COMPUTABLE NUMBERS

1. Introduction

Turing's paper is "On **computable numbers**, with an application to the Entscheidungsproblem." (Emphasis mine.) What are computable numbers?

2. The Definition

A real number is a Turing machine function

$$a : \mathbf{Z}^+ \longrightarrow \mathbf{Q} \times \mathbf{Q}_{\geq 0}, \qquad n \longmapsto (m_n, e_n).$$

The classical intuition is that for each positive integer n, the interval

$$I_n(a) = [m_n - e_n, m_n + e_n]$$

contains the "limit" quantity a. But really a is a function. And it must satisfy two conditions:

- $I_n(a) \cap I_{n'}(a) \neq \emptyset$ for all $n, n' \in \mathbf{Z}^+$.
- For any $E > 0$, it is certain that some $n_0 \in \mathbf{Z}^+$ exists such that $e_n < E$ for all $n > n_0$.

3. Equality

Two real numbers a and b are equal if

$$I_n(a) \cap I_{n'}(b) \neq \emptyset \quad \text{for all } n, n' \in \mathbf{Z}^+.$$

The proofs that equality is reflexive and symmetric are immediate. The proof that equality is transitive requires a small argument.

So in fact a real number is specified by an entire *equivalence class* of functions.

Another small argument shows that trichotomy holds for real numbers.

4. Algebra

One can use *interval arithmetic* to define addition, subtraction, multiplication, and division for real numbers. Examples of interval arithmetic are

$$(m_1 \pm e_1) + (m_2 \pm e_2) = (m_1 + m_2) \pm (e_1 + e_2)$$

and

$$(m_1 \pm e_1)(m_2 \pm e_2) = (m_1 m_2) \pm (|m_1|e_2 + |m_2|e_1 + e_1 e_2).$$

5. An Undecidable Problem

For each Turing maching M define a corresponding number a_M as follows:

$$a_M(n) = \begin{cases} 0 \pm 10^{-n} & \text{if } M \text{ does not halt within } n \text{ steps,} \\ 10^{-h} \pm 10^{-n} & \text{if } M \text{ halts in } h \text{ steps, where } h \leq n. \end{cases}$$

Then a_M is zero if and only if M does not halt.

Suppose that a Turing machine D decides whether real numbers are zero or not. Then D solves the halting problem. This is impossible. Therefore, the problem of whether an arbitrary real number os zero is undecidable.

6. Other Nonsolvable Problems

- Are two real numbers equal?
- Is one real number bigger than another?
- Is one real number at least as big as another?
- Given two real numbers, which of the three trichotomous relations between them holds?

As an example of this last, consider two real numbers a and b where

$$a(n) = 0 \pm 0 \quad \text{for all } n$$

and

$$\begin{aligned} b(1) &= 0.0 \pm 1 \quad \text{i.e., } b(1) = [-0.1, 0.1] \\ b(2) &= 0.00 \pm 1 \quad \text{i.e., } b(2) = [-0.01, 0.01] \\ b(3) &= 0.000 \pm 1 \quad \text{i.e., } b(3) = [-0.001, 0.001] \\ &\vdots \end{aligned}$$

Is b positive, negative, or zero? We don't know. It could be the case that

$$b(n) = 0 \pm 10^{-n} \quad \text{for all } n,$$

in which case $b = a = 0$. But, for example, it could also be the case that for some large N,

$$b(n) = 10^{-N} \pm 10^{-n} \quad \text{for all } n \geq N,$$

in which case $b = 10^{-N} > 0$. And similarly, b could well be -10^{-N} for some large N.

7. Solvable Problems

- Given real numbers a, b, and a positive integer k, choose a true statement among the three possibilities
  $$a > b, \quad |a - b| < \pm 10^{-k}, \quad a < b.$$
  Note that the possibilities are not mutually exclusive.