{1} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.
\]
(b) Integrate \( f \) over the parametrized curve
\[
\gamma : [0, 2\pi] \rightarrow \mathbb{R}^3, \quad \gamma(t) = (\cos t, \sin t, t).
\]

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Since a curve is a 1-surface, Definition 9.1.3 specializes to
\[
\int_{\gamma} f = \int_{t=0}^{2\pi} (f \circ \gamma) \text{length}(\gamma') = \int_{t=0}^{2\pi} (f \circ \gamma) |\gamma'|.
\]
Compute that
\[(f \circ \gamma)(t) = \cos^2 t + t \sin t,
\]
and that
\[|\gamma'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}.
\]
Therefore the integral is
\[
\int_{\gamma} f = \sqrt{2} \int_{t=0}^{2\pi} (\cos^2 t + t \sin t)
\]
A standard trick is that since sine and cosine are translates and both have period 2\pi,
\[
\int_{0}^{2\pi} \cos^2 t = \int_{0}^{2\pi} \sin^2 t = \frac{1}{2} \int_{0}^{2\pi} (\cos^2 + \sin^2) = \frac{1}{2} \int_{0}^{2\pi} 1 = \pi.
\]
And integration by parts with \(u = t\) and \(v' = \sin t\) gives
\[
\int_{t=0}^{2\pi} t \sin t = -t \cos t|_{t=0}^{2\pi} + \int_{t=0}^{2\pi} \cos t = -2\pi.
\]
Thus the entire integral is
\[
\int_{\gamma} f = \sqrt{2}(\pi - 2\pi) = -\sqrt{2} \pi.
\]
(c) Integrate \(f\) over the parametrized surface
\[S : [0,1]^2 \rightarrow \mathbb{R}^3, \quad S(u,v) = (u + v, u - v, v).
\]
The derivative matrix of \(S\) is
\[
S'(u,v) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}.
\]
Therefore,
\[
S'(u,v)^t S'(u,v) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.
\]
This has determinant 6, showing that the volume factor in Definition 9.1.3 is
\[
\text{area}(P(D_1S, D_2S)) = \sqrt{6}.
\]
(One can compute this factor in other ways as well, since we are in the particular case of $k = 2$ and $n = 3$.) So the integral is

$$\int_S f = \sqrt{6} \int_{[0,1]^2} (f \circ S)$$

But $(f \circ S)(u, v) = (u + v)^2 + (u - v)v = u^2 + 3uv$, and so the integral is

$$\int_S f = \sqrt{6} \int_{u=0}^{1} \int_{v=0}^{1} (u^2 + 3uv) = \sqrt{6} \int_{u=0}^{1} (u^2 + \frac{3}{2}u) = \sqrt{6}(\frac{1}{3} + \frac{3}{4}) = \frac{13\sqrt{6}}{12}.$$

(d) Integrate $f$ over the parametrized solid

$$V : [0,1]^3 \rightarrow \mathbb{R}^3, \quad V(u, v, w) = (u + v, v - w, u + w).$$

The derivative matrix of $V$ is

$$V'(u, v, w) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$ 

This has determinant 0, and since the volume factor in Definition 9.1.3 is

$$\text{vol}(P(D_1V, D_2V, D_3V)) = |\det V'| = 0,$$

the integral is $0$.

(In parts (b), (c), and (d) of this exercise, the volume factor worked out to a constant, and the constant even was 0 for (d). This is all flukish: in general the volume factor depends on the parameters.)

9.2.2. Derive equations (9.6) and (9.8) from equation (9.12).

To get (9.6), substitute $n = 2$ and write out the terms. To get (9.8), substitute $n = 3$ and write out the terms.