Mathematics 211: Homework Solutions

Preface

0.0.1. (a) *Consider two surfaces in space, each surface having a tangent plane and therefore a normal line at each of its points, and consider pairs of points, one on each surface. Conjecture a geometric condition, phrased in terms of tangent planes and/or normal lines, about the closest pair of points.*

There needn't be a closest pair of points at all. If there is a closest pair of points, it needn't be unique. If the two surfaces meet then a shared point certainly is a "closest pair," but no particular geometric condition need hold at the point. All of this detritus aside, the case of interest is when the two surfaces don't meet and there is at least one closest pair of points.

In this case, call the surfaces A and B, and call the points a and b. Geometric intuition says that the line containing the two points a and b is normal to both surfaces A and B. So:

- The normal line to A at a is equal to the normal line to B at b.

Consequently:

- The tangent planes to A at a and to B at b are parallel.
- The normal line to A at a is orthogonal to the tangent plane to B at b, and conversely.

(Here and elsewhere in this answer, "normal" and "orthogonal" are synonyms, each being used in different places to try to make the ideas easier to understand.) Note that the first bullet is a stronger condition than the second and the third. Also note that in all three cases, the geometric condition is necessary but not sufficient. For example, it also applies to the farthest pair of points, and it can apply to pairs of points that are neither nearest nor farthest. By analogy from one-variable calculus, a horizontal tangent is necessary at each maximum and each minimum of a smooth curve (not thinking about endpoints), but a horizontal tangent does not imply a maximum or a minimum.

In this context, it is worth mentioning that the ideas of "parallel" and "orthogonal" deserve some rethinking in space. In the plane, the condition that two lines never meet and the condition that two lines are everywhere equidistant from one another mean the same thing, and this is our notion of parallel. But in space, the first condition is weaker, the second stronger: two lines in space can be *skew*, meaning that they never meet but nor are they parallel in the sense of being everywhere equidistant. Similarly, for a pair of lines in space to be orthogonal, not only should their directions form a right angle, but the lines should meet. In the plane, every two nonparallel lines meet, and so this is a nonissue.

Two planes in space are equal, or they are parallel, or they share one line. When they share one line, we can visualize them in a cross-sectional plane at right angles to the shared line, and what we see of the planes is two lines crossing, a sort of “X” shape. The planes are orthogonal when the two lines of the “X” are orthogonal in the planar sense. Note how this process reduces a three-dimensional issue to two dimensions.

(b) *Consider a surface in space and a curve in space, the curve having a tangent line and therefore a normal plane at each of its points, and consider pairs of points, one on the surface and one on the curve. Make a conjecture about the nearest pair of points.*

After again making all of the disclaimers, the condition is that the line containing the two points a and b is normal to the surface A and to the curve B. One can reach four conclusions from this, the first two stronger than the last two, but in any case all necessary rather than sufficient:

- The normal line to A at a lies in the normal plane to B at b.
- The normal line to A at a is orthogonal to the tangent line to B at b.
- The tangent plane to A at a is orthogonal to the normal plane to B at b.
- The tangent plane to A at a contains a line parallel to the tangent line to B at b.

(c) *Make a conjecture about the nearest pair of points on two curves.*

The key beginning observation is again that the line between the two points a and b is normal to both curves A and B. In the terms of the problem, this can be expressed in three ways:

- The normal planes to A at a and to B at b share a line.
- The tangent lines to A at a and to B at b are skew and possibly parallel.
- The normal plane to A at a contains a line orthogonal to the tangent line to B at b, and conversely.

You might think about whether any of these conditions is stronger than any other, but in any case none of them is sufficient.

The answers to (a) and (c) have three phrasings each, while the answer to (b) has four. Why is this?

0.0.2. (a) *Assume that the factorial of a half-integer makes sense, and grant the general formula for the volume of a ball in n dimensions. Explain why it follows that* $(1/2)! = \sqrt{\pi}/2$.

The general formula is

$$\operatorname{vol}(B_n(r)) = \frac{\pi^{n/2}}{(n/2)!}\, r^n, \quad n = 1, 2, 3, 4, \dots$$

So in particular, for $n = 1$ we have

$$2r = \frac{\sqrt{\pi}}{(1/2)!}\, r,$$

and the result follows by basic algebra.

Further assume that the half-integral factorial function satisfies the relation

$$x! = x \cdot (x-1)! \quad \textit{for } x = 3/2, 5/2, 7/2, \ldots$$

Subject to these assumptions, verify that the volume of the ball of radius r in three dimensions is $\frac{4}{3}\pi r^3$ as claimed.

The formula for $n = 3$ is

$$\begin{aligned} \operatorname{vol}(B_3(r)) &= \frac{\pi^{3/2}}{(3/2)!}\, r^3 = \frac{\pi^{3/2}}{(3/2)\cdot(1/2)!}\, r^3 \\ &= \frac{\pi^{3/2}}{(3/2)\cdot\sqrt{\pi}/2}\, r^3 = \frac{\pi}{(3/4)}\, r^3 = \boxed{\frac{4}{3}\pi r^3}. \end{aligned}$$

What is the volume of the ball of radius r in five dimensions?

Similarly,

$$\begin{aligned} \operatorname{vol}(B_5(r)) &= \frac{\pi^{5/2}}{(5/2)!}\, r^5 = \frac{\pi^{5/2}}{(5/2)\cdot(3/2)\cdot\sqrt{\pi}/2}\, r^5 \\ &= \frac{\pi^2}{(15/8)}\, r^5 = \boxed{\frac{8}{15}\pi^2 r^5}. \end{aligned}$$

(b) *The ball of radius r in n dimensions sits inside a circumscribing box of sides $2r$. Draw pictures of this configuration for $n = 1, 2, 3$. Determine what portion of the box is filled by the ball in the limit as the dimension n gets large. That is, find*

$$\lim_{n\to\infty} \frac{\operatorname{vol}(B_n(r))}{(2r)^n}.$$

The limit is

$$\lim_{n\to\infty} \frac{\operatorname{vol}(B_n(r))}{(2r)^n} = \lim_{n\to\infty} \frac{\pi^{n/2} r^n/(n/2)!}{(2r)^n} = \lim_{n\to\infty} \frac{(\sqrt{\pi}/2)^n}{(n/2)!}.$$

Thus, perhaps surprisingly, the limit is zero. As the dimension grows large, the ball tends to filling an insignificant portion of its circumscribing box.

Chapter 1

1.1.2. *Prove that in any ordered field, $0 < 1$.*

We know that $0 \neq 1$ because 0 and 1 are assumed to be distinct. So either 1 is positive or 1 is negative. In the latter case, -1 is positive; but then $1 = (-1)(-1)$

is positive as well, contradicting the assumption that 1 is negative. So 1 is positive, i.e., $1 > 0$.

Prove that the complex number field $\mathbf{C}$ *can not be made an ordered field.*

First we argue that nonzero squares are positive in any ordered field. Let b be a nonzero square. Then $b = a^2$ where $a \neq 0$ (else $b = 0$). If $a < 0$ then $-a > 0$, and also $b = (-a)^2$, so in any case, b is the square of a positive element and hence b is positive.

But in $\mathbf{C}$ we have that $-1 = i^2$ is a nonzero square, so that if $\mathbf{C}$ is an ordered field then $-1 > 0$. This is incompatible with part (a).

1.1.4. (a) *Prove by induction that*

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{for all } n \in \mathbf{Z}^+.$$

For each $n \in \mathbf{Z}^+$ let $P(n)$ be the proposition that the displayed equality holds for that value of n. It suffices to show that $P(1)$ holds and that for each $n \in \mathbf{Z}^+$, if $P(n)$ holds then consequently so does $P(n+1)$.

To confirm $P(1)$, simply compute that

$$\sum_{i=1}^{1} i^2 = 1^2 = 1 \quad \text{and} \quad \frac{1(1+1)(2\cdot 1+1)}{6} = 1.$$

Thus the two sides of the desired equality are indeed equal.

Now let $n \in \mathbf{Z}^+$ be arbitrary, and assume that $P(n)$ holds. Compute that therefore

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 && \text{by definition of summation} \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 && \text{since } P(n) \text{ holds} \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} && \text{by adding fractions} \\
&= \frac{(n+1)(2n^2+7n+6)}{6} && \text{by algebra} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} && \text{by algebra} \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} && \text{by algebra.}
\end{aligned}$$

This establishes $P(n+1)$, completing the induction.

(b) *For any real number* $r \geq -1$, *prove* **Bernoulli's Inequality** *by induction,*

$$(1+r)^n \geq 1 + rn \quad \text{for all } n \in \mathbf{N}.$$

Here the wrinkle is that the problem involves the real variable r. The idea is to carry out induction on n independently of the value of r, using only the condition that $r \geq -1$. For $n = 0$, compute that

$$(1+r)^0 = 1 \quad \text{and} \quad 1 + r \cdot 0 = 1,$$

so indeed $(1+r)^n \geq 1+nr$ when $n = 0$. Here the fact that $r \geq -1$ is irrelevant. Next, assume that the inequality holds for some $n \in \mathbf{N}$, and compute that therefore,

$$\begin{aligned}
(1+r)^{n+1} &= (1+r)^n(1+r) && \text{by definition of the } (n+1)\text{st power} \\
&\geq (1+nr)(1+r) && \text{since } (1+r)^n \geq 1+nr \text{ and since } 1+r \geq 0 \\
&= 1+(n+1)r+nr^2 && \text{by algebra} \\
&\geq 1+(n+1)r && \text{since } nr^2 \geq 0 \text{ regardless of the value of } r.
\end{aligned}$$

(Note where the condition $r \geq -1$ is used.) This establishes the equality for $n+1$, completing the induction.

(c) *For what positive integers n is $2^n > n^3$?*

Although the problem didn't specificallly request a proof, it should go without saying that one is called for, especially since the answer here is a bit tricky.

The idea is that 2^n grows exponentially in n, while n^3 grows as a polynomial in n. Thus we expect the left side to be larger eventually. However, although the inequality $2^n > n^3$ holds for $n = 1$, it fails for $n = 2$ and then continues to fail for several more values of n. Nonetheless, it becomes true again at $n = 10$ since $2^{10} = 1024 > 1000 = 10^3$, and we expect it to remain true from then on. So we do an induction argument starting at $n = 10$: Suppose that $n \geq 10$ and that $2^n > n^3$. Then

$$\begin{aligned}
2^{n+1} &= 2 \cdot 2^n && \text{by definition of the } (n+1)\text{st power} \\
&> 2n^3 && \text{since } 2^n > n^3 \\
&= n^3 + n \cdot n^2 && \text{by algebra} \\
&> n^3 + 3n^2 + 3n^2 + n^2 && \text{since } n > 7 \\
&> n^3 + 3n^2 + 3n + 1 && \text{since } n > 1, \text{ making } n^2 > n \text{ and } n^2 > 1 \\
&= (n+1)^3 && \text{by algebra.}
\end{aligned}$$

This completes the induction. Note that the induction step works only for $n \geq 7$, so that an induction starting at $n = 1$ fails, and the base case fails at $n = 7$, so that an induction starting at $n = 7$ fails. The smallest starting value of n for which the induction can succeed is $n = 10$.

1.2.1. *Use the Intermediate Value Theorem to show that 2 has a positive square root.*

Define a function

$$f : [1,2] \longrightarrow \mathbf{R}, \quad f(x) = x^2.$$

This function is continuous. Note that $f(1) = 1 < 2$ and $f(2) = 4 > 2$. By the Intermediate Value Theorem $f(s) = 2$ for some s between 1 and 2. This value s is a positive square root of 2.

One can not use $f(x) = \sqrt{x}$ to solve this problem, as this function's very existence presupposes that square roots exist in the first place.

1.2.2. *Let $f : [0,1] \longrightarrow [0,1]$ be continuous. Use the Intermediate Value Theorem to show that $f(x) = x$ for some $x \in [0,1]$.*

This problem asks us to prove that the graph of f meets a diagonal line. On the other hand, the Intermediate Value Theorem says that under certain conditions the graph of a function g meets a horizontal line. So the idea is to reduce the diagonal problem to the horizontal one.

Let $g : [0,1] \longrightarrow \mathbf{R}$ be $g(x) = f(x) - x$. The original problem is equivalent to showing that $g(x) = 0$ for some x. The function g is continuous since the function f and the identity function are continuous and the difference of any two continuous functions is again continuous. Compute that since the values of $f(x)$ lie in $[0,1]$,

$$g(0) = f(0) - 0 \geq 0 \quad \text{and} \quad g(1) = f(1) - 1 \leq 0.$$

If $g(0) = 0$ or if $g(1) = 0$ then we are done. Otherwise, $g(0) > 0$ and $g(1) < 0$, so g changes sign, and so by the Intermediate Value Theorem $g(x) = 0$ for some $x \in [0,1]$, and again we are done.

1.2.3. *Let a and b be real numbers with $a < b$. Suppose that $f : [a,b] \longrightarrow \mathbf{R}$ is continuous and that f is differentiable on the open subinterval (a,b). Use the Mean Value Theorem to show that if $f' > 0$ on (a,b) then f is strictly increasing on $[a,b]$.*

Let x and x' lie in $[a,b]$, with $x' > x$. Then f is continuous on $[x,x']$ and differentiable on (x,x'). By the Mean Value Theorem,

$$\frac{f(x') - f(x)}{x' - x} = f'(c) \quad \text{for some } c \in (x,x').$$

Since $f' > 0$ on (a,b), this means that the quotient in the display is positive. Since the denominator $x' - x$ is also positive, it follows that so is the numerator, $f(x') - f(x) > 0$. That is,

$$\text{For all } x, x' \in [a,b], \quad x' > x \implies f(x') > f(x).$$

This is the desired result.

1.3.1. As (essentially) shown in class, if n is odd then the nth degree Taylor polynomial for sin at 0 is

$$T_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{(n-1)/2}\frac{x^n}{n!}.$$

If n is even then the polynomial stops at $n-1$ instead,

$$T_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{(n-2)/2}\frac{x^{n-1}}{(n-1)!}.$$

If n is odd, the remainder is

$$R_n(x) = \pm\frac{\sin(c)x^{n+1}}{(n+1)!} \quad \text{for some } c \text{ between } 0 \text{ and } x,$$

and if n is even, the remainder is

$$R_n(x) = \pm\frac{\cos(c)x^{n+1}}{(n+1)!} \quad \text{for some } c \text{ between } 0 \text{ and } x.$$

In either case, since $|\sin(c)| \le 1$ and $|\cos(c)| \le 1$ for all c,

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}.$$

For any fixed x, as $n \to \infty$, the factorial $(n+1)!$ dominates the exponential $|x|^{n+1}$, giving $\lim_{n\to\infty} R_n(x) = 0$. Therefore, since $f(x) = T_n(x) + R_n(x)$,

$$\lim_{n\to\infty} T_n(x) = f(x).$$

If n is even then the nth degree Taylor polynomial for cos at 0 is

$$T_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n/2}\frac{x^n}{n!}.$$

If n is odd then the polynomial stops at $n-1$ instead,

$$T_n(x) = 1 - \frac{2^3}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{(n-1)/2}\frac{x^{n-1}}{(n-1)!}.$$

The roles of sine and cosine are interchanged in the ensuing analysis of the remainder, but as before, all that matters is the estimates $|\sin(c)| \le 1$ and $|\cos(c)| \le 1$, after which the argument is identical.

1.3.2. (a) *What is the nth degree Taylor polynomial $T_n(x)$ for the function $f(x) = \arctan x$ at 0?*

The derivative of $\arctan x$ is $1/(1+x^2)$. There are several ways to proceed from here:

1. We can start computing derivatives of $1/(1+x^2)$ and quickly get discouraged by the messy algebra.

2. We can carry out a partial fractions analysis of $1/(1+x^2)$, using the complex numbers,

$$\frac{1}{1+x^2} = \frac{1}{2}\left(\frac{1}{1-ix} + \frac{1}{1+ix}\right),$$

and then compute the derivatives of the right side a little more systematically:

k	$f^{(k)}(x)$	$\dfrac{f^{(k)}(0)}{k!}$
0	$\arctan x$	0
1	$\frac{1}{2}\left(\frac{1}{1-ix}+\frac{1}{1+ix}\right)$	1
2	$\frac{1}{2}\left(\frac{i}{(1-ix)^2}-\frac{i}{(1+ix)^2}\right)$	0
3	$\frac{1}{2}\left(\frac{2i^2}{(1-ix)^3}+\frac{2i^2}{(1+ix)^3}\right)$	$-\frac{1}{3}$
4	$\frac{1}{2}\left(\frac{3!i^3}{(1-ix)^4}-\frac{3!i^3}{(1+ix)^4}\right)$	0
5	$\frac{1}{2}\left(\frac{4!i^4}{(1-ix)^5}+\frac{4!i^4}{(1+ix)^5}\right)$	$\frac{1}{5}$
$\vdots$	$\vdots$	$\vdots$

Thus the Taylor polynomials are the truncations of the series

$$T(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

3. We can recognize the derivative as a geometric series $1/(1-r)$ where $r = -x^2$. Then by the geometric series formula,

$$\arctan' x = 1 - x^2 + x^4 - x^6 + \cdots .$$

This expansion is valid for $|r| < 1$, i.e., for $|-x^2| < 1$, i.e., for $|x| < 1$. Integrating term-by-term gives that for some constant C,

$$\arctan x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

We hope that this is still valid for $|x| < 1$, but we don't know that it is unless we invoke the Differentiation Theorem from the end of Ray Mayer's Math 112 notes, and somehow that is too powerful a tool for this problem. Proceeding blithely on in any case, substitute $x = 0$ to see that $C = 0$, giving the same answer as in method 2. Although this method is appealing by virtue of involving by far the least algebra, it either has a logical gap or relies on a substantial theorem.

(b) $f(x) = (1+x)^\alpha$ *where* $\alpha \in \mathbf{R}$.
The first few derivatives of f are

$$\begin{aligned} f(x) &= (1+x)^\alpha, \\ f'(x) &= \alpha(1+x)^{\alpha-1}, \\ f(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, \end{aligned}$$

and so on. That is, for each $k \in \mathbf{N}$,

$$f^{(k)}(x) = \alpha(\alpha-1)\cdots(\alpha-k+1)(1+x)^{\alpha-k},$$

For convenience introduce the binomial coefficient symbol,

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad \text{for } \alpha \in \mathbf{R} \text{ and } k \in \mathbf{N}.$$

(By definition, this means 1 when $k = 0$.) Then for each $k \in \mathbf{N}$,

$$\frac{f^{(k)}(0)}{k!} = \binom{\alpha}{k},$$

and thus the nth degree Taylor polynomial is

$$T_n(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + \binom{\alpha}{n}x^n.$$

The interesting feature of this problem is that *if* $\alpha \in \mathbf{N}$ (*i.e.,* α *is* 0 *or* 1 *or* 2 *or...*) *then the binomial coefficient* (α, k) *is* 0 *for all* $k > \alpha$. This is because the product in its numerator contains a 0. In this case we recover the binomial theorem from high school algebra: For every natural number $\alpha \in \mathbf{N}$ there is a finite expansion of $(1+x)^\alpha$,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + \binom{\alpha}{\alpha}x^\alpha.$$

For example,

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

On the other hand, if $\alpha \notin \mathbf{N}$ then the Taylor series is infinite. For example, the Taylor series for $(1+x)^{1/2}$ is

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots.$$

1.3.3 *In figure 1.1, identify the graphs of* T_1 *through* T_5 *and the graph of* ln *near* $x = 0$ *and near* $x = 2$.

We have

$$\begin{aligned}
T_1(x) &= x - 1, \\
T_2(x) &= (x-1) - \frac{(x-1)^2}{2}, \\
T_3(x) &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}, \\
T_4(x) &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}, \\
T_4(x) &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}.
\end{aligned}$$

At $x = 0$ the function $\ln(x)$ is undefined (and $\ln(x)$ tends to $-\infty$ as $x \to 0$), but the Taylor polynomials are perfectly well behaved. Specifically,

$$\begin{aligned}
T_1(0) &= -1, \\
T_2(0) &= -1 - \frac{1}{2}, \\
T_3(0) &= -1 - \frac{1}{2} - \frac{1}{3}, \\
T_4(0) &= -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}, \\
T_5(0) &= -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5}.
\end{aligned}$$

Thus toward the left of the figure, the graphs from top to bottom show $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$, $T_5(x)$, and then $\ln(x)$

At $x = 2$ the function $\ln(2)$, the Taylor polynomials take the values

$$\begin{aligned}
T_1(2) &= 1, \\
T_2(2) &= 1 - \frac{1}{2}, \\
T_3(2) &= 1 - \frac{1}{2} + \frac{1}{3}, \\
T_4(2) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}, \\
T_5(2) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}.
\end{aligned}$$

That is, $T_1(2)$ is the greatest, $T_2(2)$ drops to the lowest, $T_3(2)$ climbs back to the second-greatest, $T_4(2)$ drops back to the second-lowest, and then $T_5(2)$ climbs again, with the Taylor polynomial values oscillating around the actual value $\ln(2)$. Thus toward the right of the figure, the graphs from top to bottom show $T_1(x)$, $T_3(x)$, $T_5(x)$, $\ln(x)$, $T_4(x)$, and finally $T_2(x)$.

1.3.5 *Use a second degree Taylor polynomial to approximate $\sqrt{4.2}$ and to estimate the accuracy of the approximation.*

Let $f(x) = \sqrt{x}$ on the interval $[0, \infty)$, and let $a = 4$. Then

$$f(x) = x^{1/2}, \quad f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2}, \quad f'''(x) = \frac{3}{8}x^{-5/2}.$$

It follows that

$$f(a) = 2, \quad f'(a) = \frac{1}{4}, \quad \frac{f''(a)}{2!} = -\frac{1}{64},$$

and

$$\frac{f'''(c)}{3!} = \frac{1}{16c^{5/2}}.$$

Thus the second degree Taylor polynomial is

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2,$$

and the remainder is

$$R_2(x) = \frac{1}{16c^{5/2}}(x-4)^3 \quad \text{for some } c \text{ between 4 and } x.$$

Substitute $x = 4.2$:

$$T_2(4.2) = 2 + \frac{1}{4}(0.2) - \frac{1}{64}(0.2)^2 = 2 + \frac{1}{20} - \frac{1}{1600} = 2.049375,$$

and

$$|R_2(4.2)| \le \frac{(0.2)^3}{16 \cdot 32} = \frac{1}{512 \cdot 125} < \frac{1}{50000} = 0.00002.$$

These calculations show that

$$2.049355 < \sqrt{4.2} < 2.049395.$$

Chapter 2

2.1.3. *Verify that $\mathbf{R}^n$ satisfies vector space axioms* (A2), (A3), (D1).

These are very similar to the argument in the text for (M1), each reducing to its field axiom counterpart. For (A2), let $x = (x_1, \dots, x_n)$ and likewise for y and z. Then

$$\begin{aligned}
&(x+y)+z \\
&\quad = ((x_1,\dots,x_n)+(y_1,\dots,y_n))+(z_1,\dots,z_n) && \text{by definition of } x,\ y, \text{ and } z \\
&\quad = (x_1+y_1,\dots,x_n+y_n)+(z_1,\dots,z_n) && \text{by definition of vector addition} \\
&\quad = ((x_1+y_1)+z_1,\dots,(x_n+y_n)+z_n) && \text{by definition of vector addition} \\
&\quad = (x_1+(y_1+z_1),\dots,x_n+(y_n+z_n)) && \text{by } n \text{ applications of (a1) in } \mathbf{R} \\
&\quad = (x_1,\dots,x_n)+(y_1+z_1,\dots,y_n+z_n) && \text{by definition of vector addition} \\
&\quad = x+(y+z) && \text{by definition of vector addition.}
\end{aligned}$$

The other two proofs are virtually identical.

2.1.5. (*Throughout this exercise, the solution can proceed in coordinates by reducing problems in $\mathbf{R}^n$ to problems in $\mathbf{R}$ that have been solved, or it can proceed intrinsically by repeating in $\mathbf{R}^n$ the symbol-pattern of the solution from $\mathbf{R}$ with no reference to coordinates.*)

Show that $\mathbf{0}$ is the unique additive identity in $\mathbf{R}^n$.

A solution that reduces to $\mathbf{R}$ is as follows: Let $z = (z_1, \dots, z_n)$ be an additive identity in $\mathbf{R}^n$. That is, $x+z = x$ for all $x \in \mathbf{R}^n$. In terms of component scalars, this means that

$$(x_1+z_1, \dots, x_n+z_n) = (x_1, \dots, x_n) \quad \text{for all } x_1, \dots, x_n \in \mathbf{R}.$$

Thus each of $z_1, \dots, z_n$ is an additive identity in $\mathbf{R}$. But 0 is the unique additive identity in $\mathbf{R}$, so each of $z_1, \dots, z_n$ is 0, and therefore $z = \mathbf{0}$ as desired.

A solution that repeats the symbol-pattern of the solution in $\mathbf{R}$ is as follows: Let z be an additive identity in $\mathbf{R}^n$. Then

$$\begin{aligned} z &= z + \mathbf{0} && \text{since } \mathbf{0} \text{ is an additive identity} \\ &= \mathbf{0} + z && \text{since vector addition is commutative} \\ &= \mathbf{0} && \text{since } z \text{ is an additive identity.} \end{aligned}$$

Show that each vector $x \in \mathbf{R}^n$ has a unique additive inverse, which can therefore be denoted $-x$.

This is very similar. A solution that reduces to $\mathbf{R}$ is as follows: If $y = (y_1, \ldots, y_n)$ is an additive inverse of x then the vector equation $x + y = \mathbf{0}$ is

$$(x_1 + y_1, \ldots, x_n + y_n) = (0, \ldots, 0).$$

This forces each y_j to be an additive inverse of x_j in $\mathbf{R}$, determining y_j uniquely since additive inverses are unique in $\mathbf{R}$. Thus y is determined uniquely.

A solution that repeats the symbol-pattern of the solution in $\mathbf{R}$ is as follows: Let y and z additive inverses of x. Then

$$\begin{aligned} y &= y + \mathbf{0} && \text{since } \mathbf{0} \text{ is the additive identity} \\ &= y + (x + z) && \text{since } z \text{ is an additive inverse of } x \\ &= (y + x) + z && \text{since vector addition is associative} \\ &= (x + y) + z && \text{since vector addition is commutative} \\ &= \mathbf{0} + z && \text{since } y \text{ is an additive inverse of } x \\ &= z + \mathbf{0} && \text{since vector addition is commutative} \\ &= z && \text{since } \mathbf{0} \text{ is the additive identity.} \end{aligned}$$

Show that $0x = \mathbf{0}$ for all $x \in \mathbf{R}^n$.

A solution that reduces to $\mathbf{R}$ is as follows: Let $x = (x_1, \ldots, x_n)$. Then by definition of scalar multiplication,

$$0x = (0x_1, \ldots, 0x_n),$$

and this is $(0, \ldots, 0)$ since we know that 0 is a multiplicative annihilator in $\mathbf{R}$.

A solution that repeats the symbol-pattern of the solution in $\mathbf{R}$ is as follows: Compute that

$$\begin{aligned} 0x &= (0 + 0)x && \text{since } 0 \text{ is the additive identity in } \mathbf{R} \\ &= 0x + 0x && \text{by the distributivity axiom (D1) for } \mathbf{R}^n. \end{aligned}$$

Add the additive inverse of $0x$ to both sides to get $\mathbf{0} = 0x$.

2.1.7. *Show the uniqueness of additive identity and additive inverse using only* (A1), (A2), (A3).

To show uniqueness of additive identity, suppose that 0 and $0'$ are both additive identities on the right. That is, suppose that

$$x + 0 = x + 0' = x \quad \text{for all } x.$$

By (A3), $0'$ has an additive inverse on the right, i.e., there exists some y such that

$$0' + y = 0.$$

(Throughout this solution, we don't bother using boldface notation for zero since no reference will be made to scalars.) It follows that

$$\begin{aligned}
0' &= 0' + 0 && \text{by (A2), since 0 is an additive identity} \\
&= 0' + (0' + y) && \text{from the previous display} \\
&= (0' + 0') + y && \text{by (A1)} \\
&= 0' + y && \text{by (A2), since } 0' \text{ is an additive identity} \\
&= 0 && \text{from the previous display.}
\end{aligned}$$

This completes the argument.

To show uniqueness of right inverses, suppose that some x has two right inverses a and b. That is,

$$x + a = x + b = 0.$$

By (A3), a has a right inverse α, i.e.,

$$a + \alpha = 0.$$

Consequently,

$$\begin{aligned}
a + x &= (a + x) + 0 \\
&= (a + x) + (a + \alpha) \\
&= a + (x + (a + \alpha)) && \text{by (A1)} \\
&= a + ((x + a) + \alpha) && \text{by (A1)} \\
&= a + (0 + \alpha) && \text{since } x + a = 0 \\
&= (a + 0) + \alpha && \text{by (A1)} \\
&= a + \alpha && \text{by (A2)} \\
&= 0 && \text{since } a + \alpha = 0.
\end{aligned}$$

That is, $a + x = 0$. And similarly,

$$b + x = 0.$$

So

$$a + x = b + x,$$

and consequently,

$$(a + x) + a = (b + x) + a,$$

so that by (A1),

$$a + (x + a) = b + (x + a).$$

That is,

$$a + 0 = b + 0,$$

so that finally, by (A2),

$$a = b.$$

2.1.9. *Which of the following sets are bases of* $\mathbf{R}^3$*?*
$S_1 = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis since the equation

$$(x, y, z) = a(1,0,0) + b(1,1,0) + c(1,1,1)$$

has unique solution $a = x - y$, $b = y - z$, $c = z$.

$S_2 = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$ is a not a basis since the fourth vector is a linear combination of the first three, giving a nonunique linear combination of the elements of S_2.

$S_3 = \{(1,1,0),(0,1,1)\}$ is not a basis since, for example, $(0,1,0)$ can not be expressed as a linear combination of its elements.

$S_4 = \{(1,1,0),(0,1,1),(1,0,-1)\}$ is not a basis since the third vector is the first minus the second; or it is not a basis because, for example, $(1,0,0)$ can not be expressed as a linear combination of its elements.

Conjecturally a basis of $\mathbf{R}^n$ must have n elements. Bases of $\mathbf{R}^2$ consist of pairs of nonzero noncollinear vectors, like the hands of a clock excluding "midnight" and "six o'clock" scenarios. Bases of $\mathbf{R}^3$ consist of triples of nonzero noncoplanar vectors.

2.1.10. A basis of $\mathbf{C}^n$ over $\mathbf{R}$ is

$$\{(1,0,\dots,0),(i,0,\dots,0),(0,1,\dots,0),(0,i,\dots,0),\dots,(0,0,\dots,1),(0,0,\dots,i)\}.$$

Note that this basis contains $2n$ elements. On the other hand, scalar multiplication by complex numbers is more powerful than scalar multiplication by real numbers only, so a basis of $\mathbf{C}^n$ over $\mathbf{C}$ requires only n elements,

$$\{(1,0,\dots,0),(0,1,\dots,0),\dots,(0,0,\dots,1)\}.$$

2.2.2. *Show that* $x = (2,-1,3,1)$*,* $y = (4,2,1,4)$*, and* $z = (1,3,6,1)$ *form the vertices of a triangle in* $\mathbf{R}^4$ *with two equal angles.*

Compute that the sides of the triangle are

$$x - y = (-2,-3,2,-3), \qquad y - z = (3,-1,-5,3), \qquad z - x = (-1,4,3,0).$$

Consequently, $|x - y|^2 = |z - x|^2 = 26$. That is, the sides that meet at x have the same length, suggesting that the angles at y and z are equal. The cosine of the angle at y is

$$\frac{\langle x - y, z - y\rangle}{|x - y|\,|z - y|} = \frac{6 - 3 + 10 + 9}{\sqrt{26}\sqrt{44}} = \frac{22}{\sqrt{26 \cdot 44}},$$

and the cosine of the angle at z is

$$\frac{\langle x - z, y - z\rangle}{|x - z|\,|y - z|} = \frac{3 + 4 + 15 + 0}{\sqrt{26}\sqrt{44}} = \frac{22}{\sqrt{26 \cdot 44}}.$$

Since the angles lie between 0 and π and they have the same cosines, they are equal.

2.2.3. *Prove that* $x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j$.

This is a matter of unwinding the notation. Compute that

$$\langle x, e_j \rangle = \langle (x_1, \dots, x_j, \dots, x_n), (0, \dots, 1, \dots, 0) \rangle = x_j,$$

so that $\langle x, e_j \rangle e_j$ is the vector with x_j in the jth slot and all other entries 0. Summing these vectors clearly gives x.

2.2.4. *Prove the Inner Product Properties.*

For (IP1), compute that for any $x \in \mathbf{R}^n$,

$$\langle x, x \rangle = \sum_{i=1}^{n} x_i^2.$$

This is a sum of squares. By ordered field properties of the real number system, each square x_i^2 is zero or positive, zero if and only $x_i = 0$. A sum of such numbers is zero or positive, zero if and only if $x_i = 0$ for $i = 1, \dots, n$, i.e., if and only if $x = \mathbf{0}$.

For (IP2), compute that for any $x, y \in \mathbf{R}^n$,

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = \langle y, x \rangle.$$

For the first part of (IP3), compute that for any $x, x', y \in \mathbf{R}^n$,

$$\begin{aligned}
\langle x + x', y \rangle &= \sum_{i=1}^{n} (x + x')_i y_i && \text{by definition of inner product} \\
&= \sum_{i=1}^{n} (x_i + x'_i) y_i && \text{by definition of vector addition} \\
&= \sum_{i=1}^{n} (x_i y_i + x'_i y_i) && \text{by the distributive field axiom} \\
&= \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x'_i y_i && \text{by the commutativity field axiom} \\
&= \langle x, y \rangle + \langle x', y \rangle.
\end{aligned}$$

For the second part of (IP3), compute that for any $a \in \mathbf{R}$ and any $x, y \in \mathbf{R}^n$,

$$\begin{aligned}
\langle ax, y\rangle &= \sum_{i=1}^n (ax)_i y_i && \text{by definition of inner product}\\
&= \sum_{i=1}^n a x_i y_i && \text{by definition of scalar–vector multiplication}\\
&= a \sum_{i=1}^n x_i y_i && \text{by the distributive field axiom}\\
&= a\langle x, y\rangle.
\end{aligned}$$

The third and fourth parts of (IP3) follow from the first two parts via (IP2):

$$\langle x, y+y'\rangle = \langle y+y', x\rangle = \langle y, x\rangle + \langle y', x\rangle = \langle x, y\rangle + \langle x, y'\rangle,$$

and

$$\langle x, by\rangle = \langle by, x\rangle = b\langle y, x\rangle = b\langle x, y\rangle.$$

2.2.5. *Prove that $|x| \geq 0$ for all $x \in \mathbf{R}^n$, with equality if and only if $x = \mathbf{0}$.*
The definition

$$|x| = \sqrt{\langle x, x\rangle},$$

the property (IP1) that $\langle x, x\rangle \geq 0$ with equality if and only if $x = \mathbf{0}$, and the fact that the square root is nonnegative and is zero if and only if $\langle x, x\rangle = 0$ all combine to give the result immediately.

Prove that $|ax| = |a|\,|x|$ for all $a \in \mathbf{R}$ and $x \in \mathbf{R}^n$.
Compute, using results from the section, that

$$|ax| = \sqrt{\langle ax, ax\rangle} = \sqrt{a^2\langle x, x\rangle}.$$

Each of a^2 and $\langle x, x\rangle$ is a nonnegative real number, so

$$\sqrt{a^2\langle x, x\rangle} = \sqrt{a^2}\sqrt{\langle x, x\rangle}.$$

But $\sqrt{a^2} = |a|$ by work in the real number system (the square root of a^2 is not necessarily a), and $\sqrt{\langle x, x\rangle} = |x|$ by definition. In sum, we have the desired result,

$$|ax| = |a|\,|x|.$$

2.2.7. *Starting from the basic Triangle Inequality,*

$$|x+y| \leq |x| + |y| \quad \text{for all } x, y \in \mathbf{R}^n,$$

derive the full Triangle Inequality,

$$|\,|x| - |y|\,| \leq |x \pm y| \leq |x| + |y| \quad \text{for all } x, y \in \mathbf{R}^n.$$

For any x and y, substitute $-y$ for y in the basic inequality to see that $|x + (-y)| \leq |x| + |-y|$. That is,

$$|x - y| \leq |x| + |y|.$$

Along with the basic inquality, this gives the right side of the full inequality. Now, for any x and y, note that by the right side of the full inequality,

$$|x| = |(x + y) - y| \leq |x + y| + |y|,$$

so that subtracting $|y|$ from both sides gives

$$|x| - |y| \leq |x + y|.$$

Reverse the roles of x and y to see that also

$$|y| - |x| \leq |x + y|.$$

Since $|\,|x| - |y|\,|$ is one of $|x| - |y|$ or $|y| - |x|$, we now have

$$|\,|x| - |y|\,| \leq |x + y|.$$

Finally, replace y by $-y$ to complete the proof.

2.2.8. *Prove the Size Bounds.*

First, compute for each $j \in \{1, \ldots, n\}$ by the fact that $x_j = \langle x, e_j \rangle$, by the Cauchy–Schwarz inequality, and by the fact that $|e_j| = 1$ that

$$|x_j| = |\langle x, e_j \rangle| \leq |x||e_j| = |x|.$$

Equality holds in the Cauchy–Schwarz inequality exactly when the two vectors involved are parallel. So in our case here, equality holds exactly when x is parallel to e_j, i.e., $x = x_j e_j$ has all but its jth component equal to 0 (and possibly $x_j = 0$ as well).

Second, compute by the Triangle Inequality, by the second part of exercise 2.2.5, and by the fact that $|e_j| = 1$ for all j that

$$\begin{aligned} |x| = |x_1 e_1 + \cdots + x_n e_n| &\leq |x_1 e_1| + \cdots + |x_n e_n| \\ &= |x_1|\,|e_1| + \cdots + |x_n|\,|e_n| \\ &= |x_1| + \cdots + |x_n|. \end{aligned}$$

Equality holds in the triangle inequality when all the vectors involved are parallel and point in the same direction. So in our case here, since the vectors are $x_i e_i$, equality holds exactly when x has at most one nonzero component.

2.2.10. *Use the Law of Cosines to derive the formula for $\cos\theta$ in the plane.*

With x, y, and θ defined as in the exercise, the Law of Cosines is

$$|x - y|^2 = |x|^2 + |y|^2 - 2|x|\,|y| \cos\theta.$$

The left side is $\langle x-y, x-y\rangle$, which, by bilinearity, expands out to $|x|^2-2\langle x,y\rangle+|y|^2$. Thus, after some cancellation the display becomes

$$|x|^2 + |y|^2 - 2\langle x, y\rangle = |x|^2 + |y|^2 - 2|x|\,|y|\cos\theta.$$

The result follows immediately from a little algebra.

2.2.12. *Show that two nonzero vectors x and y are orthogonal if and only if $|x+y|^2 = |x|^2 + |y|^2$.*

Similarly to the work in exercise 2.2.10, compute that

$$|x+y|^2 = |x|^2 + 2\langle x, y\rangle + |y|^2.$$

The result follows immediately.

2.2.13. *Show that the diagonals of a parallelogram are orthogonal if and only if the parallelogram is a rhombus.*

Let the parallelogram have sides x and y. Then the diagonals are $x+y$ and $x-y$. Compute, using properties of the inner product, that

$$\langle x+y, x-y\rangle = |x|^2 - |y|^2.$$

This shows that the diagonals are orthogonal if and only if the sides are equal. This is the desired result.

2.3.4. *Define an inner product and a modulus on $\mathcal{C}([0,1],\mathbf{R})$ by*

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt, \quad |f| = \sqrt{\langle f, f\rangle}.$$

How much of the material on inner product and modulus in $\mathbf{R}^n$ carries over to $\mathcal{C}([0,1],\mathbf{R})$? Express the Cauchy–Schwarz inequality as a relation between integrals.

All of the material carries through. The point of the exercise is that to establish this, *all one has to do is establish the inner product properties for this new inner product*, because the rest of the results of the section were derived solely from the inner product properties.

Beyond parsing language, the subtle point in the inner product properties for this new inner product is that $\langle f, f\rangle = 0$ only if f is identically 0. The proof of this is where it matters that the functions in question are continuous. The argument proceeds as follows: If f is not identically 0 then $f(t)^2 > 0$ for some $t_0 \in [0,1]$. Give $f(t_0)^2$ the name r, so that $r > 0$. In the Persistence of Inequality principle (Proposition 2.3.9), let the f in the principle be the f^2 here, let $a = t_0$ and let $b = r/2$. The principle then says that $f(t)^2 > r/2$ for all $t \in [0,1]$ within some positive distance ε of t_0. Even if t_0 is an endpoint, this shows that $\int_0^1 f(t)^2\,dt \geq r\varepsilon/4 > 0$.

The Cauchy–Schwarz inequality in this context is

$$\left(\int_0^1 f(t)g(t)\,dt\right)^2 \leq \int_0^1 f(t)^2\,dt \cdot \int_0^1 g(t)^2\,dt.$$

2.3.5. *Prove the componentwise nature of convergence.*

Let $\{x_\nu\}$ be a sequence of vectors in $\mathbf{R}^n$, and let $a \in \mathbf{R}^n$ be a fixed vector. The text has shown that the sequence $\{x_\nu\}$ converges to a if and only if the sequence $\{x_\nu - a\}$ is null. That is, $\{x_\nu\}$ converges to a if and only if the sequence

$$\{(x_{1,\nu} - a_1, \dots, x_{n,\nu} - a_n)\}$$

is null. By Lemma 2.3.2, this holds if and only if each scalar sequence $\{x_{j,\nu} - a_j\}$ is null, and by definitions and/or results from one variable, this is equivalent to each scalar sequence $\{x_{j,\nu}\}$ converging to a_j.

2.3.9. *Which of the following functions on $\mathbf{R}^2$ can be defined continuously at $\mathbf{0}$?*

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq \mathbf{0}, \\ b & \text{if } (x,y) = \mathbf{0}, \end{cases}$$

Let $x \neq 0$ and let $y = mx$, and compute that in this case

$$f(x,mx) = \frac{(1-m^2)x^2}{(1+m^2)x^2} = \frac{(1-m^2)}{(1+m^2)}.$$

This shows that f is a "spiral staircase" function as in the text, so it can not be made continuoous at $\mathbf{0}$.

$$g(x,y) = \begin{cases} \frac{x^2-y^3}{x^2+y^2} & \text{if } (x,y) \neq \mathbf{0}, \\ b & \text{if } (x,y) = \mathbf{0}, \end{cases}$$

Compute that $g(x,0) = 1$ for $x \neq 0$ but $g(0,y) = -y$ for $y \neq 0$. Thus $g(x,0) \to 1$ as $x \to 0$, while $g(0,y) \to 0$ as $y \to 0$, showing that g can not be made continuous at $\mathbf{0}$.

$$h(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^{1/2}} & \text{if } (x,y) \neq \mathbf{0}, \\ b & \text{if } (x,y) = \mathbf{0}, \end{cases}$$

Compute that for $(x,y) \neq \mathbf{0}$,

$$|h(x,y)| = \frac{|x^2-y^2|}{|(x,y)|} \leq \frac{|x|^2+|y|^2}{|(x,y)|} \leq \frac{|(x,y)|^2+|(x,y)|^2}{|(x,y)|} = 2|(x,y)|.$$

Since $2|(x,y)| \to 0$ as $(x,y) \to \mathbf{0}$, it follows that $|h(x,y)| \to 0$ as $(x,y) \to \mathbf{0}$, and therefore $h(x,y) \to \mathbf{0}$ as $(x,y) \to \mathbf{0}$. So define $h(0,0) = 0$ to make h continuous at $\mathbf{0}$.

2.3.11. *Let $f, g \in \mathcal{M}(\mathbf{R}^n, \mathbf{R})$ be such that $f+g$ and fg are continuous. Are f and g necessarily continuous?*

No. For instance, let $n = 1$, let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ -1 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

and let $g(x) = -f(x)$. Neither f nor g is continuous, but $f+g$ is the constant function 0 and fg is the constant function -1, both of which are continuous.

2.4.1. (a) The set $B(\mathbf{0}, 1)$ is not closed: for example, e_1 is a limit point of the set that is not in the set. The set is bounded.

(b) The set $\{(x, y) \in \mathbf{R}^2 : y - x^2 = 0\}$ is closed. It is not bounded: for any positive number R the set contains the point (R, R^2) whose modulus is greater than R.

(c) The set $\{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 - 1 = 0\}$ is closed and bounded, and hence it is compact.

(d) The set $\{x : f(x) = \mathbf{0}_m\}$ where $f \in \mathcal{M}(\mathbf{R}^n, \mathbf{R}^m)$ is continuous is closed. To see this, let a be a limit point of the set. Then there is a sequence $\{x_\nu\}$ in the set such that $\{x_\nu\}$ approaches a. By the definition of the set, $f(x_\nu) = \mathbf{0}_m$ for each ν. By the continuity of f it follows that $f(a) = \mathbf{0}_m$ as well, i.e., a lies in the set.

As shown by (b) and (c), we can not determine whether the set is bounded unless we know more about f.

(e) The set $\mathbf{Q}^n$ is neither closed nor bounded. For example, $\sqrt{2}e_1$ is a limit point of the set that is not in the set. The set is unbounded: any positive real number R there is an integer $n > R$ by Archimedes's Principle, and so $ne_1 \in \mathbf{Q}^n$ has modulus greater than R.

(f) The set $\{(x_1, \ldots, x_n) : x_1 + \cdots + x_n > 0\}$ is neither closed nor bounded. For example, $\mathbf{0}$ is a limit point of the set that does not belong to the set, and for any positive real number R the point $(R+1)e_1$ lies in the set and has modulus greater than R.

2.4.2. *Give a set $A \subset \mathbf{R}^n$ and limit point b of A such that $b \notin A$.*

Let $A = B(\mathbf{0}, 1)$ and let $b = (1, 0, \ldots, 0)$.

Give a set $A \subset \mathbf{R}^n$ and a point $a \in A$ such that a is not a limit point of A.

Let $A = \{\mathbf{0}\}$ and let $a = \mathbf{0}$.

2.4.5. *Prove that any ball $B(p, \varepsilon)$ is bounded in $\mathbf{R}^n$.*

Let $R = |p| + \varepsilon$. For any $x \in B(p, \varepsilon)$ we have by the triangle inequality

$$|x| = |p + x - p| \leq |p| + |x - p| < |p| + \varepsilon = R.$$

This shows that $B(p, \varepsilon) \subset B(\mathbf{0}, R)$.

2.4.7. *Show by example that a closed set need not satisfy the sequential characterization of bounded sets, and that a bounded set need not satisfy the sequential characterization of closed sets.*

For the sake of simple examples, let $n = 1$.

Consider the closed set $A = \mathbf{R}$. The sequence $\{x_\nu\} = \{1, 2, 3, \cdots\}$ has all of its entries in A but it has no subsequence that converges in $\mathbf{R}$.

Consider the bounded set $A = (0, 1]$. The sequence $\{x_\nu\} = \{1, 1/2, 1/3, \cdots\}$ has all of its entries in A and it converges; but its limit, 0, does not lie in A.

2.4.8. *Show by example that the continuous image of a closed set need not be closed.*

The closed set $[1, \infty)$ is taken by the continuous function $f(x) = 1/x$ to the nonclosed set $(0, 1]$.

Show that the continuous image of a closed set need not be bounded.

The closed set $\mathbf{R}$ is taken by the continuous function $f(x) = x$ to the unbounded set $\mathbf{R}$.

Show that the continuous image of a bounded set need not be closed.

The bounded set $(0, 1)$ is taken by the continuous function $f(x) = x$ to the nonclosed set $(0, 1)$.

Show that the continuous image of a bounded set need not be bounded.

The bounded set $(0, 1]$ is taken by the continuous function $f(x) = 1/x$ to the unbounded set $[1, \infty)$.

2.4.9. *A subset A of $\mathbf{R}^n$ is called* **discrete** *of each of its points is isolated. Is discreteness a topological property? That is, need the continuous image of a discrete set be discrete?*

No. Scrutinizing the definition of continuity shows that *every* mapping whose domain is discrete must be continuous. Especially, the function

$$f : \mathbf{N} \longrightarrow \mathbf{R}$$

given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x > 0 \end{cases}$$

is continuous. And this function takes the discrete set $\mathbf{N}$ to the set

$$\{0\} \cup \{1, 1/2, 1/3, \dots\},$$

a set in which 0 is a nonisolated point.

Chapter 3

3.1.1. *Prove that $T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ is linear if and only if it satisfies* (3.1) *and* (3.2).

If T is linear then it satisfies the displayed condition in Definition 3.1.1,

$$T\left(\sum_{i=1}^{k} \alpha_i x_i\right) = \sum_{i=1}^{k} \alpha_i T(x_i),$$

for all $k \in \mathbf{Z}^+$, $\alpha_1, \dots, \alpha_k \in \mathbf{R}$, and $x_1, \dots, x_k \in \mathbf{R}^n$. Specialize to $k = 2$ and $\alpha_1 = \alpha_2 = 1$ to get (3.1). Specialize to $k = 1$ to get (3.2).

Conversely, if T satisfies (3.1) and (3.2) then we prove that it satisfies the condition in Definition 3.1.1 by induction on k. For $k = 1$, the condition in the definition is (3.1), which we know that T satisfies. Now let $k \in \mathbf{Z}^+$ be arbitrary, and assume that T satisfies the condition in the definition for k. We need to show that T satisfies the condition for $k+1$. Compute that for any $\alpha_1, \dots, \alpha_{k+1} \in \mathbf{R}$ and $x_1, \dots, x_{k+1} \in \mathbf{R}^n$,

$$\begin{aligned}
T\left(\sum_{i=1}^{k+1} \alpha_i x_i\right) &= T\left(\sum_{i=1}^{k} \alpha_i x_i + \alpha_{k+1} x_{k+1}\right) && \text{by def'n of summation} \\
&= T\left(\sum_{i=1}^{k} \alpha_i x_i\right) + T(\alpha_{k+1} x_{k+1}) && \text{since } T \text{ satisfies (3.1)} \\
&= \sum_{i=1}^{k} \alpha_i T(x_i) + \alpha_{k+1} T(x_{k+1}) && \text{by ind. hyp. and (3.2)} \\
&= \sum_{i=1}^{k+1} \alpha_i T(x_i) && \text{by def'n of summation.}
\end{aligned}$$

This completes the induction. Note how neatly the induction step uses (3.1) and (3.2) once each.

3.1.2. *Suppose that $T \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$. Show that $T(\mathbf{0}_n) = \mathbf{0}_m$.*

Compute that $T(\mathbf{0}_n) = T(\mathbf{0}_n + \mathbf{0}_n) = T(\mathbf{0}_n) + T(\mathbf{0}_n)$, and now adding $-T(\mathbf{0}_n)$ to both sides gives the result.

3.1.3. *Fix a vector $a \in \mathbf{R}^n$. Show that the mapping $T : \mathbf{R}^n \longrightarrow \mathbf{R}$ given by $T(x) = \langle a, x\rangle$ is linear, and that $T(e_j) = a_j$ for $j = 1, \dots, n$.*

To prove linearity it suffices to establish (3.1) and (3.2). For (3.1), compute that for any $x_1, x_2 \in \mathbf{R}^n$,

$$\begin{aligned}
T(x_1 + x_2) &= \langle a, x_1 + x_2\rangle && \text{by definition of } T \\
&= \langle a, x_1\rangle + \langle a, x_2\rangle && \text{since the inner product is bilinear} \\
&= T(x_1) + T(x_2) && \text{by definition of } T.
\end{aligned}$$

For (3.2), compute that for any $\alpha \in \mathbf{R}$ and $x \in \mathbf{R}^n$,

$$\begin{aligned}
T(\alpha x) &= \langle a, \alpha x\rangle && \text{by definition of } T \\
&= \alpha\langle a, x\rangle && \text{since the inner product is bilinear} \\
&= \alpha T(x) && \text{by definition of } T.
\end{aligned}$$

By exercise 3.1.1, this is enough to show that T is linear.

For the second part of the exercise, take any $j \in \{1, \dots, n\}$ and compute

$$T(e_j) = \langle a, e_j\rangle = \langle (a_1, \dots, a_j, \dots, a_n), (0, \dots, 1, \dots, 0)\rangle = a_j.$$

3.1.5. *Complete the proof of the componentwise nature of linearity.*

Let $T = (T_1, \dots, T_m) : \mathbf{R}^n \longrightarrow \mathbf{R}^m$. We need to show that T satisfies (3.2) if and only if each T_j does. Compute that for any $\alpha \in \mathbf{R}$ and any $x \in \mathbf{R}^n$,

$$T(\alpha x) = (T_1(\alpha x), \quad \dots, \quad T_m(\alpha x))$$

and

$$\begin{aligned}\alpha T(x) &= \alpha(T_1(x), \quad \dots, \quad T_m(x)) \\ &= (\alpha T_1(x), \quad \dots, \quad \alpha T_m(x)).\end{aligned}$$

But T satisfies (3.2) exactly when the left sides are equal, the left sides are equal exactly when the right sides are equal, and the right sides are equal exactly when each T_i satisfies (3.2). This completes the proof.

3.1.6. *Carry out the matrix-by-vector multiplies.*

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \\ ex+fy \end{bmatrix},$$

$$\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + \cdots + x_ny_n = \langle x, y \rangle,$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

3.1.8. *Let θ denote a fixed but generic angle. Argue geometrically that the mapping $R : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ given by counterclockwise rotation by θ is linear, and then find its matrix.*

The argument is virtually identical to the particular case $\theta = \pi/6$ in the text. The matrix is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

3.1.13. *If $S \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$ and $T \in \mathcal{L}(\mathbf{R}^p, \mathbf{R}^n)$, show that $S \circ T : \mathbf{R}^p \longrightarrow \mathbf{R}^m$ lies in $\mathcal{L}(\mathbf{R}^p, \mathbf{R}^m)$.*

It suffices to show that $S \circ T$ satisfies (3.1) and (3.2). For (3.1), take any $x_1, x_2 \in \mathbf{R}^p$ and compute that

$$\begin{aligned}(S \circ T)(x_1 + x_2) &= S(T(x_1 + x_2)) && \text{by definition of composition} \\ &= S(T(x_1) + T(x_2)) && \text{since } T \text{ satisfies (3.1)} \\ &= S(T(x_1)) + S(T(x_2)) && \text{since } S \text{ satisfies (3.1)} \\ &= (S \circ T)(x_1)) + (S \circ T)(x_2) && \text{by definition of composition.}\end{aligned}$$

The proof that $S \circ T$ satisfies (3.2) is similar.

3.2.2. *Carry out the matrix multiplies.*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad-bc)I_2,$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \begin{bmatrix} a_1x_1+a_2x_2+a_3x_3 & b_1x_1+b_2x_2+b_3x_3 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

3.2.4. *Let $A = [a_{ij}]$ be a matrix in $\mathrm{M}_{m,n}(\mathbf{R})$. Its* **transpose** $A^t \in \mathrm{M}_{n,m}(\mathbf{R})$ *is the matrix obtained by flipping A about its Northwest–Southeast diagonal. Thus the rows of A^t are the columns of A, the columns of A^t are the rows of A, and the (i,j)th entry of A^t is a_{ji}. Show that*

$$(AB)^t = B^tA^t \quad \textit{for all } A \in \mathrm{M}_{m,n}(\mathbf{R}) \textit{ and } B \in \mathrm{M}_{n,p}(\mathbf{R}).$$

First note that AB is an m-by-p matrix, so that $(AB)^t$ is p-by-m. On the other hand, B^tA^t is the product of a p-by-n matrix and an n-by-m matrix, also giving a p-by-m matrix. So at least $(AB)^t$ and B^tA^t have the same dimensions.

For any $i \in \{1, \cdots, p\}$ and any $j \in \{1, \cdots, m\}$, the (i,j)th entry of $(AB)^t$ is the (j,i)th entry of AB, which is

$$\langle j\text{th row of } A, i\text{th column of } B\rangle,$$

which in turn is

$$\langle i\text{th row of } B^t, j\text{th column of } A^t\rangle,$$

and this is the (i,j)th row of B^tA^t.

3.3.1. *Write down the following 3-by-3 elementary matrices and their inverses: $R_{3;2,\pi}$, $S_{3,3}$, $T_{3;2}$, $T_{2;3}$.*

Solution:

$$R_{3;2,\pi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \pi & 1 \end{bmatrix}, \quad R_{3;2,\pi}^{-1} = R_{3;2,-\pi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\pi & 1 \end{bmatrix},$$

and

$$S_{3,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad S_{3,3}^{-1} = S_{3,1/3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix},$$

and

$$T_{3;2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad T_{3;2}^{-1} = T_{3;2},$$

and $T_{2,3} = T_{3,2}$ so that $T_{2,3}^{-1} = T_{2,3} = T_{3,2}$.

3.3.3. *Let* $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{smallmatrix}\right]$. *Evaluate the following products without actually multiplying matrices:* $R_{3;2,\pi}A$, $S_{3,3}A$, $T_{3;2}A$, $T_{2;3}A$.

Since $T_{2,3} = T_{3,2}$, we only need to compute the first three. They are

$$R_{3;2,\pi}A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5+3\pi & 6+4\pi \end{bmatrix}, \quad S_{3,3}A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 15 & 18 \end{bmatrix}, \quad T_{3;2}A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 3 & 4 \end{bmatrix}.$$

3.3.7. *Are the following matrices echelon? For each matrix M, solve the equation $Mx = \mathbf{0}$.*

The matrix

$$M = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is not echelon. Its echelon form is I_3, and so the only solution of the equation $Mx = \mathbf{0}$ is $x = \mathbf{0}$. The matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is echelon. The equation $Mx = \mathbf{0}$ is solved by all vectors $(x_1, x_2, x_3, 0)$ where x_1, x_2, and x_3 are free. The matrix

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

is echelon. The equation $Mx = \mathbf{0}$ has solutions $(-x_2, x_2, -x_4, x_4)$ where x_2 and x_4 are free. The matrix

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not echelon because its nonzero rows are not at the top. The only solution of the equation $Mx = \mathbf{0}$ is $x = \mathbf{0}$. The matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is not echelon because the leading 1 in the bottom row doesn't have all 0's above it. But it becomes echelon after its third row is subtracted from its second, and then the equation $Mx = \mathbf{0}$ has solutions $(0, 0, 0, x_4)$ where x_4 is free. The matrix

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

is not echelon, but it becomes echelon after its first two rows are transposed. The equation $Mx = \mathbf{0}$ has solutions $(-3x_3, -x_3, x_3)$.

3.3.8. *For each matrix A solve the equation $Ax = \mathbf{0}$.*

$$\begin{bmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ 1 & 2 & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & 1 \\ 2 & 6 & -1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

The echelon form of the first matrix is

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

and so in this case the equation $Ax = \mathbf{0}$ has solutions $(x_3, -3x_3, x_3)$ where x_3 is free. The echelon form of the second matrix is

$$E = \begin{bmatrix} 1 & 0 & 0 & 17/5 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & -9/5 \end{bmatrix},$$

and so the equation $Ax = \mathbf{0}$ has solutions $(-(17/5)x_4, (3/5)x_4, (9/5)x_4, x_4)$ where x_4 is free. The echelon form of the third matrix is I_3, and so the equation $Ax = \mathbf{0}$ has only the trivial solution $x = \mathbf{0}$.

3.3.9. *Balance the chemical equation*

$$\mathrm{Ca} + \mathrm{H_3PO_4} \longrightarrow \mathrm{Ca_3P_2O_8} + \mathrm{H_2}.$$

Of course this is silly. One can pretty much do it on sight. But also one can set it up as a linear algebra problem, exhibiting a systematic method that applies as well to situations too complicated to do in one's head. Let x be the number of Ca molecules, y the number of H_3PO_4 molecules, z the number of $Ca_3P_2O_8$ molecules, and w the number of H_2 molecules. Then the conditions for balancing the equation are that the number of Ca atoms be the same on both sides, and similarly for the number of H atoms, P atoms, and O atoms. This gives four equations in the unknowns:

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 1 & -2 & 0 \\ 0 & 4 & -8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \mathbf{0}_4.$$

Putting this equation into echelon form gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \mathbf{0}_4.$$

So $x = w$, $y = (2/3)w$, and $z = (1/3)w$. Set $w = 3$: then $x = 3$ and $y = 2$ and $z = 1$,

$$3\ \mathrm{Ca} + 2\ \mathrm{H_3PO_4} \longrightarrow \mathrm{Ca_3P_2O_8} + 3\ \mathrm{H_2}.$$

3.5.4. *The square matrix A is* **orthogonal** *if $A^tA = I$. Show that if A is orthogonal then* $\det A = \pm 1$. *Give an example with determinant -1.*

By Theorem 3.5.4 (whose proof is exercise 3.5.3), $\det A^t = \det A$. Thus, if $A^tA = I$ then

$$1 = \det I = \det A^tA = \det A^t \det A = (\det A)^2,$$

and so $\det A = \pm 1$. An example with determinant -1 is $A = [-1]$.

3.5.5. *The matrix A is* **skew symmetric** *if $A^t = -A$. Show that if A is n-by-n skew symmetric with n odd then* $\det A = 0$.

By Theorem 3.5.4 (whose proof is exercise 3.5.3), $\det A = \det(A^t)$. That is, $\det A = \det(-A)$. But $-A$ is the matrix whose n rows are the n rows of A each multiplied by -1. Thus, by the multilinearity of the determinant, $\det(-A) = (-1)^n \det A$. Since n is odd, this says that $\det(-A) = -\det A$. Concatenating the various results so far, we have

$$\det A = \det(A^t) = \det(-A) = (-1)^n \det A = -\det A.$$

Since $\det A$ is a real number and equal to its additive inverse, it is 0.

3.6.2. *Use the desired determinant properties to obtain the formula in the section for 1-by-1 and the 3-by-3 determinant. Verify that the 1-by-1 formula satisfies the properties.*

For the 1-by-1 case, the only permutation is (1), and so the formula is simply $\det[a] = a$. This is multilinear because

$$\det[\alpha a + \alpha' a'] = \alpha a + \alpha' a' = \alpha \det[a] + \alpha' \det[a].$$

It is vacuously skew-symmetric. And it is normalized because $\det[1] = 1$.

For the 3-by-3 case, the relevant permutations are $(1,2,3)$, $(2,3,1)$, and $(3,1,2)$, with plus signs, and $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$, with minus signs. The rook-placement interpretation of the determinant formula now gives the result.

3.6.3. *For each permutation, count the inversions and compute the sign:* $(2,3,4,1)$, $(3,4,1,2)$, $(5,1,4,2,3)$.

The permutation $(2,3,4,1)$ has the three inversions $(2,1)$, $(3,1)$, and $(4,1)$, so its sign is negative.

The permutation $(3,4,1,2)$ has the four inversions $(3,1)$, $(3,2)$, $(4,1)$, and $(4,2)$, so its sign is positive.

The permutation $(5,1,4,2,3)$ has the six inversions $(5,1)$, $(5,4)$, $(5,2)$, $(5,3)$, $(4,2)$, $(4,3)$, and so its sign is positive.

3.6.8. *Prove that the determinant of a triangular matrix is the product of its diagonal entries.*

The only rook-placement that can give a nonzero term in the general formula for the determinant is the placement of the rooks down the diagonal. This placement has no upward slopes, so its sign is positive. This gives the result.

3.6.9. *Calculate the determinants of the following matrices:*

$$\begin{bmatrix} 4 & 3 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 4 & 1 \\ 2 & 0 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 0 & 2 \\ 4 & 1 & -1 & -1 \\ 1 & 2 & 3 & 0 \end{bmatrix}.$$

The determinants are 9 and 128 respectively. In each case there are many ways to do it, but the idea is to use row and column operations to put the matrix into triangular form, and then the determinant is product of the diagonal elements of the triangular matrix, times the reciprocals of the row and column scale-factors that were used, times the parity of the number of transpositions that were used.

3.8.2. *Describe the geometric effect of multiplying by the matrices R' and S' in the text. Describe the effect of multiplying by R and S if $a < 0$.*

Multiplying by R' produces a shear in the y-direction, taking $(1,0)$ to $(1,a)$ and taking $(0,1)$ back to itself.

Multiplying by S' produces a scale in the y-direction, taking $(1,0)$ back to itself and taking $(0,1)$ to $(0,a)$.

Multiplying by R but with $a < 0$ produces a negative shear in the x-direction, and multiplying by S but with $a < 0$ scales the box in the negative x-direction.

3.8.3. *Describe the geometric effect of multiplying by the 3-by-3 elementary matrices $R_{2;3,1}$, $R_{3;1,2}$, and $S_{2,-3}$.*

Multiplying by $R_{2;3,1}$ preserves $(1,0,0)$ and $(0,1,0)$ but takes $(0,0,1)$ to $(0,1,1)$. The effect is to shear the positive z-axis 45 degrees toward the positive y-axis. (The negative z-axis is sheared the other way, of course.)

Multiplying by $R_{3;1,2}$ shears the positive x-axis toward the positive z-axis, taking $(1,0,0)$ to $(1,0,2)$.

Multiplying by $S_{2,-3}$ negates and triples in the y-direction while preserving the x and z-directions.

3.8.4. *Describe counterclockwise rotation of the plane by angle $\pi/2$ as a composition of shears and scales.*

The relevant matrix is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Compute that

$$R_{1;2,1}R_{2;1,-1}R_{1;2,1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = I.$$

It follows that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = R_{1;2,-1}R_{2;1,1}R_{1;2,-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

That is, the counterclockwise rotation through angle $\pi/2$ is a composition shearing the positive y-axis counterclockwise through angle $\pi/4$, then shearing the positive x-axis counterclockwise through angle $\pi/4$, then again shearing the positive y-axis counterclockwise through angle $\pi/4$.

3.8.6. *In* $\mathbf{R}^3$*, describe the linear mapping that takes* e_1 *to* e_2*,* e_2 *to* e_3*, and* e_3 *to* e_1 *as a composition of shears, scales, and transpositions.*

The relevant matrix is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Compute that

$$T_{2;3}T_{1;2}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = I.$$

It follows that

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = T_{1;2}T_{2;3}.$$

That is, the mapping is a composition of transposing the second and third axes followed by transposing the first and second.

3.9.1. *Any invertible mapping* $T : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ *is a composition of scales, shears and transpositions. Give conditions on such a composition to make the mapping orientation-preserving, orientation-reversing.*

The condition for the mapping to preserve orientation is that the determinant be positive, and similarly for reversing orientation and negative determinant. Since each shear has determinant 1, shears preserve orientation. Scaling in any direction by a positive number preserves orientation, while scaling in any direction by a negative number reverses it. Every transposition reverses orientation. Thus the condition for the mapping to preserve orientation is that the combined number of negative scales and transpositions must be even, and the condition for the mapping to reverse orientation is that the combined number of negative scales and transpositions must be odd.

3.10.1. *Evaluate* $(2, 0, -1) \times (1, -3, 2)$.

By the formula in the section, the cross product is $(-3, -5, -6)$.

3.10.2. *Suppose that* $v \times e_1 = v \times e_2 = \mathbf{0}$*. Describe* v*.*

The given conditions imply that v is parallel to e_1 and to e_2. It follows that v is $\mathbf{0}_3$.

3.10.3. *True or false: For all u, v, w in $\mathbf{R}^3$, $(u \times v) \times w = u \times (v \times w)$.*

This is false. For instance, $(e_1 \times e_1) \times e_2 = \mathbf{0}_3 \times e_2 = \mathbf{0}_3$, but $e_1 \times (e_1 \times e_2) = e_1 \times e_3 = -e_2$.

3.10.4. *Express $(u + v) \times (u - v)$ as a scalar multiple of $u \times v$.*

Compute, using various properties of the cross product, that

$$(u + v) \times (u - v) = u \times u - u \times v + v \times u - v \times v = -2(u \times v).$$

3.10.5. *For fixed u, v in $\mathbf{R}^3$ with $u \neq \mathbf{0}$, describe the vectors w satisfying the condition $u \times v = u \times w$.*

The condition is $u \times (w - v) = \mathbf{0}_3$. That is $w - v$ is parallel to u, and since $u \neq \mathbf{0}$ this condition is that $w - v = cu$ for some $c \in \mathbf{R}$, or $w = v + cu$ for some $c \in \mathbf{R}$. Another way to say this is that $w \in \ell(v, u)$.

3.10.9. *What can you conclude about the lines*

$$\frac{x - x_p}{x_d} = \frac{y - y_p}{y_d} = \frac{z - z_p}{z_d} \qquad \text{and} \qquad \frac{x - x_p}{x_D} = \frac{y - y_p}{y_D} = \frac{z - z_p}{z_D}$$

given that $x_d x_D + y_d y_D + z_d z_D = 0$?

The lines have the common point p and orthogonal directions. That is, they are normal.

What can you conclude if $x_d/x_D = y_d/y_D = z_d/z_D$?

The lines have the common point p and parallel directions. That is, they are equal.

3.10.11. *Use vector geometry to show that the distance from the point q to the line $\ell(p, d)$ is*

$$\frac{|(q - p) \times d|}{|d|}.$$

By results in the text, $|(q - p) \times d|$ is the area of the parallelogram spanned by $q - p$ and d. View d as the base of the parallelogram, making the height of the parallelogram exactly the distance we seek. But the height is the area divided by the base, $|(q - p) \times d|/|d|$. So we are done.

3.10.14. *Where does the plane $x/a + y/b + z/c = 1$ intersect each axis?*

The points of intersection are $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

3.10.16. *Use vector geometry to show that the distance from the point q to the plane $P(p, n)$ is*

$$\frac{|\langle q - p, n\rangle|}{|n|}.$$

We want the modulus of $(q - p)_{\|n}$. From exercise 2.2.16,

$$(q - p)_{\|n} = \frac{\langle q - p, n\rangle}{|n|^2} n,$$

and this is

$$(q-p)_{\|n} = \frac{\langle q-p, n\rangle}{|n|}\frac{n}{|n|}.$$

Since $n/|n|$ is a unit vector, the modulus of the displayed vector is $|\langle q-p, n\rangle|/|n|$ as desired.

Chapter 4

4.1.2. *Give a geometric interpretation of the derivative when $n = m = 2$. Give a geometric interpretation of the derivative when $n = 1$ and $m = 2$.*

When $m = n = 2$, we can conceive of the map f from (some of) the plane back to the plane as distorting a grid in some nonuniform, curvy way, as in figure 2.9 from the notes. The derivative T_a distorts a grid into another grid, as in figure 3.6. The interpretation of the derivative is that near the point a where the derivative is taken, the distortion of the grid under f is very closely approximated by the translation to $f(a)$ of the distortion of the grid under T near the origin.

When $n = 1$ and $m = 2$, we can conceive of the map f from (some of) the line into the plane as motion along a curve. Similarly, the derivative T_a can be viewed as motion along a straight line through the origin in the plane. The interpretation of the derivative is that near the point a where the derivative is taken, the motion along the curve is very closely approximated by the translation to $f(a)$ of the motion along the straight line near the origin.

4.1.4. *Prove the componentwise nature of differentiability: Let $f : A \longrightarrow \mathbf{R}^m$ (where $A \subset \mathbf{R}^n$) have component functions $f_1, \cdots, f_m$, and let a be a point of A. Let $T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ be a linear mapping with component functions $T_1, \cdots, T_m$. Show that f is differentiable at a with derivative T if and only if each component f_i is differentiable at a with derivative T_i.*

Let a be an interior point of A. (At any other point a, neither f nor any of its component functions can be differentiable, so the condition that f be differentiable is equivalent to the condition that each f_i be differentiable, since both conditions are flat-out false.) Recall that vectors converge to the zero-vector exactly when their magnitudes converge to the zero-scalar. Therefore, the scalar condition that f is differentiable at a with derivative T,

$$\lim_{h\to\mathbf{0}_n} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0_{\mathbf{R}},$$

is equivalent to a vector condition,

$$\lim_{h\to\mathbf{0}_n} \frac{f(a+h) - f(a) - T(h)}{|h|} = \mathbf{0}_m.$$

By the componentwise nature of convergence, the vector condition is equivalent to m scalar conditions in turn,

$$\lim_{h\to\mathbf{0}_n} \frac{f_i(a+h) - f_i(a) - T_i(h)}{|h|} = 0_{\mathbf{R}}, \quad i = 1, \cdots, m.$$

And this last condition says precisely that each f_i is differentiable at a with derivative T_i.

4.1.5. *Let $f(x, y) = (x^2 - y^2, 2xy)$. Show that $Df_{(a,b)}(h, k) = (2ah - 2bk, 2bh + 2ak)$ for all $(a, b) \in \mathbf{R}^2$.*

We may work componentwise. The first component was discussed in the section. For the second component, compute that for any $(a, b) \in \mathbf{R}^2$ and any $(h, k) \in \mathbf{R}^2$, a little algebra gives

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{|f_2(a + h, b + k) - f_2(a, b) - (2bh - 2ak)|}{|(h, k)|} \\
&= \lim_{(h,k)\to(0,0)} \frac{|2(a + h)(b + k)^2 - 2ab - 2bh + 2ak|}{|(h, k)|} \\
&= \lim_{(h,k)\to(0,0)} \frac{|2hk|}{|(h, k)|} \leq \lim_{(h,k)\to(0,0)} \frac{2|(h, k)|^2}{|(h, k)|} = \lim_{(h,k)\to(0,0)} 2|(h, k)| = 0.
\end{aligned}$$

That is, the defining condition of the derivative is satisfied.

4.1.6. *Let $g(x, y) = xe^y$. Show that $Dg_{(a,b)}(h, k) = he^b + kae^b$ for all $(a, b) \in \mathbf{R}^2$.*

Compute that

$$\begin{aligned}
&g(a + h, b + k) - g(a, b) - he^b - kae^b \\
&\qquad = (a + h)e^{b+k} - ae^b - he^b - kae^b && \text{by definition of } g \\
&\qquad = (a + h)e^b(e^k - 1) - kae^b && \text{by algebra} \\
&\qquad = (a + h)e^b(e^k - e^0) - kae^b.
\end{aligned}$$

But by the Mean Value Theorem, $e^k - e^0 = (k - 0)e^\kappa$ for some κ between 0 and k. That is,

$$g(a + h, b + k) - g(a, b) - he^b - kae^b = (a + h)e^b ke^\kappa - kae^b.$$

It follows from this and Size Bounds that

$$\begin{aligned}
|g(a + h, b + k) - g(a, b) - he^b - kae^b| &\leq |k|\,|(a + h)e^b e^\kappa - ae^b| \\
&\leq |(h, k)|\,|(a + h)e^b e^\kappa - ae^b|,
\end{aligned}$$

and consequently for $(h, k) \neq \mathbf{0}$,

$$\frac{|g(a + h, b + k) - g(a, b) - he^b - kae^b|}{|(h, k)|} \leq |(a + h)e^b e^\kappa - ae^b|.$$

As $(h, k) \to \mathbf{0}$, also $a + h \to a$ and $e^\kappa \to 1$. This gives the equality at the second step of the following calculation,

$$\begin{aligned}
&\lim_{(h,k)\to\mathbf{0}} \frac{|g(a + h, b + k) - g(a, b) - he^b - kae^b|}{|(h, k)|} \\
&\leq \lim_{(h,k)\to\mathbf{0}} |(a + h)e^b e^\kappa - ae^b| \\
&= 0,
\end{aligned}$$

and the defining condition of the derivative is satisfied.

4.1.7. *Show that if $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ satisfies $|f(x)| \le |x|^2$ for all $x \in \mathbf{R}^n$ then f is differentiable at $\mathbf{0}_n$.*

Since the graph of f is trapped in a parabolic envelope, the only reasonable candidate for the derivative $Df_{\mathbf{0}}$ is the zero function $T(h) = 0$ for all h. Note that the condition $|f(x)| \le |x|^2$ specializes for $x = 0$ to $|f(\mathbf{0})| \le 0$, i.e., $f(\mathbf{0}) = 0$. Compute that indeed,

$$\lim_{h \to \mathbf{0}} \frac{|f(\mathbf{0}+h) - f(\mathbf{0}) - 0|}{|h|} = \lim_{h \to \mathbf{0}} \frac{|f(h)|}{|h|} \le \lim_{h \to \mathbf{0}} \frac{|h|^2}{|h|} = \lim_{h \to \mathbf{0}} |h| = 0,$$

and the defining condition of the derivative is satisfied.

4.2.2. *Prove part (2) of Proposition 4.2.2.*

We are assuming that f is differentiable at a with derivative Df_a. That is, we are assuming that

$$\lim_{h \to \mathbf{0}_n} \frac{|f(a+h) - f(a) - Df_a(h)|}{|h|} = 0.$$

We want to show that αf is differentiable at a with derivative αDf_a. That is, we want to show that

$$\lim_{h \to \mathbf{0}_n} \frac{|(\alpha f)(a+h) - (\alpha f)(a) - (\alpha Df_a)(h)|}{|h|} = 0.$$

By definition of scalar multiplication of mappings, this means showing that

$$\lim_{h \to \mathbf{0}_n} \frac{|\alpha \cdot f(a+h) - \alpha \cdot f(a) - \alpha \cdot Df_a(h)|}{|h|} = 0.$$

(Here the dots denote scalar-by-vector multiplication in $\mathbf{R}^m$. They do not denote inner product.) By vector algebra and then the fact that constants pass through limits, this means showing that

$$|\alpha| \lim_{h \to \mathbf{0}_n} \frac{|f(a+h) - f(a) - Df_a(h)|}{|h|} = 0.$$

But now the left side is a constant multiple of the limit that we know to be 0, and so we are done.

4.2.4. *Prove part (2) of Lemma 4.2.6.*

Let a be a nonzero real number. Then a is an interior point of the domain $\mathbf{R} - \{0\}$ of the reciprocal function r, so the derivative Dr_a at least might exist. Compute that for all h small enough,

$$\begin{aligned} r(a+h) - r(a) + \frac{h}{a^2} &= \frac{1}{a+h} - \frac{1}{a} + \frac{h}{a^2} = \frac{a^2 - a(a+h) + h(a+h)}{a^2(a+h)} \\ &= \frac{h^2}{a^2(a+h)}. \end{aligned}$$

It follows that

$$\lim_{h\to 0}\frac{|r(a+h)-r(a)+h/a^2|}{|h|}\le \lim_{h\to 0}\frac{|h|}{a^2|a+h|}=\frac{|0|}{a^2|a+0|}=0.$$

This shows that the linear mapping $T:\mathbf{R}\longrightarrow\mathbf{R}$ given by $T(h)=-h/a^2$ satisfies the defining condition of Dr_a.

(Alternatively, since r is a function of one variable, we may quote from one-variable calculus that

$$r'(a)=-\frac{1}{a^2},\quad a\neq 0.$$

By the discussion at the very beginning of the chapter, generalizing the one-variable derivative to many variables, it follows that the derivative of r at a in our new sense is $T(h)=-h/a^2$.)

4.2.6. *Let $f(x,y,z)=xyz$. Find $Df_{(a,b,c)}$ for arbitrary $(a,b,c)\in\mathbf{R}^3$.*

Use the product rule, the fact that X, Y, and Z are linear, and the fact that the derivative of a linear map is itself:

$$\begin{aligned}Df_{(a,b,c)}&=D((XY)Z)_{(a,b,c)}=(XY)(a,b,c)DZ_{(a,b,c)}+Z(a,b,c)D(XY)_{(a,b,c)}\\&=abZ+c(X(a,b,c)DY_{(a,b,c)}+Y(a,b,c)DX_{(a,b,c)})\\&=abZ+c(aY+bX)=abZ+bcX+caY.\end{aligned}$$

That is,

$$Df_{(a,b,c)}(h,k.\ell)=bch+cak+ab\ell.$$

4.2.8. *Recall that a function*

$$f:\mathbf{R}^n\times\mathbf{R}^n\longrightarrow\mathbf{R}$$

is called bilinear if for all $x,x',y,y'\in\mathbf{R}^n$ and all $\alpha\in\mathbf{R}$,

$$\begin{aligned}f(x+x',y)&=f(x,y)+f(x',y),\\f(x,y+y')&=f(x,y)+f(x,y'),\\f(\alpha x,y)&=\alpha f(x,y)=f(x,\alpha y).\end{aligned}$$

(a) *Show that if f is bilinear then* $\displaystyle\lim_{(h,k)\to(\mathbf{0}_n,\mathbf{0}_n)}\frac{|f(h,k)|}{|(h,k)|}=0.$

If $h=\mathbf{0}_n$ then

$$f(h,k)=f(0\cdot\mathbf{0}_n,k)=0\cdot f(\mathbf{0}_n,k)=0,$$

and similarly, if $k=\mathbf{0}_n$ then $f(h,k)=0$. So we may assume that h and k are nonzero. Let $\hat{h}=h/|h|$ and $\hat{k}=k/|k|$, so that $h=|h|\hat{h}$ and $k=|k|\hat{k}$. Then

$$f(h,k)=f(|h|\hat{h},|k|\hat{k})=|h|\,|k|\,f(\hat{h},\hat{k}),$$

and so

$$|f(h,k)| = |h|\,|k|\,|f(\hat{h},\hat{k})| \le |(h,k)|^2\,|f(\hat{h},\hat{k})|.$$

Thus

$$\frac{|f(h,k)|}{|(h,k)|} \le |(h,k)|\,|f(\hat{h},\hat{k})|,$$

and so it suffices to bound $f(\hat{h},\hat{k})$. But

$$\hat{h} = \sum_{i=1}^{n} h_i e_i \quad \text{where each } |h_i| \le 1,$$

and similarly for $\hat{k} = \sum_j k_j e_j$. Thus

$$\begin{aligned}
|f(\hat{h},\hat{k})| &= |f\Big(\sum_i h_i e_i, \sum_j k_j e_j\Big)| \\
&= |\sum_{i,j} h_i k_j f(e_i,e_j)| \\
&\le \sum_{i,j} |h_i|\,|k_j|\,|f(e_i,e_j)| \\
&\le \sum_{i,j} |f(e_i,e_j)| \\
&\overset{\text{call}}{=} C.
\end{aligned}$$

This completes the argument.

(b) *Show that if f is bilinear then f is differentiable with $Df_{(a,b)}(h,k) = f(a,k) + f(h,b)$.*

Compute that by bilinearity

$$\frac{|f(a+h,b+k) - f(a,b) - f(a,k) - f(h,b)|}{|(h,k)|} = \frac{|f(h,k)|}{|(h,k)|}.$$

By part (a), this tends to 0 as $(h,k) \to \mathbf{0}_{2n}$.

(c) *What does this exercise say about the inner product?*

The inner product function is differentiable. Its derivative is

$$D\langle,\rangle_{(a,b)}(h,k) = \langle a,k\rangle + \langle h,b\rangle.$$

4.3.3. *Define $f : \mathbf{R} \longrightarrow \mathbf{R}$ by*

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f'(x)$ exists for all x but that f' is discontinuous at 0. Explain how this disproves the converse of Theorem 4.3.3.

For $x \neq 0$, the methods of one variable calculus give

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad x \neq 0.$$

For $x = 0$, the quick solution is to recognize that exercise 4.1.8 applies here, showing that that the linear mapping derivative of f at 0 (i.e., Df_0) is the zero mapping, and hence $f'(0)$ is the 1-by-1 matrix with entry 0, i.e., $f'(0) = 0$. Alternatively, we can go back to the definition of one variable derivative:

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin 1/h - 0}{h} = \lim_{h \to 0} (h \sin 1/h).$$

But $h \to 0$ while $\sin 1/h$ is bounded, so this limit is 0. That is, regardless of which method we use,

$$f'(0) = 0.$$

So $f'(x)$ exists for all x. But the limit

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} (2x \sin 1/x - \cos 1/x)$$

does not exist: both $\sin 1/x$ and $\cos 1/x$ oscillate faster and faster near $x = 0$, but although the first term $2x \sin 1/x$ is a dampened oscillation that goes to 0, the second term $\cos 1/x$ oscillates without being dampened. So f' is discontinuous at 0.

The converse of Theorem 4.3.3 is the statement that if f is differentiable at a then all partial derivatives exist at a and about a, and they are continuous at a. The example here (with $n = m = 1$) is differentiable at 0, and all partial derivatives (i.e., the derivative itself in the one variable sense of derivative) exist at and about 0, but some partial derivative (again, i.e., the derivative itself in the one variable sense) fails to be continuous.

4.3.4. *Discuss the derivatives of the following mappings at the following points.*

(a) $f(x,y) = \frac{x^2 - y}{y+1}$ *on* $\{(x,y) \in \mathbf{R}^2 : y \neq -1\}$ *at generic* (a,b) *with* $b \neq -1$.

Each such (a,b) is an interior point of the domain of f. Since f is a rational function, the partial derivatives exist and are continuous at every point of its domain. Therefore Theorem 4.3.3 applies, and then Theorem 4.3.2. That is, we may simply compute the Jacobian matrix of partial derivatives of f at (a,b):

$$f'(a,b) = [D_1 f(a,b), D_2 f(a,b)] = \left[\frac{2a}{b+1}, -\frac{a^2+1}{(b+1)^2} \right].$$

The derivative is the corresponding linear map,

$$Df_{(a,b)}(h,k) = \frac{2a}{b+1} h - \frac{a^2+1}{(b+1)^2} k.$$

(Compare the ease of this method to the far more laborious derivation of the same formula from first principles at the end of section 4.2.)

(b) $f(x,y) = \frac{xy^2}{y-1}$ *on* $\{(x,y) \in \mathbf{R}^2 : y \neq 1\}$ *at generic* (a,b) *with* $b \neq 1$.

We may reason similarly to part (a) and then proceed to the calculation of the Jacobian matrix,

$$f'(a,b) = [D_1 f(a,b), D_2 f(a,b)] = \left[\frac{b^2}{b-1}, \frac{ab(b-2)}{(b-1)^2}\right].$$

Again the derivative is the corresponding linear map,

$$Df_{(a,b)}(h,k) = \frac{b^2}{b-1}h + \frac{ab(b-2)}{(b-1)^2}k.$$

(c) $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$ *at generic* $(a,b) \neq (0,0)$ *and at* $(0,0)$.

Away from $(0,0)$ we may proceed as in parts (a) and (b) since all partial derivatives exist around each such point and are continuous at each such point. The Jacobian matrix at a point $(a,b) \neq (0,0)$ is

$$f'(a,b) = [D_1 f(a,b), D_2 f(a,b)] = \left[\frac{b^3}{(a^2+b^2)^{3/2}}, \frac{a^3}{(a^2+b^2)^{3/2}}\right],$$

and so the derivative is

$$Df_{(a,b)}(h,k) = \frac{b^3}{(a^2+b^2)^{3/2}}h + \frac{a^3}{(a^2+b^2)^{3/2}}k.$$

At $(0,0)$ we can not quote general results from calculus since the function is defined casewise. But it is easy to compute that the partial derivatives at $(0,0)$ are both 0:

$$D_1 f(0,0) = \lim_{t\to 0} \frac{f(0+t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = \lim_{t\to 0} 0 = 0,$$

and similarly $D_2 f(0,0) = 0$. Therefore, by Theorem 4.3.2, the only possible candidate for $Df_{(0,0)}$ is the zero map, and we need to check whether it works. That is, we need to study

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - 0|}{|(h,k)|} = \lim_{(h,k)\to(0,0)} \frac{|hk|}{h^2+k^2},$$

to see whether it is 0. Let $(h,k) \to (0,0)$ along the 45-degree line, i.e., let $h = k$, and compute that

$$\frac{|f(h,h) - f(0,0) - 0|}{|(h,h)|^2} = \frac{h^2}{2h^2} = \frac{1}{2}.$$

This does not go to 0 as $h \to 0$. Therefore $Df_{(0,0)}$ does not exist.

4.3.6. *For what differentiable mappings $f : A \longrightarrow \mathbf{R}^m$ is $f'(a)$ a diagonal matrix for all $a \in A$?*

This depends a bit on what we mean by "diagonal matrix," but the main idea is that we want all off-diagonal entries of $f'(a)$ to be 0. For this to happen on the first row, whose entries are the partial derivatives of the first component function f_1 of f, in fact f_1 must be a function only of x_1. Similarly, the second component function f_2 of f must be a function only of x_2, and so on. That is,

$$f(x) = (f_1(x_1), \ldots, f_m(x_m)).$$

If we further insist that a diagonal matrix must be square then we are requiring that $m = n$. If not, then when $m > n$, the previous display connotes that last component functions f_{n+1} through f_m must be constant functions.

4.3.8. *Let $w = F(xz, yz)$. Show that $x \cdot w_x + y \cdot w_y = z \cdot w_z$.*

Compute by the Chain Rule that

$$\begin{aligned}
w_x &= D_1F(xz, yz) \cdot z + D_2F(xz, yz) \cdot 0 = zD_1F(xz, yz), \\
w_y &= D_1F(xz, yz) \cdot 0 + D_2F(xz, yz) \cdot z = zD_2F(xz, yz), \\
w_z &= D_1F(xz, yz) \cdot x + D_2F(xz, yz) \cdot y.
\end{aligned}$$

The result follows immediately.

4.3.10. *The function $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ is called* **homogeneous of degree** k *if $f(tx, ty) = t^k f(x, y)$ for all scalars t and vectors (x, y). Show that such f satisfies the differential equation*

$$xf_1(x, y) + yf_2(x, y) = kf(x, y).$$

Since $f(tx, ty) = t^k f(x, y)$, the partial derivatives of both sides with respect to t are equal,

$$f_1(tx, ty) \cdot x + f_2(tx, ty) \cdot y = kt^{k-1} f(x, y).$$

Now set $t = 1$ to get the result.

Alternatively, one can let $w = f(tx, ty)$ and then cite exercise 4.3.8 with t in place of z to get

$$xw_x + yw_y = tw_t.$$

But also $w = t^k f(x, y)$, so that $w_x = t^k f_1(x, y)$ and $w_y = t^k f_2(x, y)$ and $w_t = kt^{k-1} f(x, y)$. Thus the previous display becomes

$$xt^k f_1(x, y) + yt^k f_2(x, y) = kt^k f(x, y).$$

In particular, this holds for $t = 1$, giving the result.

4.4.2. *Suppose u, as a function of x and y, satisfies the differential equation $u_{xx} - u_{yy} = 0$. Make the change of variables $x = s + t$, $y = s - t$. What corresponding differential equation does u satisfy when viewed as a function of s and t?*

Compute that since $x_s = 1$ and $y_s = 1$,

$$u_s = u_x x_s + u_y y_s = u_x + u_y,$$

and then, since also $x_t = 1$ and $y_t = -1$,

$$u_{st} = u_{xx}x_t + u_{xy}y_t + u_{yx}x_t + u_{yy}y_t = u_{xx} - u_{yy}$$

But we are given that this is 0, so the new differential equation is

$$u_{st} = 0.$$

4.4.3. *Show that if $u = F(x - ct) + G(x + ct)$ then $c^2 u_{xx} = u_{tt}$.*
Compute that

$$u_x = F'(x - ct) + G'(x + ct), \qquad u_{xx} = F''(x - ct) + G''(x + ct),$$

and compute that

$$u_t = -cF'(x - ct) + cG'(x + ct), \qquad u_{tt} = (-c)^2 F''(x - ct) + c^2 G''(x + ct),$$

The result follows.

4.4.4. *Show that the substitution $x = e^s$, $y = e^t$ converts the equation*

$$x^2 u_{xx} + y^2 u_{yy} + x u_x + y u_y = 0$$

into Laplace's equation $u_{ss} + u_{tt} = 0$.
Compute that since $x_s = x$ and y is independent of s, the chain rule simplifies to

$$u_s = x u_x,$$

so that by the product rule and then the chain rule again,

$$u_{ss} = (x u_x)_s = x_s u_x + x(u_x)_s = x u_x + x^2 u_{xx}.$$

Symmetrical calculations with t and y in place of s and x give

$$u_t = y u_y, \qquad u_{tt} = y u_y + y^2 u_{yy}.$$

Thus the original given equation is precisely $u_{ss} + u_{tt} = 0$.