CONGRUENT NUMBERS

The Congruent Number problem is: *Let n be a positive integer. Is there a right triangle, with the lengths of all three of its sides being rational numbers, that has area n?*

Fermat showed that $n = 1$ is not a congruent number. Euler showed that $n = 7$ is a congruent number. Eventually, it became known that $n = 1, 2, 3, 4$ are not congruent, but $n = 5, 6, 7$ are. But an efficient procedure to answer the question for general n seemed hopelessly out of reach.

A terribly inefficient procedure does exist that will eventually answer "Yes" if n is congruent and run forever if not. But if we ask the procedure whether some value of n (say, $n = 157$) is a congruent number, and the procedure then runs for a century without giving us an answer, then we don't know if this is because $n = 157$ congruent but the procedure hasn't figured this out yet, or because $n = 157$ is not congruent.

In fact, $n = 157$ *is* congruent. Let

$$a = \frac{411340519227716149383203}{21666555693714761309610}$$

and

$$b = \frac{6803298487826435051217540}{411340519227716149383203}$$

and

$$c = \frac{224403517704336969924557513090674863160948472041}{8912332268928859588025535178967163570016480830}.$$

Then

$$a^2 + b^2 = c^2,$$

and

$$\tfrac{1}{2}ab = 157.$$

But an example this gigantic must be using ideas beyond brute-force search.

In fact, the following result holds:

Let n be an odd squarefree natural number. Consider the following two conditions:

- *The number n is congruent.*
- *The number of integer triples (x, y, z) satisfying the equation*
$$2x^2 + y^2 + 8z^2 = n$$
is equal to twice the number of integer triples satisfying the equation
$$2x^2 + y^2 + 32z^2 = n.$$

Then the first condition implies the second; and if a weak form of the so-called Birch–Swinnerton-Dyer conjecture is true then also the second condition implies the first.

Note that checking the second condition for a given n terminates in a finite number of steps, and in fact in a number of steps that we can bound explicitly.

And a similar result holds for even n.