EULER'S PROOF OF INFINITELY MANY PRIMES

1. Bound From Euclid's Proof

Recall Euclid's proof that there exist infinitely many primes: *If p_1 through p_n are prime then the number*

$$q = 1 + \prod_{i=1}^{n} p_i$$

is not divisible by any p_i. According to this argument, the next prime after p_1 through p_n could be as large as q. The overestimate is astronomical. Specifically, compute that for $n \geq 3$, since

$$p_n \leq 1 + p_1 \cdots p_{n-1} \leq (7/6) p_1 \cdots p_{n-1},$$

it follows that

$$\begin{aligned} p_n &\leq (7/6) p_1 \cdots p_{n-1} \\ &\leq (7/6)^2 (p_1 \cdots p_{n-2})^2 \\ &\leq (7/6)^4 (p_1 \cdots p_{n-3})^4 \\ &\leq \cdots \\ &\leq (7/6)^{2^{n-3}} (p_1 p_2)^{2^{n-3}} \\ &= 7^{2^{n-3}} \quad \text{(since } p_1 p_2 = 6) \\ &< e^{2^{n-2}}. \end{aligned}$$

So, for example, the tenth prime p_{10} satisfies $p_{10} < 1.51143 \times 10^{11}$. Since in fact $p_{10} = 29$, we see how little Euclid's argument tells us.

2. Euler's Result on the Density of Primes

By contrast, Euler argued that

$$\sum_{p \in \mathcal{P}} \frac{1}{p} \quad \textit{diverges},$$

and in fact his argument shows more,

$$\sum_{p \in \mathcal{P}} \frac{1}{p^s} \sim \log\left(\frac{1}{s-1}\right) \quad \text{as } s \to 1^+.$$

Here the "$\sim$" symbol means that the ratio of the two quantities goes to 1 in the limit.

The argument proceeds as follows. Define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1.$$

Comparing box-areas and the area under the curve $y = 1/x^s$ gives

$$\frac{1}{s-1} = \int_1^\infty x^{-s}\,\mathrm{d}x < \zeta(s) < 1 + \int_1^\infty x^{-s}\,\mathrm{d}x = 1 + \frac{1}{s-1}.$$

This shows that

$$\zeta(s) \sim \frac{1}{s-1} \quad \text{as } s \to 1^+,$$

and so

$$\log(\zeta(s)) \sim \log\left(\frac{1}{s-1}\right) \quad \text{as } s \to 1^+.$$

So to prove Euler's result, we need to show that

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s} \sim \log(\zeta(s)) \quad \text{as } s \to 1^+.$$

We establish a product expansion of the sum $\zeta(s)$. Sum over values of n with steadily more prime factors, as follows.

$$\begin{aligned}
\sum_{n=2^{e_2}} n^{-s} &= \sum_{e_2=0}^\infty (2^{-s})^{e_2} = (1-2^{-s})^{-1}, \\
\sum_{n=2^{e_2}3^{e_3}} n^{-s} &= \sum_{e_2=0}^\infty (2^{-s})^{e_2} \sum_{e_3=0}^\infty (3^{-s})^{e_3} = (1-2^{-s})^{-1}(1-3^{-s})^{-1}, \\
&\vdots \\
\sum_{n=2^{e_2}\cdots p^{e_p}} n^{-s} &= (1-2^{-s})^{-1}\cdots(1-p^{-s})^{-1}.
\end{aligned}$$

And so, being very casual about convergence, it is essentially a restatement of unique factorization that the zeta function also has an infinite product expression,

$$\zeta(s) = \sum_{n\in\mathbb{Z}^+} n^{-s} = \prod_{p\in\mathcal{P}} (1-p^{-s})^{-1}.$$

From the general series

$$\log(1-X)^{-1} = \sum_{n=1}^\infty X^n/n, \quad |X| < 1,$$

we have (again being very casual about convergence)

$$\log\zeta(s) = \log\prod_{p\in\mathcal{P}}(1-p^{-s})^{-1} = \sum_{p\in\mathcal{P}}\log(1-p^{-s})^{-1} = \sum_{p\in\mathcal{P}}\sum_{n=1}^\infty p^{-ns}/n.$$

Separating out the $n = 1$ part of the last double sum,

$$\log\zeta(s) = \sum_{p\in\mathcal{P}} p^{-s} + \sum_{p\in\mathcal{P}}\sum_{n=2}^\infty p^{-ns}/n.$$

The previous display contains the two quantities $\log\zeta(s)$ and $\sum_{p\in\mathcal{P}} p^{-s}$ that we are trying to show grow at the same rate as $s \to 1^+$. To do so, we show that their difference is small,

$$\sum_{p\in\mathcal{P}}\sum_{n=2}^\infty p^{-ns}/n < \sum_{p\in\mathcal{P}}\sum_{n=2}^\infty p^{-ns} = \sum_{p\in\mathcal{P}} p^{-2s}(1-p^{-s})^{-1} = \sum_{p\in\mathcal{P}} \frac{1}{p^s(p^s-1)}\,,$$

and

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s(p^s-1)} < \sum_{n=2}^{\infty} \frac{1}{n^s(n^s-1)} < \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1.$$

Thus

$$\left|\sum_{p\in\mathcal{P}} p^{-s} - \log\zeta(s)\right| < 1.$$

And so, as desired,

$$\sum_{p\in\mathcal{P}} p^{-s} \sim \log\zeta(s) \sim \log\left(\frac{1}{s-1}\right) \quad \text{as } s\to 1^+.$$

meaning that

$$\lim_{s\to 1^+} \frac{\sum_p p^{-s}}{\log\left(\frac{1}{s-1}\right)} = 1.$$

Euler's result is much stronger than Euclid's, and it illustrates *analytic number theory*.