EULER'S PROOF OF INFINITELY MANY PRIMES

Recall Euclid's proof that there exist infinitely many primes: *If p_1 through p_n are prime then the number*

$$q = 1 + \prod_{i=1}^{n} p_i$$

is not divisible by any p_i. According to this argument, the next prime after p_1 through p_n could be as large as q. The overestimate is astronomical. Specifically, compute that for $n \geq 3$, since

$$p_n \leq 1 + p_1 \cdots p_{n-1} \leq (7/6)p_1 \cdots p_{n-1},$$

it follows that

$$\begin{aligned}
p_n &\leq (7/6)p_1 \cdots p_{n-1} \\
&\leq (7/6)^2 (p_1 \cdots p_{n-2})^2 \\
&\leq (7/6)^4 (p_1 \cdots p_{n-3})^4 \\
&\leq \cdots \\
&\leq (7/6)^{2^{n-3}} (p_1 p_2)^{2^{n-3}} \\
&= 7^{2^{n-3}} \quad \text{(since } p_1 p_2 = 6) \\
&< e^{2^{n-2}}.
\end{aligned}$$

So, for example, the tenth prime p_{10} satisfies $p_{10} < 1.51143 \times 10^{11}$. Since in fact $p_{10} = 29$, we see how little Euclid's argument tells us.

By contrast, Euler argued that

$$\sum_{p \in \mathcal{P}} \frac{1}{p} \quad \textit{diverges},$$

and in fact his argument shows more. The argument proceeds as follows. Define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \quad s > 1.$$

Note that

$$\lim_{s \to 1^+} \zeta(s) = \infty,$$

so that also

$$\lim_{s \to 1^+} \log \zeta(s) = \infty,$$

(The logarithm is natural, of course.)

Now, summing over values of n with steadily more prime factors gives

$$\begin{aligned}
\sum_{n=2^{e_2}} n^{-s} &= \sum_{e_2=0}^{\infty} (2^{-s})^{e_2} = (1-2^{-s})^{-1}, \\
\sum_{n=2^{e_2}3^{e_3}} n^{-s} &= \sum_{e_2=0}^{\infty} (2^{-s})^{e_2} \sum_{e_3=0}^{\infty} (3^{-s})^{e_3} = (1-2^{-s})^{-1}(1-3^{-s})^{-1}, \\
&\vdots \\
\sum_{n=2^{e_2}\cdots p^{e_p}} n^{-s} &= (1-2^{-s})^{-1}\cdots(1-p^{-s})^{-1}.
\end{aligned}$$

And so, being very casual about convergence, it is essentially a restatement of unique factorization that the zeta function also has an infinite product expression,

$$\zeta(s) = \sum_{n\in\mathbb{Z}^+} n^{-s} = \prod_{p\in\mathcal{P}} (1-p^{-s})^{-1}.$$

From the general series

$$\log(1-X)^{-1} = \sum_{n=1}^{\infty} X^n/n, \quad |X|<1,$$

we have (again being very casual about convergence)

$$\begin{aligned}
\log\zeta(s) = \log\prod_{p\in\mathcal{P}}(1-p^{-s})^{-1} &= \sum_{p\in\mathcal{P}} \log(1-p^{-s})^{-1} = \sum_{p\in\mathcal{P}}\sum_{n=1}^{\infty} p^{-ns}/n \\
&= \sum_{p\in\mathcal{P}} p^{-s} + \sum_{p\in\mathcal{P}}\sum_{n=2}^{\infty} p^{-ns}/n.
\end{aligned}$$

From above, we know that $\lim_{s\to 1^+}\log\zeta(s)=\infty$, while one term on the right side of the previous display is $\sum_{p\in\mathcal{P}} p^{-s}$, which we want to understand as s tends to 1. As for the other term,

$$\sum_{p\in\mathcal{P}}\sum_{n=2}^{\infty} p^{-ns}/n < \sum_{p\in\mathcal{P}}\sum_{n=2}^{\infty} p^{-ns} = \sum_{p\in\mathcal{P}} p^{-2s}(1-p^{-s})^{-1} = \sum_{p\in\mathcal{P}} \frac{1}{p^s(p^s-1)},$$

and

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s(p^s-1)} < \sum_{n=2}^{\infty} \frac{1}{n^s(n^s-1)} < \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1.$$

Thus the quantity that we want to understand is bounded on both sides by quantities that we do understand,

$$\log\zeta(s) - 1 < \sum_{p\in\mathcal{P}} p^{-s} < \log\zeta(s).$$

And so

$$\lim_{s\to 1^+} \sum_{p\in\mathcal{P}} p^{-s} = \infty,$$

and more specifically,

$$\lim_{s\to 1^+} \frac{\sum_p p^{-s}}{\log\zeta(s)} = 1.$$

Thus *the sum of prime reciprocals grows asymptotically as the logarithm of the harmonic series*. Recall that the partial sums of the harmonic series themselves grow logarithmically, so that the sum of prime reciprocals grows very slowly. Euler's result is much stronger than Euclid's, and it illustrates *analytic number theory*.