Rees Valuations

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Abstract This expository paper contains history, definitions, constructions, and the basic properties of Rees valuations of ideals. A section is devoted to one-fibered ideals, that is, ideals with only one Rees valuation. Cutkosky [5] proved that there exists a two-dimensional complete Noetherian local integrally closed domain in which no zero-dimensional ideal is one-fibered. However, no concrete ring of this form has been found. An emphasis in this paper is on bounding the number of Rees valuations of ideals. The last section is about the Izumi–Rees Theorem, which establishes comparability of Rees valuations with the same center. More on Rees valuations can be done via the projective equivalence of ideals, and there have been many articles along that line. See the latest article by Heinzer, Ratliff, and Rush [11], in this volume.

All rings in this paper are commutative with identity, and most are Noetherian domains. The following notation will be used throughout:

- $Q(R)$ denotes the field of fractions of a domain R.
- For any prime ideal P in a ring R, $\kappa(P)$ denotes the field of fractions of R/P.
- If V is a valuation ring, $\mathfrak{m}_V$ denotes its unique maximal ideal, and v denotes an element of the equivalence class of valuations naturally determined by V.
- We say that a Noetherian valuation is **normalized** if its value group is a subset of $\mathbb{Z}$ whose greatest common divisor is 1.
- If R is a ring and V is a valuation overring, then the **center** of V on R is $\mathfrak{m}_V \cap R$.
- A valuation ring V (or a corresponding valuation v) is said to be **divisorial** with respect to a subdomain R if $Q(R) = Q(V)$ and if $\text{tr.deg}_{\kappa(p)}\kappa(\mathfrak{m}_V) =$

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ht $p-1$, where $p = \mathfrak{m}_V \cap R$. It is a fact that every divisorial valuation with respect to a Noetherian ring R is essentially of finite type over R.
- If I is an ideal in a ring R, the I-adic **order** is the function $\text{ord}_I : R \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ given by

$$\text{ord}_I(x) = \sup\{n \in \mathbb{Z} : x \in I^n\}.$$

- If R is a domain and v is a valuation on $Q(R)$, we allow v to be defined on all of R by setting $v(0) = \infty$.
- For any Noetherian valuation v that is non-negative on a domain R, and for any ideal I in R, $v(I)$ is $\min\{v(r) : r \in I\}$.

Many of the omitted proofs can be found in Chapter 10 in [17].

1 Introduction

David Rees was the first to systematically study the valuations associated to an ideal that were later called Rees valuations. Rees proved their existence, uniqueness, he proved that they determine the integral closures of the powers of the given ideal, and over the years he and others proved many applications.

Recall that if I is an ideal in a ring R, we denote by $\bar{I}$ the integral closure of I, namely,

$$\bar{I} = \{r \in R : r^n + a_1 r^{n-1} + \cdots + a_n = 0 \text{ for some positive } n \text{ and some } a_j \in I^j\}.$$

It is well-known that this equals

$$\bar{I} = \bigcap_V IV \cap R,$$

as V varies over the valuation rings that are R-algebras, or alternatively, as V varies over the valuation rings between R/P and $Q(R/P)$, and P varies over the set $\text{Min}(R)$ of minimal prime ideals in R. In case R is Noetherian, the valuation rings may all be restricted to be Noetherian valuations rings. If R/I is Artinian, by the descending chain property, there exists a finite set S of valuation rings such that $\bar{I} = \bigcap_{V \in S} IV \cap R$. Finiteness of the needed set of valuations is a desirable property in general, as it simplifies existence proofs and algorithmic computations. Rees valuations aim for more: there are finitely many valuations that suffice in the sense above not just for I but also for $I, I^2, I^3, I^4, \ldots$ simultaneously. Here is a formal definition:

Definition 1.1. Let R be a ring and I an ideal in R. A set of **Rees valuation rings** of I is a set $\{V_1, \ldots, V_s\}$ consisting of valuation rings, subject to the following conditions:

1. Each V_i is Noetherian and is not a field.

2. For each $i = 1, \dots, s$, there exists a minimal prime ideal P_i of R such that V_i is a ring between R/P_i and $Q(R/P_i)$.
3. For all $n \in \mathbb{N}$, $\overline{I^n} = \cap_{i=1}^{s} (I^n V_i) \cap R$.
4. The set $\{V_1, \dots, V_s\}$ satisfying the previous conditions is minimal possible.

By a slight abuse of notation, the notation $\mathcal{RV}(I)$ stands for a set of Rees valuation rings of I – even though the set is in general not uniquely determined. (Uniqueness is discussed in Section 2.)

For each valuation ring there is a natural corresponding equivalence class of valuations. A set of representatives of valuations for a set of Rees valuation rings is called a set of **Rees valuations**. Typically, we take the normalized representatives.

By the standard abuse of notation, each valuation v is defined on the whole ring R and takes on in addition the value ∞, with $\{r \in R : v(r) = \infty\}$ being a prime ideal. If a valuation v_i corresponds to a Rees valuation ring V_i, then $\{r \in R : v_i(r) = \infty\}$ is precisely the minimal prime ideal P_i of R as in condition 2. above. With this notation, condition 3. translates to:

$$\overline{I^n} = \{r \in R : v_1(r) \geq n v_1(I), \dots, v_s(r) \geq n v_s(I)\} \text{ for all } n \geq 0.$$

All Rees valuations are constructed as localizations of the integral closures of finitely many finitely generated R-algebras contained in $Q(R)$, as we explain in Section 2. One idea of where Rees valuations might be found is contained in the following observation: If $\{V_1, \dots, V_s\}$ is a set of Rees valuation rings of I (unique or not), then for all n,

$$\overline{I^n} = \bigcap_{j=1}^{s} (I^n V_j \cap R)$$

is a (possibly redundant) primary decomposition of $\overline{I^n}$, and thus

$$\begin{aligned} \bigcup_{n \geq 1} \operatorname{Ass}(R/\overline{I^n}) &\subseteq \{\{r \in R : rV_j \neq V_j\} : j = 1, \dots, s\} & (1) \\ &= \{\mathfrak{m}_{V_j} \cap R : j = 1, \dots, s\} & (2) \end{aligned}$$

is a finite set.

It is straightforward to verify that for all ideals I, $\mathcal{RV}(I) = \mathcal{RV}(\overline{I})$.

A basic property of Rees valuations of an ideal is that they localize, in the sense that for any multiplicatively closed set W in R, $\mathcal{RV}(W^{-1}I) = \{V \in \mathcal{RV}(I) : \mathfrak{m}_V \cap W = \emptyset\}$. This follows in a straightforward way from the definitions.

How do Rees valuations behave under extending the ideal to an overring? Let $R \to S$ be a ring homomorphism of rings such that S is either faithfully flat over R or S is integral over R. Then for any ideal I in R, $\overline{I} = \overline{IS} \cap R$

(proofs can be found in Propositions 1.6.1 and 1.6.2 of [17]). If Rees valuations exist for IS, this implies that

$$\mathcal{RV}(I) \subseteq \{V \cap Q(R) : V \in \mathcal{RV}(IS)\}.$$

If S is the integral closure of the Noetherian domain R in its field of fractions, even equality holds, i.e., $\mathcal{RV}(I) = \{V \cap Q(R) : V \in \mathcal{RV}(IS)\} = \mathcal{RV}(IS)$. Furthermore, if $(R, \mathfrak{m})$ is a Noetherian local ring and I is an $\mathfrak{m}$-primary ideal of I, then $\overline{I}\widehat{R}$ is the integral closure of $I\widehat{R}$, whence also in this case, $\mathcal{RV}(I) = \{V \cap Q(R) : V \in \mathcal{RV}(I\widehat{R})\}$. If in addition $\widehat{R}$ is a domain, no two Noetherian valuations on $Q(\widehat{R})$ centered on $m\widehat{R}$ contract to the same valuation on $Q(R)$, so in that case the numbers of Rees valuations of I and of $I\widehat{R}$ are the same if I is $\mathfrak{m}$-primary. More generally, Katz and Validashti [21, Theorem 5.3] proved the following:

Theorem 1.2. *(Katz and Validashti [21, Theorem 5.3]) Let I be an ideal in a Noetherian local ring $(R, \mathfrak{m})$ that is not contained in any minimal prime ideal. Let w be a Rees valuation of $I\widehat{R}$ with center $\mathfrak{m}\widehat{R}$, and let Q be the corresponding minimal prime ideal in $\widehat{R}$ such that w is a valuation on $\kappa(Q)$. Then w restricted to $\kappa(Q \cap R)$ is a Rees valuation of I with center $\mathfrak{m}$. The function*

$$w \mapsto w|_{\kappa(\{r \in R : w(r) = \infty\})}$$

from Rees valuations of $I\widehat{R}$ with center on $\mathfrak{m}\widehat{R}$ to Rees valuations of I with center on $\mathfrak{m}$ is a one-to-one and onto function.

The results above were in the direction of where to search for Rees valuations of a given ideal; the following result searches for an ideal for which a given valuation is a Rees valuation:

Proposition 1.3. *Let R be a Noetherian domain. Let V be a divisorial valuation ring with respect to R. Then there exists an ideal I in R, primary to $P = \mathfrak{m}_V \cap R$, such that V is one of its Rees valuation rings.*

Conversely, let J be an ideal and W a Rees valuation ring of J. Set $P = \mathfrak{m}_W \cap R$ and assume that R_P is formally equidimensional. Then W is a divisorial valuation ring with respect to R and R_P.

(A proof can be found for example in [17, Propositions 10.4.3 and 10.4.4].)

For this reason, on Noetherian locally formally equidimensional domains, the Rees valuations of non-zero ideals are the same as the divisorial valuations with respect to R.

Examples

1. A maximal ideal $\mathfrak{m}$ in a regular ring has only one Rees valuation, namely the $\mathfrak{m}$-adic valuation. The $\mathfrak{m}$-adic valuation ring equals $R[\frac{\mathfrak{m}}{x}]_{(x)R[\frac{\mathfrak{m}}{x}]}$ for any $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.

2. Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring over a field k. For any monomial ideal I in R, the convex hull of the set $\{(a_1, \ldots, a_d) \in \mathbb{N}^d : X_1^{a_1} \cdots X_d^{a_d} \in I\}$ in $\mathbb{R}^d$ is called the **Newton polyhedron** of I, and is denoted $\mathrm{NP}(I)$. The Newton polyhedron of I contains the information on the integral closure of I:
$$\bar{I} = (X_1^{a_1} \cdots X_d^{a_d} \mid (a_1, \ldots, a_d) \in \mathrm{NP}(I) \cap \mathbb{N}^d).$$
By Carathéodory's Theorem the convex hull is bounded by the coordinate hyperplanes and by finitely many other faces/hyperplanes, each of the form $c_1X_1 + \cdots + c_dX_d = 1$ for some $c_i \in \mathbb{Q}$. The corresponding hyperplane bounding the Newton polyhedron of I^n has the form $c_1X_1+\cdots+c_dX_d = n$. Thus
$$\begin{aligned}\overline{I^n} = (X_1^{a_1} \cdots X_d^{a_d} : c_1a_1 + \cdots + c_da_d \geq n,&\\ \text{as } (c_1, \ldots, c_d) \text{ varies over the hyperplane coefficients as above}).&\end{aligned}$$
For each such irredundant bounding hyperplane $c_1X_1 + \cdots + c_dX_d = 1$, define a **monomial valuation** $v : R \to \mathbb{Q}$ by $v(X_1^{a_1} \cdots X_d^{a_d}) = c_1a_1 + \cdots + c_da_d$ and by $v(\sum_{i=1}^n m_i) = \min\{v(m_i) : i = 1, \ldots, n\}$, where the m_i are products of non-zero elements of k with distinct monomials. By above, these valuations are the Rees valuations of I. (Hübl and the author generalized this in [16] to all ideals generated by monomials in a regular system of parameters.)
3. In particular, in a polynomial ring $k[X_1, \ldots, X_n]$, every ideal of the form $(X_1^{a_1}, \ldots, X_m^{a_m})$ has only one Rees valuation.
4. The above can easily be worked on the monomial ideal (X, Y) in the polynomial ring $R = k[X, Y, Z]$ over a field k, to get that the only Rees valuation of (X, Y) is the monomial valuation v_1, which takes value 1 on X and Y and value 0 on Z.
Similarly, the only Rees valuation of $(X, Z)k[X, Y, Z]$ is the monomial valuation v_2 taking value 1 on X and Z and value 0 on Y.
The valuations v_1 and v_2 are not the only Rees valuations of the product of the two ideals (X, Y) and (X, Z), by the following reasoning: $v_1((X, Y)(X, Z)) = 1$ since $v_1(XZ) = 1$, and similarly $v_2((X, Y)(X, Z)) = 1$. However, $v_1(X) = v_2(X) = 1$ as well, but $X \notin \overline{(X, Y)(X, Z)}$.

Related to the last example is the following:

Proposition 1.4. *(Cf. Muhly–Sakuma [29]) For any non-zero ideals I and J in a Noetherian domain R, $\mathcal{RV}(I) \cup \mathcal{RV}(J) \subseteq \mathcal{RV}(IJ)$. If I is locally principal or if R is Noetherian locally formally equidimensional of dimension at most 2, then $\mathcal{RV}(I) \cup \mathcal{RV}(J) = \mathcal{RV}(IJ)$.*

The example above the proposition shows that the inclusion $\mathcal{RV}(I) \cup \mathcal{RV}(J) \subseteq \mathcal{RV}(IJ)$ may be proper.

On page 3 it was mentioned that $\mathcal{RV}(I) = \mathcal{RV}(J)$ if the integral closures of I and J coincide. It is similarly clear that for ideals $I \subseteq J$, $\overline{I} = \overline{J}$ if and only if for every Rees valuation ring V of I, $IV = JV$. It is not much harder to prove that for every positive integer n, $\mathcal{RV}(I) = \mathcal{RV}(I^n)$. Furthermore, if $\overline{I^m} = \overline{J^n}$ for some positive integers m, n, then $\mathcal{RV}(I) = \mathcal{RV}(J)$. However, the converse may fail, namely $\mathcal{RV}(I) = \mathcal{RV}(J)$ does not imply that $\overline{I^m} = \overline{J^n}$ for some positive integers m, n. For example, let $I = (X^2, Y)^5 \cap (X, Y^2)^4$ and $J = (X^2, Y)^4 \cap (X, Y^2)^5$, and use the monomial ideal method above for finding the Rees valuations. The two ideals I and J have the same two monomial Rees valuations, yet the integral closures of powers of I do not coincide with the integral closures of powers of J.

We recall some more vocabulary: an ideal J is a **reduction** of an ideal I if $J \subseteq I$ and $\overline{J} = \overline{I}$. The first crucial paper on reductions is [31], by Northcott and Rees.

2 Existence and uniqueness

There is the question of existence and uniqueness of Rees valuation rings. For the zero ideal in a domain, any one Noetherian valuation ring V between R and $Q(R)$ is the Rees valuation ring. Thus we have existence but not uniqueness in this case.

The first case of the existence and uniqueness of Rees valuations was proved by David Rees in [32], for zero-dimensional ideals in equicharacteristic Noetherian local rings. The second case was proved by Rees in [34] for arbitrary ideals I in Noetherian domains for which the following Artin–Rees-like assumption holds: there exists an integer t such that for all sufficiently large n, $\overline{I^{t+n+1}} \cap I^n \subseteq I^{n+1}$. Neither of the two cases of existence in [32] and [34] is covered by the other. The general existence theorem, for all ideals in Noetherian rings, was proved by Rees in [35]. Uniqueness was proved in [32]. Here is a summary general result (for a proof, see for example [17, Theorems 10.1.6 and 10.2.2]):

Theorem 2.1. *(Existence and uniqueness of Rees valuations) Let R be a Noetherian ring. Then for any ideal I of R, there exists a set of Rees valuation rings, and if I is not contained in any minimal prime ideal of R, then the set of Rees valuation rings is uniquely determined.*

The main case of the proof of the existence is actually when R is a Noetherian domain and I is a non-zero principal ideal. In that case, by the Mori–Nagata Theorem, the integral closure $\overline{R}$ of R is a Krull domain, so that the associated primes of $I^n\overline{R}$, as n varies, are all minimal over I, there are only finitely many of them, and the localizations of $\overline{R}$ at these primes are Noetherian valuation domains. These finitely many valuation rings are then the Rees valuation rings of $I\overline{R}$, and hence of I.

The reduction of the existence proof in general to the non-zero principal ideal case is via the extended Rees algebra $R[It, t^{-1}]$ (cf. [17, Exercise 10.6]):

$$\mathcal{RV}(I) = \{V \cap Q(R) : V \in \mathcal{RV}(t^{-1}R[It, t^{-1}])\}.$$

The reduction to the domain case relies on the fact that the integral closure of ideals is determined by the integral closures when passing modulo the minimal primes.

It turns out that in a Noetherian ring, all minimal prime ideals play a role in the Rees valuations of all ideals of positive height, in the sense that for every ideal I of positive height and every $P \in \mathrm{Min}(R)$ there exists a Rees valuation v of I such that $\{r \in R : v(r) = \infty\} = P$. But even more is true (and does not seem to be in the literature):

Proposition 2.2. *Let R be a Noetherian ring and I an ideal in R not contained in any minimal prime ideal. For each $P \in \mathrm{Min}(R)$, let T_P be the set of Rees valuations of $I(R/P)$. By abuse of notation, these valuations are also valuations on R, with $\{r \in R : v(r) = \infty\} = P$. Then $\cup_P T_P$ is the set of Rees valuations of I.*

Proof. The standard proofs of the existence of Rees valuations show that $\mathcal{RV}(I) \subseteq \cup_P T_P$. We need to prove that no valuation in $\cup_P T_P$ is redundant.

Let $Q \in \mathrm{Min}(R)$ and $v \in T_Q$. By the minimality of Rees valuations of $I(R/Q)$, there exist $n \in \mathbb{N}$ and $r \in R$ such that for all $w \in T_Q \setminus \{v\}$, $w(r) \geq nw(I)$, yet $r \notin \overline{I^n}(R/Q)$ (i.e., $v(r) < nv(I)$). Let r' be an element of R that lies in precisely those minimal prime ideals that do not contain r. Then $r + r'$ is not contained in any minimal prime ideal, for all $w \in T_Q \setminus \{v\}$, $w(r + r') \geq nw(I)$, and $v(r + r') < nv(I))$. Let J' be the intersection of all the minimal primes other than Q, let J'' be the intersection of all the centers of $w \in T_Q$, and let $s \in J' \cap J'' \setminus Q$. By assumption on r, there exists a positive integer k such that for all $w \in T_Q \setminus \{v\}$,

$$\frac{v(s)}{v(I)} - \frac{w(s)}{w(I)} + 1 < k\left(\frac{w(r + r')}{w(I)} - \frac{v(r + r')}{v(I)}\right).$$

Note that for all $w \in \cup_{P \neq Q} S_P$, $w(s) = \infty$. Thus for all $w \in \cup_P T_P \setminus \{v\}$, $\frac{v(s)}{v(I)} - \frac{w(s)}{w(I)} + 1 < k\left(\frac{w(r+r')}{w(I)} - \frac{v(r+r')}{v(I)}\right)$. Then with $m = \lfloor \frac{v(s(r+r')^k)}{v(I)} \rfloor$, $s(r + r')^k \notin \overline{I^{m+1}}$, yet for all $w \in \cup_P T_P \setminus \{v\}$, $w(s(r + r')^k) \geq (m + 1)w(I)$. This proves that v is not redundant. □

How does one construct the Rees valuation rings in practice? The steps indicated above of computing the integral closure of $R[It, t^{-1}]$ require an additional variable over R, and afterwards one needs to take the intersections of the obtained valuation rings with $Q(R)$. There is an **alternative construction** that eliminates these two steps of extending and intersecting, namely a construction using blowups: if $I = (a_1, \ldots, a_r)$, then

$$\mathcal{RV}\,(I) = \bigcup_{j=1}^{r} \mathcal{RV}\,\left(a_j R\left[\frac{I}{a_j}\right]\right).$$

Just as the first construction, this one also reduces to the case of principal ideals. As announced, this construction avoids introducing a new indeterminate and then intersecting the valuation rings with a field, but it instead requires the computation of r integral closures of rings. This can also be computationally daunting. If there is a way of making r smaller, the task gets a bit easier. Since $\mathcal{RV}\,(J) = \mathcal{RV}\,(I)$ whenever $\overline{I} = \overline{J}$, one can replace I in this alternative construction with J, if J has fewer generators than I. A standard choice for J is a minimal reduction of I, or even a minimal reduction of a power of I. It is known that in a Noetherian local ring R, every ideal has a power that has a reduction generated by at most $\dim R$ elements. In case the residue field is infinite, the ideal itself has a reduction generated by at most $\dim R$ elements (see [31] or [17, Propositions 8.3.7, 8.3.8]).

Even better, if R contains an infinite field, or if R is local with an infinite residue field, there exists a "sufficiently general" element $a \in I$ such that $\mathcal{RV}\,(I) = \mathcal{RV}\,(aR[\frac{I}{a}])$. In fact, $\mathcal{RV}\,(I) = \mathcal{RV}\,(aR[\frac{I}{a}])$ whenever $aV = IV$ for all $V \in \mathcal{RV}\,(I)$. Unfortunately, the "sufficient generality" is not so easily determined, and might be doable only after the Rees valuations have already been found.

Sally proved a determinate case when only one affine piece of the blowup suffices for finding the Rees valuations:

Theorem 2.3. *(Sally [40, page 438]) Let $(R, \mathfrak{m})$ be a Noetherian formally equidimensional local domain of dimension $d > 0$, and let I be an $\mathfrak{m}$-primary ideal minimally generated by d elements. Then for any $a \in I$ that is part of a minimal generating set, any Rees valuation ring V of I is the localization of the integral closure of $R[\frac{I}{a}]$ at a height one prime ideal minimal over a.*

Similarly, if $x_1, \ldots, x_r$ is a regular sequence in a Noetherian domain R, then for every Rees valuation ring V of $I = (x_1, \ldots, x_r)$, and for every $i = 1, \ldots, r$, $x_i V = IV$, and V is the localization of the integral closure of $S = R[\frac{I}{x_i}]$ at a height one prime ideal containing x_i.

Thus there are occasions when the alternative construction of Rees valuations using blowups only requires the computation of the integral closure of one ring, and sometimes this one ring is known a priori.

There is yet **another construction** of Rees valuations, this one via the (ordinary) Rees algebra $R[It]$: the set of Rees valuation rings of I equals the set of all $\overline{R[It]}_P \cap Q(R)$, as P varies over the prime ideals in $\overline{R[It]}$ that are minimal over $I\overline{R[It]}$.

A consequence of this formulation is a criterion for recognizing when the associated graded ring and the associated "integral" graded ring are reduced:

Theorem 2.4. *(Hübl–Swanson [15]) If R is integrally closed, then $\mathrm{gr}_I(R)$ is reduced if and only if all the powers of I are integrally closed and if for each* *(normalized integer-valued) Rees valuation v of I, $v(I) = 1$. Also, $R/\overline{I} \oplus \overline{I}/\overline{I^2} \oplus \overline{I^2}/\overline{I^3} \oplus \cdots$ is a reduced ring if and only if for each (integer-valued) Rees valuation v of I, $v(I) = 1$.*

Here is an example illustrating this result. Let X, Y, Z be variables over $\mathbb{C}$, and $R = \mathbb{C}[X, Y, Z]/(X^2 + Y^3 + Z^5)$. Then R is an integrally closed domain. By Flenner [8, 3.10], R is a rational singularity ring, so that by Lipman [22], all the powers of the maximal ideal $\mathfrak{m} = (X, Y, Z)R$ are integrally closed, and the blowup rings in the construction of Rees valuations of $\mathfrak{m}$ are integrally closed. As $X \in \overline{(Y, Z)R}$, it follows that $\mathcal{RV}(\mathfrak{m}) = \mathcal{RV}((Y, Z))$, and by Sally's Theorem 2.3 above, $\mathcal{RV}(\mathfrak{m}) = \mathcal{RV}(YR[\frac{(Y,Z)}{Y}])$. Certainly $\frac{X}{Y}$ is integral over $R[\frac{(Y,Z)}{Y}]$, so that $\mathcal{RV}(\mathfrak{m}) = \mathcal{RV}(YR[\frac{(X,Y,Z)}{Y}])$. By the cited Lipman's result, with $X' = \frac{X}{Y}$ and $Z' = \frac{Z}{Y}$,

$$R\left[\frac{(X, Y, Z)}{Y}\right] \cong R\left[\frac{\mathfrak{m}}{Y}\right] \cong \frac{\mathbb{C}[X', Y, Z']}{((X')^2 + Y + Y^3(Z')^5)}$$

is integrally closed, and there is only one minimal prime over (Y), namely (X', Y), so that $\mathfrak{m}$ has only one Rees valuation. Locally at (X', Y), the maximal ideal is generated by X', so that if v is the natural corresponding valuation, $v(X') = 1$, $v(Z') = 0$, from $Y(1 + Y^2(Z')^5) = -(X')^2$ we get that $v(Y) = 2$, hence $v(X) = v(X') + v(Y) = 3$, $v(Z) = v(Z') + v(Y) = 2$, whence $v(\mathfrak{m}) \geq 2$. Notice that $\operatorname{gr}_{\mathfrak{m}}(R) = R/\overline{\mathfrak{m}} \oplus \overline{\mathfrak{m}}/\overline{\mathfrak{m}^2} \oplus \cdots \cong \mathbb{C}[x, y, z]/(x^2)$ is not reduced.

The listed constructions of Rees valuations above give several methods for finding the unique set. All the methods require passing to finitely generated ring extensions, then taking the integral closure of the ring, followed by finding minimal primes over some height one ideals. The whole procedure may be fairly challenging: the integral closures of rings and primary decompositions of ideals are in practice hard to compute.

The following theorem gives a method for finding the centers of the Rees valuations without going through the full construction of the valuations. Recall (see [31]) that the **analytic spread** of an ideal I in a Noetherian local ring $(R, \mathfrak{m})$ is the Krull dimension of $(R/\mathfrak{m}) \oplus (I/\mathfrak{m}I) \oplus (I^2/\mathfrak{m}I^2) \oplus \cdots$. If $R/\mathfrak{m}$ is infinite, this number is the same as the number of generators of any ideal minimal reduction of I, and in general, there always exists a power of I with a reduction generated minimally by this number of elements.

Theorem 2.5. *(Burch [1], McAdam [25]) Let R be a Noetherian ring, I an ideal in R and P a prime ideal in R. If the analytic spread of IR_P equals* $\dim(R_P)$*, then P is the center of a Rees valuation of I. If R is locally formally equidimensional and P is the center of a Rees valuation of I, then the analytic spread of IR_P equals* $\dim(R_P)$.

I end this section with another example of where Rees valuations appear. In [37], Rees defined the **degree function** of an $\mathfrak{m}$-primary ideal I in $(R, \mathfrak{m})$

as follows: for any $x \in \mathfrak{m}$ such that $\dim(R/(x)) = \dim(R) - 1$, set $d_I(x) = e_{R/(x)}(I(R/(x)))$, i.e., $d_I(x)$ is the multiplicity of the ideal I in the ring $R/(x)$. Rees proved that for all allowed x,

$$d_I(x) = \sum_{v \in \mathcal{RV}'(I)} d(I, v) v(x)$$

for some positive integers $d(I, v)$ depending only on I and v, where $\mathcal{RV}'(I)$ is the set of those Rees valuations of I that are divisorial with respect to R. (Rees avoided the name "Rees valuation".)

3 One-fibered ideals

Definition 3.1. An ideal I is called **one-fibered** if $\mathcal{RV}(I)$ has exactly one element.

By the constructions of Rees valuations, this means that I is one-fibered if and only if the radical of $t^{-1}\overline{R[It, t^{-1}]}$ is a prime ideal, which holds if and only if the radical of $I\overline{R[It]}$ is a prime ideal.

Zariski [43] proved that if $(R, \mathfrak{m})$ is a two-dimensional regular local ring, then for every divisorial valuation v on R that is centered on $\mathfrak{m}$, there exists an ideal I such that v is the only Rees valuation of I. Namely, by Proposition 1.3, v is a Rees valuation of some ideal I of height two. This ideal may be assumed to be integrally closed, since $\mathcal{RV}(I) = \mathcal{RV}(\bar{I})$. Zariski proved that in a two-dimensional regular local ring the product of any two integrally closed ideals is integrally closed, and each integrally closed ideal factors uniquely (up to order) into a product of simple integrally closed ideals. Then by Proposition 1.4, v is a Rees valuation of some simple integrally closed ideal I of height two. It is a fact that a simple integrally closed $\mathfrak{m}$-primary ideal in a two-dimensional regular local ring has only one Rees valuation, so v is the only Rees valuation of I.

Lipman [22] generalized Zariski's result to all two-dimensional local rational singularity rings, and Göhner [G] proved it for two-dimensional complete integrally closed rings with torsion class group. Muhly [28] showed that there are two-dimensional analytically irreducible local domains for which Zariski's conclusion fails. Recall that a Noetherian local ring $(R, \mathfrak{m})$ is **analytically irreducible** if the $\mathfrak{m}$-adic completion of R is a domain.

More strongly, Cutkosky [5] proved that there exists a two-dimensional complete integrally closed local domain $(R, \mathfrak{m})$ in which every $\mathfrak{m}$-primary ideal has at least two Rees valuations. However, no concrete example of such a ring with no one-fibered zero-dimensional ideals has been found. One of the quests, alas unfulfilled, of the following section, is to find such a ring.

We discuss in this section what, if any, restrictions on the ring does the existence of a one-fibered ideal impose, and we also discuss various criteria for one-fiberedness.

Theorem 3.2. *(Sally [40]) Let $(R, \mathfrak{m})$ be a Noetherian local ring whose $\mathfrak{m}$-adic completion is reduced (i.e., R is* **analytically unramified***), and that has a one-fibered $\mathfrak{m}$-primary ideal I. Then R is analytically irreducible, i.e., the $\mathfrak{m}$-adic completion of R is a domain.*

Katz more generally proved in [20] that if $(R, \mathfrak{m})$ is a formally equidimensional Noetherian local ring, then for any $\mathfrak{m}$-primary ideal I, the number of Rees valuations is bounded below by the number of minimal prime ideals in the $\mathfrak{m}$-adic completion of R. Thus if R has a one-fibered $\mathfrak{m}$-primary ideal, then $\hat{R}$ has only one minimal prime ideal. The converse fails by Cutkosky's example [5] mentioned earlier in this section.

By the more recent work of Katz and Validashti, see Theorem 1.2, if $(R, \mathfrak{m})$ is a Noetherian local ring of positive dimension, the number of Rees valuations of an $\mathfrak{m}$-primary ideal I is the same as the number of Rees valuations of $I\widehat{R}$. By Proposition 2.2, the number of Rees valuations of $I\widehat{R}$ is at least the number of the minimal primes in $\widehat{R}$, so that if R has a one-fibered ideal, the completion must have only one minimal prime ideal. If in addition the completion is assumed to be reduced, this forces the completion to be a domain, thus proving Theorem 3.2.

Another proof of Theorem 3.2, without assuming Theorem 1.2 and Proposition 2.2, goes as follows: Rees proved in [36] that since R is analytically unramifed, there exists an integer k such that for all n, $\overline{I^{n+k}} \subseteq I^n$. Let V be the Rees valuation ring of I, and let r be an integer such that $IV = \mathfrak{m}_V^r$. As I is $\mathfrak{m}$-primary, $\mathfrak{m}$ is the center of V on R. Then for all n, $\mathfrak{m}^{r(n+k)} \subseteq \mathfrak{m}_V^{r(n+k)} V \cap R = I^{n+k} V \cap R = \overline{I^{n+k}} \subseteq I^n \subseteq \mathfrak{m}^n$, so that the $\mathfrak{m}$-adic completion of R is contained in the $\mathfrak{m}_V$-adic completion of V. But the latter is a domain since V is regular.

An arbitrary Noetherian local ring may have a zero-dimensional one-fibered ideal, yet not be analytically irreducible or even analytically unramified (of course the example is due to Nagata, see [17, Exercise 4.11]): Let k_0 be a perfect field of characteristic 2, let $X, Y, X_1, Y_1, X_2, Y_2, \ldots$ be variables over k_0, let $k = k_0(X_1, Y_1, X_2, Y_2, \ldots)$, $f = \sum_{i=1}^{\infty}(X_i X^i + Y_i Y^i)$, and $R = k^2[[X, Y]][k][f]$. Then R is an integrally closed Noetherian local domain whose completion $\hat{R}$ is isomorphic to $k[[X, Y, Z]]/(Z^2)$. The integral closures of powers of $(X, Y)R$ are contracted from the integral closures of powers of $(X, Y)\hat{R}$, hence clearly $(X, Y)R$ has only one Rees valuation.

As proved in Fedder–Huneke–Hübl [7, Lemma 1.3], the following are equivalent for an analytically unramified one-dimensional local domain R:

1. The integral closure $\overline{R}$ of R is local.
2. R has a non-zero one-fibered ideal.
3. Every non-zero ideal in R is one-fibered.

How does one find one-fibered ideals in arbitrary Noetherian local domains? In case $\hat{R}$ is a domain, R has a one-fibered $\mathfrak{m}$-primary ideal if and only if the integral closure of $\hat{R}$ has a one-fibered zero-dimensional ideal (see Sally [40, page 440]).

If $(R, \mathfrak{m})$ is a Noetherian local analytically irreducible domain, and I an $\mathfrak{m}$-primary ideal, then I is one-fibered if and only if there exists an integer b such that for all positive integers n and all $x, y \in R$, $xy \in I^{2n+b}$ implies that either x or y lies in I^n (see Hübl–Swanson [15]). A question that appeared in the same paper and has not yet been answered may be worth repeating:

Question. Let I be an $\mathfrak{m}$-primary ideal in an analytically irreducible Noetherian local domain $(R, \mathfrak{m})$. Suppose that for all $n \in \mathbb{N}$ and all x, y such that $xy \in I^{2n}$, either x or y lies in I^n. Or even suppose that for all $x \in R$ such that $x^2 \in I^{2n}$, necessarily $x \in I^n$. Are all the powers of I then integrally closed?

Another criterion of one-fiberedness was observed first by Sally [7, page 323] in dimension one, and the more general case below appeared in [14, page 3510]:

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian d-dimensional analytically unramified local ring, and let l be a positive integer satisfying the following:*

If $f \in \mathfrak{m} \setminus I^n$, then there exist $g_2, \ldots, g_d \in I$ such that $I^{n+l} \subseteq (f, g_2, \ldots, g_d)$, and for all Rees valuations v of I, $v(I^{n+l}) \geq v(f)$.

Then I is one-fibered.

Remark 3.4. Lipman [22] proved that the quadratic transformations of two-dimensional rational singularity rings are integrally closed. However, this does not mean that the maximal ideal has only one Rees valuation. For example, let $c \geq 3$ and take R to be the localization of $\frac{\mathbb{C}[X,Y,Z]}{(X^2+Y^2+Z^c)}$ at (X, Y, Z). By Flenner [8, 3.10], R is a rational singularity ring. The quadratic transformation $S = R[\frac{x}{z}, \frac{y}{z}]$ is isomorphic to a localization of $\frac{\mathbb{C}[Z,A,B]}{(A^2+B^2+Z^{c-2})}$ and is integrally closed, so that the primes in S minimal over ZS are $(Z, A + iB)$, $(Z, A - iB)$. By the blowup construction of Rees valuations, this says that (X, Y, Z) has at least two Rees valuations. In fact, since (Y, Z) is a minimal reduction of (X, Y, Z), Sally's result [40, page 438] shows that (X, Y, Z) has exactly two Rees valuations, the two arising from the two obtained prime ideals.

Muhly and Sakuma [29, Lemma 4.1] proved the following result on one-fibered ideals $I_1, \ldots, I_r$ in a two-dimensional universally catenary Noetherian integral domain R: If for $j = 1, \ldots, r$, $\mathcal{RV}(I_j) = \{V_j\}$, and the corresponding valuations $v_1, \ldots, v_r$ are pairwise not equivalent, then $\det(v_i(I_j)_{i,j}) \neq 0$. Here is a sketch of the proof. Let A be the $r \times r$ matrix $(v_i(I_j)_{i,j})$. Suppose that $\det A = 0$. Then the columns of A are linearly dependent over $\mathbb{Q}$, and we can find integers $a_1, \ldots, a_r$, not all zero, such that for all i, $\sum_j a_j v_i(I_j) = 0$. By changing indices, we assume that $a_1, \ldots, a_t, -a_{t+1}, \ldots, -a_r$ are non-negative integers. Let $I = I_1^{a_1} \cdots I_t^{a_t}$, $J = I_{t+1}^{-a_{t+1}} \cdots I_r^{-a_r}$ for some positive integer $t < r$. By Proposition 1.4, $\mathcal{RV}(I) = \{V_1, \ldots, V_t\}$, $\mathcal{RV}(J) = \{V_{t+1}, \ldots, V_r\}$. Since for all $i = 1, \ldots, r$, $v_i(I) = v_i(J)$, we have $\overline{I^n} = \overline{J^n}$ for all n, so I and J have the same Rees valuations, which is a contradiction.

4 Upper bounds on the number of Rees valuations

The main goal of this section is to bound above the number of Rees valuations of ideals in an arbitrary Noetherian local ring $(R, \mathfrak{m})$, and to find $\mathfrak{m}$-primary ideals with only one Rees valuation, if possible.

If R is a one-dimensional Noetherian semi-local integral domain, then it follows easily from the constructions of Rees valuations that $\{\overline{R}_P : P \in \operatorname{Max} \overline{R}\} = \cup_I \mathcal{RV}(I)$ is finite, as I varies over all the ideals of R. Thus the total number of Rees valuations of all possible ideals in a one-dimensional Noetherian semi-local ring is finite, and this number is a desired upper bound on the number of Rees valuations of any one ideal.

In higher dimensions, there is no upper bound on the number of all possible Rees valuations of ideals. For one thing, there are infinitely many prime ideals of height one, and each one of these has at least one Rees valuation centered on the prime itself. But more importantly, even if we restrict the ideals to $\mathfrak{m}$-primary ideals, there is no upper bound on the number of Rees valuations:

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension $d > 1$. Let $(x_1, \ldots, x_d)$ be an $\mathfrak{m}$-primary ideal. Then $\cup_n \mathcal{RV}(x_1^n, x_2, \ldots, x_d)$ is not a finite set. Furthermore, there is no upper bound on $|\mathcal{RV}(I)|$ as I varies over $\mathfrak{m}$-primary ideals.*

Proof. The last statement follows from the first one, by Proposition 1.4.

Suppose that the set S of all Rees valuations of $(x_1^n, x_2, \ldots, x_d)$, as n varies, is finite. Let N be a positive integer such that for all $v \in S$, $Nv(x_1) \geq v(x_2), \ldots, v(x_d)$. Then for all $n \geq N$, the integral closure of $(x_1^n, x_2, \ldots, x_d)$ is independent of n, whence $x_1^N \in \overline{(x_1^n, x_2, \ldots, x_d)}$. Let $\_'$ denote the images modulo $(x_2, \ldots, x_d)$. Then in the one-dimensional Noetherian ring R', x_1' is a parameter, and $x_1'^N \in \overline{(x_1')^n}$ for all $n \geq N$. We may even pass to the completion of R' and then go modulo a minimal prime ideal to get a one-dimensional complete Noetherian local domain A and a parameter x such that for all $n \geq N$, $x^N \in \overline{(x^n)}$. Since A is analytically unramified, by Rees [36] there exists an integer l such that for all $n \geq l$, $\overline{(x^n)} \subseteq (x^{n-l})$. Thus $x^N \subseteq \cap_{n \geq N, l}(x^{n-l}) = (0)$, which is a contradiction. $\square$

As already mentioned, a maximal ideal in a regular ring has only one Rees valuation. For zero-dimensional monomial ideals (in polynomial or power series rings, or even zero-dimensional monomial ideals in a regular system of parameters in a regular local ring), the number of Rees valuations is exactly the number of bounding non-coordinate hyperplane faces of the Newton polyhedron. By Carathéodory's Theorem, each of these hyperplanes is determined by $d = \dim R$ of the exponent vectors of the generators of I. With this geometric consideration one obtains a very crude upper bound on the number of Rees valuations of a zero-dimensional monomial ideal in terms of its generators:

$$|\mathcal{RV}(I)| \le \binom{\text{number of generators of } I}{\dim(R)}.$$

In practice, this upper bound is much too generous. I thank Ezra Miller for providing the following much better upper bound: the number of Rees valuations of I is at most

$$\begin{cases} 2\left(1+m+\binom{m+1}{2}+\binom{m+2}{3}+\cdots+\binom{m-1+\frac{d-1}{2}}{\frac{d-1}{2}}\right); & \text{if } d \text{ is odd;}\\ \binom{m-1+\frac{d}{2}}{\frac{d}{2}}+2\left(1+m+\binom{m+1}{2}+\binom{m+2}{3}+\cdots+\binom{m-1+\frac{d-2}{2}}{\frac{d-2}{2}}\right); & \text{if } d \text{ is even,}\end{cases}$$

where n is the number of generators of I and $m = n-d-1$. In particular, for fixed d, this upper bound on the number of Rees valuations of monomial ideals in d variables is a polynomial in the number of generators of degree $\lfloor d/2 \rfloor$. This follows among others from the Upper Bound Theorem for simplicial complexes.The relevant ingredients using Hilbert functions can be found in Lemma 16.19, Exercise 14.34, and Definition 16.32 in [18].

In general, however, the number of Rees valuations is not a function of the number of generators: at least when R is a polynomial ring over an infinite field (see [24]), or if $(R, \mathfrak{m})$ is a Noetherian local ring with infinite residue field (see [31]), every ideal I has a minimal reduction J generated by $\dim(R)$ elements, we already know that $\mathcal{RV}(I) = \mathcal{RV}(\overline{I}) = \mathcal{RV}(\overline{J}) = \mathcal{RV}(\overline{J})$, yet the number of Rees valuations is unbounded when the ring dimension is strictly bigger than one.

We now concentrate on finding upper bounds on the number of Rees valuations of ideals in Noetherian local rings. Let $(R, \mathfrak{m})$ be a Noetherian local ring. For every ideal I in R, $\bar{I} = \overline{I\hat{R}} \cap R$, so that the number of Rees valuations of $I\hat{R}$ is an upper bound on the number of Rees valuations of I. By Proposition 2.2, it then suffices to find an upper bound on the number of Rees valuations of $I(\hat{R}/Q)$ for each $Q \in \operatorname{Min}(\hat{R})$ (and adding them), so finding bounds on the number of Rees valuations of I reduces to the ring being a complete local domain. These reductions preserve the property of ideals being primary to the maximal ideal. As established on page 8, we may replace I by its power, and in particular, by a power that has a d-generated reduction, where d is the dimension of the ring. (Or alternatively, we could first pass in the standard way to $R[X]_{\mathfrak{m}R[X]}$, which is a faithfully flat extension of $(R, \mathfrak{m})$ with an infinite residue field, to have the existence of d-generated reductions for all $\mathfrak{m}$-primary ideals.) We have thus reduced to finding upper bounds on the number of Rees valuations of a d-generated $\mathfrak{m}$-primary ideal in a complete Noetherian local domain $(R, \mathfrak{m})$ of dimension d. To simplify matters, we now **restrict** our attention to the case where R contains a field. In that case, by the Cohen Structure Theorem, there exists a regular local subring $A = k[[X_1, \ldots, X_d]]$ of R, with $k \cong R/\mathfrak{m}$ a field and $X_1, \ldots, X_d$ variables over k, such that R is module-finite over A and such that $JR = I$, where $J = (X_1, \ldots, X_d)A$. If the field of fractions of R is separable over that of A, there exists an element $z \in R$ such that $A[z] \subseteq R$ is a module-finite extension of domains with identical fields of fractions. Necessarily $A[z]$ is a hypersurface ring, and the Rees valuations of $JA[z]$ are the Rees valuations of $JR = I$ (see page 4). Under the separable assumption we have thus reduced to finding upper bounds on the number of Rees valuations of the parameter ideal $(X_1, \ldots, X_d)$ in the complete local hypersurface domain

$$R = \frac{A[Z]}{(Z^n + a_1 Z^{n-1} + \cdots + a_n)} = \frac{k[[X_1, \ldots, X_d, Z]]}{(Z^n + a_1 Z^{n-1} + \cdots + a_n)},$$

where $a_i \in A$. Since without loss of generality z may be replaced by any A-multiple of z, we may assume that z is in the integral closure of I, so that we may assume that $a_i \in J^i A$ for all i. We can even control the degree n: since $1 = e_A((X_1, \ldots, X_d)A) = e_R(I)\ [R/\mathfrak{m} : k]/[Q(R) : Q(A)] = e_R(I)/n$, we get $n = e_R(I)$. Now we handle the general case, not assuming that R is separable over A. By the standard field theory, there exists a purely inseparable field extension k' of k and a positive integer m such that the field of fractions of $B[R] = R[k'][X_1^{1/p^m}, \ldots, X_d^{1/p^m}]$ is finite and separable over the field of fractions of $B = k'[X_1^{1/p^m}, \ldots, X_d^{1/p^m}]$. Note that $B[R]$ is a module finite (hence integral) extension of R, and by page 4, an upper bound on the set of Rees valuations of $IB[R]$ is an upper bound on the set of Rees valuations of I. Thus it suffices to replace R by $B[R]$. The field of fractions of this ring is separably generated over that of B, and the extension $B \subseteq B[R]$ has the same form as the extension $A \subseteq R$, so we are in the situation as above. In this case, the degree of the integral extension from B to $B[R]$ is $[Q(B[R]) : Q(B)] = e_{B[R]}\ (IB[R])[k' : k] = e_R(I)\ [Q(B[R]) : Q(R)]$.

In summary, in all cases of Noetherian local rings containing a field, we reduce the computation of the bounds on the number of Rees valuations of I to the computation of upper bounds on the number of Rees valuations of the ideal $(X_1, \ldots, X_d)R$ in the domain

$$R = \frac{k[[X_1, \ldots, X_d, Z]]}{(Z^n + a_1 Z^{n-1} + \cdots + a_n)},$$

where $a_i \in (X_1, \ldots, X_d)^i k[[X_1, \ldots, X_d]]$. We can even control the degree n as $e_R(I)$ if R is separably generated over k, say in characteristic 0.

Proposition 4.2. *With notation as above,*

$$|\mathcal{RV}\,((X_1, \ldots, X_d)R)| \leq n.$$

Proof. Let $S = R[\frac{X_2}{X_1}, \ldots, \frac{X_d}{X_1}]$. Then

$$S \cong \frac{k[[X_1, \ldots, X_d, Z]][T_2, \ldots, T_d]}{(Z^n + b_1 Z^{n-1} + \cdots + b_n, X_1 T_2 - X_2, \ldots, X_1 T_d - X_d)},$$

for some $b_i \in X_1^i k[[X_1, \ldots, X_d,]][T_2, \ldots, T_d]$. By Theorem 2.3, all the Rees valuations of $(X_1, \ldots, X_d)R$ are of the form $(\overline{S})_P$, where $\overline{S}$ is the integral closure of S, and P is a height one prime ideal in $\overline{S}$ containing X_1.

Let P be such a prime ideal, and let $p = P \cap S$. Since R is formally equidimensional, by the Dimension Formula, $\mathrm{ht}(p) = \mathrm{ht}(P) = 1$. Necessarily p is a prime ideal in S minimal over $X_1 S$, hence $p = (X_1, \ldots, X_d, Z)$. Thus it suffices to prove that the number of prime ideals in $(\overline{S})_{S \setminus p}$ that contract to p in S is at most n. By [17, Proposition 4.8.2], it suffices to prove that the number of minimal primes in the completion $\widehat{S_p}$ of S_p is at most n. Let $T = k[[X_1, \ldots, X_d]][\frac{X_2}{X_1}, \ldots, \frac{X_d}{X_1}]$, and let $q = (X_1, \ldots, X_d)T$. Then T_q is a regular local ring of dimension 1, and the maximal ideal is generated by X_1. The q-adic completion of T_q is $T_q[[Y]]/(X_1 - Y)$, which is a regular local ring of dimension 1 with maximal ideal generated by Y. But $\widehat{S_p}$ is $T_q[Z][[Y]]/(Z^n + b_1 Z^{n-1} + \cdots + b_n, X_1 - Y)$, which has at most n minimal primes. $\square$

Here is a table for the number of Rees valuations of the maximal ideal in $R = \frac{k[[X,Y,Z]]}{(X^a+Y^b+Z^c)}$ (with $2 \le a \le b \le c$) that illustrates the proposition above, showing that the number of Rees valuations of (X, Y, Z) is at most a. By page 6, it suffices to bound the number of Rees valuations of the ideal (X, Y, Z) in the ring $R = \frac{\mathbb{C}[X,Y,Z]}{(X^a+Y^b+Z^c)}$. Some of the calculations below were done with Anna Guerrieri.

a, b, c	$\#\mathcal{RV}(X, Y, Z)$
$2, 2, 2$	1
$2, 2, c \ge 3$	1 if $k = \mathbb{R}$, 2 if $k = \mathbb{C}$
$2, 3, c \ge 3$	1
$2, 4, 4$	1
$2, 4, c \ge 5$	1 if $k = \mathbb{R}$, 2 if $k = \mathbb{C}$
$2, 5, c \ge 5$	1
$2, 6, 6$	1
$2, 6, c \ge 7$	1 if $k = \mathbb{R}$, 2 if $k = \mathbb{C}$
$3, 3, 3$	1
$3, 3, c \ge 4$	2 if $k = \mathbb{R}$, 3 if $k = \mathbb{C}$

On the list above, do all of the rings have an (X, Y, Z)-primary ideal with only one Rees valuation? Can one find examples that are generated by monomials in X, Y, Z? This is indeed the case:

Proposition 4.3. *Let* $R = \frac{\mathbb{C}[[X,Y,Z]]}{(X^a+Y^b+Z^c)}$ *or* $R = \frac{\mathbb{C}[X,Y,Z]}{(X^a+Y^b+Z^c)}$, *with* $2 \le a \le b \le c$ *integers. Then the ideal* (X^a, Y^b, Z^c) *has exactly one Rees valuation.*

Proof. By page 4, it suffices to prove the proposition for the ideal $I = (X^a, Y^b, Z^c)$ in the ring $R = \frac{\mathbb{C}[X,Y,Z]}{(X^a+Y^b+Z^c)}$. Let S be obtained from R by adjoining a (bc)th root x of X, an (ac)th root y of Y, and an (ab)th root z of Z. Then $S = \frac{\mathbb{C}[x,y,z]}{(x^{abc}+y^{abc}+z^{abc})}$ and $IS = (x^{abc}, y^{abc}, z^{abc})S$. By reductions on page 4, it suffices to prove that IS has only one Rees valuation. But $\mathcal{RV}(IS) = \mathcal{RV}(\overline{IS}) = \mathcal{RV}((x,y,z)^{abc}S) = \mathcal{RV}((x,y,z)S)$, so it suffices to prove that the ideal (x,y,z) in the ring $S = \mathbb{C}[x,y,z]/(x^d+y^d+z^d)$ has only one Rees valuation. Since (x,y) is a reduction of (x,y,z), by Theorem 2.3, all the Rees valuations of (x,y,z) are of the form $\overline{T'}_P$, as P varies over the height one prime ideals in $\overline{T'}$ that are minimal over $x\overline{T'}$, where $T' = S[y/x]$. Since clearly z/x is integral over T', all the Rees valuations of (x,y,z) are of the form $\overline{T}_P$, as P varies over the minimal prime ideals over $Y\overline{T}$, and $T = S[y/x, z/x]$. Note that

$$T \cong \frac{\mathbb{C}[x,y,z,U,V]}{(xU-y, xV-z, 1+U^d+V^d)}.$$

By the Jacobian criterion, T is an integrally closed domain, so that each Rees valuation corresponds to a prime ideal in T minimal over xT, but xT is a prime ideal. □

5 The Izumi–Rees Theorem

The Izumi–Rees theorem is a very powerful and possibly surprising theorem, saying that all divisorial valuations over a good ring R with the same center are comparable, in the sense that if v and w are such valuations, there exists a constant C such that for all $x \in R$, $v(x) \le Cw(x)$. Since over good rings divisorial valuations are the same as Rees valuations (of possibly different ideals), this theorem enables us to compare Rees valuations with the same center. The surprising part of the Izumi–Rees Theorem is the contrast with the fact that if v and w are any two non-equivalent integer-valued valuations on a field K (such as on $Q(R)$), then for any integers $n, m \in \mathbb{Z}$ there exists $x \in K$ such that $v(x) = n$ and $w(x) = m$. The difference between this result and the Izumi–Rees Theorem is that the former takes elements from the field of fractions, but the Izumi–Rees Theorem only from the (good) subring.

Izumi [19] characterized analytically irreducible local domains, in the context of analytic algebras, without passing to the completion of the domains. Rees [38] generalized Izumi's result to the following two versions:

Theorem 5.1. *(Rees [38, (C)]) A Noetherian local ring $(R, \mathfrak{m})$ is analytically irreducible if for a least one $\mathfrak{m}$-primary ideal I, and only if, for all $\mathfrak{m}$-primary ideals I, there exist constants C and C', depending only on I, such that*

$$\mathrm{ord}_I(xy) - \mathrm{ord}_I(y) \le C \, \mathrm{ord}_I(x) + C', \text{ for all non-zero } x, y \in R.$$

Theorem 5.2. *(Rees [38, (E)]) Let* $(R, \mathfrak{m})$ *be a complete Noetherian local domain and let* v, w *be divisorial valuations centered on* $\mathfrak{m}$*. Then there exists a constant* C *such that for all non-zero* $x \in R$*,* $v(x) \le Cw(x)$*.*

Rees's proof first reduces to the proof in Krull dimension two, and then uses the existence of desingularizations and intersection numbers of $\mathfrak{m}$-adic valuations: in case the intersection number $[v, w]$ of v and w is non-zero, the constant C in Rees's theorem may be taken to be $C = -[w, w]/[v, w]$.

A stronger version of Theorem 5.2 was stated in Hübl–Swanson [15]: whenever $(R, \mathfrak{m})$ is an analytically irreducible excellent local domain and whenever v is a divisorial valuation centered on $\mathfrak{m}$, there exists a constant C such that for all divisorial valuations w centered on $\mathfrak{m}$ and all non-zero $x \in R$, $v(x) \le Cw(x)$. In recent conversations with Shuzo Izumi, we removed the excellent assumption above, analytically irreducible assumption suffices.

A version of the Izumi–Rees Theorem for affine rings, with explicit bounds for comparisons of valuations in terms of MacLane key polynomials, was given by Moghaddam in [27].

One of the consequences of the Izumi–Rees Theorem is a form of control of zero divisors modulo powers of ideals:

Theorem 5.3. *(Criterion of analytic irreducibility [15, Theorem 2.6]) Let* $(R, \mathfrak{m})$ *be a Noetherian local ring. The following are equivalent:*

1. *R is analytically irreducible.*
2. *There exist integers a and b such that for all $n \in \mathbb{N}$, whenever $x, y \in R$ and $xy \in \mathfrak{m}^{an+b}$, then either $x \in \mathfrak{m}^n$ or $y \in \mathfrak{m}^n$.*
3. *For every $\mathfrak{m}$-primary ideal I there exist integers a and b such that for all $n \in \mathbb{N}$, whenever $x, y \in R$ and $xy \in I^{an+b}$, then either $x \in I^n$ or $y \in I^n$.*

In [42], the author used the Izumi–Rees Theorem to prove the following: Let R be a Noetherian ring and I, J ideals in R such that the topology determined by $\{I^n : J^\infty\}_n$ is equivalent to the I-adic topology. Then the two topologies are equivalent linearly, i.e., there exists an integer k such that for all n, $I^{kn} : J^\infty \subseteq I^n$. In particular, if I is a prime ideal for which the topology determined by the symbolic powers is equivalent to the I-adic topology, then there exists an integer k such that $I^{(kn)} \subseteq I^n$. However, one cannot read k from the proof.

Subsequently, Ein, Lazarsfeld, and Smith in [6], and Hochster and Huneke in [12] proved that in a regular ring containing a field, the constant k for the prime ideal I may be taken to be the height of I. (The two papers [6] and [12] prove much more general results.)

In short, the Izumi–Rees Theorem has proved to be a powerful tool for handling powers of ideals.

Rond used the Izumi–Rees Theorem in a very different context: he proved in [39] that the Izumi–Rees Theorem is equivalent to a bounding of the Artin functions by a special upper bound of a certain family of polynomials. Also, Rond used the Izumi–Rees Theorem to bound other Artin functions.

6 Adjoints of ideals

In this final section I present yet another construction that is related to Rees valuations, and I end with an open question.

As already mentioned, Ein, Lazarsfeld and Smith [6] proved that for any prime ideal P in a regular ring containing a field of characteristic 0, $P^{(nh)} \subseteq P^n$ for all integers n, where h is the height of P. Hochster and Huneke [12] extended this to regular rings containing a field of positive prime characteristic, but no corresponding result is known in mixed characteristic. Hochster and Huneke used tight closure, and Ein, Lazarsfeld and Smith used the multiplier ideals. A possible approach to proving such a result in mixed characteristic is to use the adjoint ideals. The adjoint and multiplier ideals agree whenever they are both defined. However, the theory of multiplier ideals has access to powerful vanishing theorems, whereas adjoint ideals do not.

Definition 6.1. (Lipman [23]) Let R be a regular domain with field of fractions K. The **adjoint** of an ideal I in R is the ideal

$$\operatorname{adj} I = \bigcap_V \{r \in K : rJ_{V/R} \subseteq IV\},$$

where V varies over all the divisorial valuations with respect to R, and $J_{V/R}$ denotes the Jacobian ideal of the essentially finite-type extension $R \subseteq V$.

The adjoint $\operatorname{adj} I$ is an integrally closed ideal in R containing the integral closure of I, and hence containing I. Also, $\operatorname{adj}(I) = \operatorname{adj}(\bar{I})$, and if $x \in R$, then $\operatorname{adj}(xI) = x \cdot \operatorname{adj}(I)$. In particular, the adjoint of every principal ideal is the ideal itself.

In general, adjoints are not easily computable. One problem is the apparent need to use infinitely many valuations in the definition. The emphasis in the rest of this section is on limiting the number of necessary valuations, and the connection with Rees valuations.

Howald [13] proved that if I is a monomial ideal in $k[X_1, \ldots, X_d]$, then $\operatorname{adj} I = (\underline{X}^{\underline{e}} : \underline{e} \in \mathbb{N}^d, \underline{e} + (1, \ldots, 1) \in \mathrm{NP}^\circ(I))$, where $\mathrm{NP}^\circ(I)$ is the interior of the Newton polyhedron of I. Hübl and Swanson [16] extended this to all ideals generated by monomials in an arbitrary permutable regular sequence $X_1, \ldots, X_d$ in a regular ring R such that for every $i_1, \ldots, i_s \in \{1, \ldots, d\}$, the ring $R/(X_{i_1}, \ldots, X_{i_s})$ is a regular domain. Furthermore, [16] proved that for such I, $\operatorname{adj} I = \bigcap_V \{r \in K : rJ_{V/R} \subseteq IV\}$, where V varies only over the finite set of Rees valuations of I.

In addition, [16] proved that for all ideals I in a two-dimensional regular local ring, $\operatorname{adj} I = \bigcap_V \{r \in K : rJ_{V/R} \subseteq IV\}$, where V varies only over the finite set of Rees valuations.

However, in general, Rees valuations do not suffice for computing the adjoints of ideals, see [16].

Question. Given an $\mathfrak{m}$-primary ideal I in a regular local ring $(R, \mathfrak{m})$, does there exist a finite set S of valuations such that the adjoint of all the (integer) powers of I can be computed by using only the valuations from S?

If there is such a set S, by [16] it always contains the set of Rees valuations of I. In general, S contains other valuations as well. There is as yet no good criterion on what the other needed valuations might be.

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