

Polarization (of monomial ideals)

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November 16, 2006

1 Regular sequences

Let R be a ring and let x_1, \dots, x_n be a permutable regular sequence in R . Typically R is a polynomial ring in variables x_1, \dots, x_n over a field k .

A **monomial ideal** in x_1, \dots, x_n is an ideal in R generated by monomials in these elements.

Lemma 1.1 (Kaplansky [Lemma 3, 2]) *Let I be an ideal that is monomial in a permutable regular sequence x_1, \dots, x_n . Suppose that x_1 does not appear in any minimal monomial generator of I . Then x_1 is a non-zerodivisor modulo I .*

If J is any ideal in R , then $(x_1J + I) : x_1 = J + I$.

Proof. It is an elementary fact that $(x_1J + I) : x_1 = J + (I : x_1)$, so the last part follows from the first.

If $I = R$, say if $n = 1$, then the conclusion follows trivially. So we may assume that $n > 1$ and that x_2 appears in some monomial generator. Write $I = x_2J + K$, where J and K are ideals generated by monomials in x_2, \dots, x_n , with the further restriction that K is monomial in x_3, \dots, x_n . Write $rx_1 = x_2s + t$ for some $s \in J$ and $t \in K$. In $R/(x_1)$, the images of x_2, \dots, x_n still form a permutable regular sequence, so by induction on n , $s \in K + (x_1)$. Write $s = t' + x_1s'$ for some $t' \in K$ and $s' \in R$. Then $x_1s' \in J + K$, and by induction on the sums of degrees of generators of $I = x_2J + K$, we deduce that $s' \in J + K$. From the substitution $rx_1 = x_1x_2s' + (x_2t' + t)$ we get that $x_1(r - x_2s') \in K$. Since x_1, x_3, \dots, x_n is a permutable regular sequence, by induction on n then $r - x_2s' \in K$. Thus $r \in x_2J + K = I$. \square

Lemma 1.2 *Let I and J be ideals that are monomial in a permutable regular sequence x_1, \dots, x_n . Then $I \cap J$ is generated by monomials $\text{lcm}(g, h)$ as g varies over the generating monomial set G of I and h varies over a generating monomial set H of J .*

Proof. First assume that $I = (\underline{x}^e)$ is principal. Say $e_1 > 0$. Write $J = x_1J_1 + J_2$ for some monomial ideals J_1, J_2 , with no x_1 appearing in any monomial generator of J_2 . An element of $I \cap J$ is of the form $r\underline{x}^e = x_1j_1 + j_2$ for some $j_i \in J_i$. By Lemma 1.1, $j_2 = x_1j'_2$ for some $j'_2 \in J_2$. By induction on the sum of degrees of monomial generators of I and J we conclude that $r\underline{x}^e/x_1$ is in the ideal generated by monomials $\text{lcm}(\underline{x}^e/x_1, g)$, as g varies over a monomial generating set of $J_1 + J_2$. Thus

$$\begin{aligned} r\underline{x}^e &\in (\text{lcm}(\underline{x}^e, x_1g) \mid g \text{ a monomial in } J_1 + J_2) \\ &= (\text{lcm}(\underline{x}^e, g) \mid g \text{ a generating monomial in } x_1J_1 + J_2). \end{aligned}$$

Now assume that I is arbitrary. Let $r \in I \cap J$. Write r as an R -linear combination of the monomial generators of I . Let g be one of these generators that appears in r non-trivially, and let I' be the monomial ideal generated by the other generators of I . Thus $r = ag + b$ for some non-zero $a \in R$ and some $b \in I'$. Then $ag \in I' + J$, so by the first case, $ag \in (g) \cap I' + (g) \cap J$. Write $ag = c + d$ for some $c \in (g) \cap I'$ and some $d \in (g) \cap J$. Then $r - d \in I \cap J$ and since $r - d = ag + b - d = c + d + b - d = b + c \in I'$, by induction on the number of generators of I we get that $r - d$ is generated by least common multiples of monomials taken one from I' and the other from J' . By the first part, since $d \in (g) \cap J$, the rest follows. \square

A consequence is that a minimal monomial generating set of a monomial ideal in a permutable regular sequence is uniquely determined. Another consequence is that on the collection of monomial ideals in a fixed permutable regular sequence, intersection commutes with sums.

Lemma 1.3 *Let I be an ideal that is monomial in a permutable regular sequence. Suppose that x and y are two of the elements in the regular sequence and that furthermore the following properties hold:*

- (1) y is a variable over a subring.
- (2) y appears in each generator with exponent at most 1.
- (3) Every monomial generator of I that is a multiple of y is also a multiple of x .
- (4) For any monomial generator g of I that is a multiple of y and any monomial generator h of I that is not a multiple of y , the x -exponent in g is greater than or equal to the x -exponent in h .

Then $y - x$ is not a zero-divisor modulo I .

Proof. Write $I = x y I_1 + x I_2 + I_3$, where I_1, I_2, I_3 are all monomial ideals, the generators of I_1, I_2, I_3 do not involve y , the generators of I_3 do not involve x , and the x -exponent of any generator of I_1 is at least as big as the the x -exponent of any generator of I_2 .

We use the \mathbb{N} -grading of R as a polynomial ring in y over a subring. Since I is homogeneous, all the associated primes of I are homogeneous. If $y - x$ is a zerodivisor modulo I , it lies in some associated prime ideal of I , whence both y and x lie in that associated prime ideal. Thus there exists $r \in R \setminus I$ such that $r(x, y) \subseteq I$. By Lemma 1.1, $rx \in I = x y I_1 + x I_2 + I_3$ shows that $r \in y I_1 + I_2 + I_3$, and $ry \in I$ shows that $r \in x I_1 + x I_2 + I_3$. Thus

$$\begin{aligned}
r &\in (y I_1 + I_2 + I_3) \cap (x I_1 + x I_2 + I_3) \\
&= (y I_1 + I_2 + I_3) \cap (x I_1) + x I_2 + I_3 \\
&= (y I_1 \cap x I_1) + I_2 \cap (x I_1) + I_3 \cap (x I_1) + x I_2 + I_3 && \text{(by Lemma 1.2)} \\
&\subseteq x y I_1 + x I_2 + I_3,
\end{aligned}$$

the latter inclusion by assumption (4). Thus $r \in I$, so that $y - x$ cannot be a zerodivisor modulo I . \square

2 Polarization

We fix a permutable regular sequence x_1, \dots, x_n . The **polarization** of a monomial ideal I in this sequence is defined as follows: first fix a minimal monomial generating set G of I (see comment after Lemma 1.2 that such a set is even unique). For each $i = 1, \dots, n$, let e_i be the maximum exponent of the variable x_i appearing in elements of G . Then $e_i \in \mathbb{Z}_{\geq 0}$, but by possibly omitting some x_i , without loss of generality e_i are all positive. Let $S = R[Y_{ij} \mid i = 1, \dots, n; j = 2, \dots, e_i]$. For each $m = x_1^{a_1} \cdots x_n^{a_n} \in G$, define

$$\tilde{m} = x_1^{\min\{a_1, 1\}} Y_{12} \cdots Y_{1, a_1} x_2^{\min\{a_2, 1\}} Y_{22} \cdots Y_{2, a_2} \cdots x_n^{\min\{a_n, 1\}} Y_{n2} \cdots Y_{n, a_n}.$$

By the choice of e_i , $\tilde{m} \in S$. The **polarization** \tilde{I} of I is the ideal in S generated by all \tilde{m} as m varies over elements of G .

Note that \tilde{I} is an ideal in S generated by monomials in the permutable regular sequence consisting of all the x_i and all the Y_{ij} , and that moreover in each of the generators, the exponent of each of these is at most 1.

Polarizations are normally defined only if R is a polynomial ring in variables x_1, \dots, x_n over a field. In that case, for any monomial ideal I , \tilde{I} is a radical monomial ideal, monomial in (many) more variables. One of the advantages to the polarization is that one can construct its Stanley-Reisner complex. Another advantage is that resolutions of radical ideals are easier to come by. Fortunately, a resolution of a polarization compacts readily to a resolution of the original ideal (see Proposition 2.2).

Proposition 2.1 *Use the set-up as above. Then*

- (1) *The sequence $\{Y_{ij} - x_i \mid i = 1, \dots, n; j = 2, \dots, e_i\}$ is a permutable regular sequence on S/\tilde{I} .*
- (2) *S/\tilde{I} modulo the regular sequence from (1) is isomorphic to R/I .*

Proof. Pick any subset A of the sequence. It suffices to prove that if $Y_{ij} - x_i \notin A$, then $Y_{ij} - x_i$ is a non-zerodivisor modulo $S/(A)$.

Let $B = \{Y_{ij} \mid Y_{ij} - x_i \in A\}$. Let $S' = R[\tilde{Y}_{ij} \mid Y_{ij} \text{ not } \in B]$ and let I' be the ideal of S' obtained from the generators of \tilde{I} by identifying all $Y_{ij} \in B$ by the corresponding x_i . In other words, $S' = S/(A)$, and I' is monomial in in the permutable regular sequence consisting of all the x_i and all the Y_{ij} in S' . By construction of polarization and passing modulo (A) , each $Y_{ij} \in S'$ appears in each minimal monomial generator with exponent at most 1, and if Y_{ij} appears, so does x_i . Furthermore, if g and h are monomial generators of I' such that g is a multiple of Y_{ij} and h is not, then the x -exponent in g is greater than or equal to the x -exponent in h . Thus by Lemma 1.3, each $Y_{ij} - x_i$ that is not in A is a non-zerodivisor on S'/I' . This proves the proposition. \square

Proposition 2.2 *Use the polarization set-up. Let J be the ideal in S generated by all the $Y_{ij} - x_i$. Assume that S/\tilde{I} has a finite S -free resolution \mathbb{F} . Then \mathbb{F} tensored with S/J is a R -free resolution of R/I .*

Proof. By the previous proposition, the given generators of J form a regular sequence on S/\tilde{I} and they clearly form a regular sequence on S . Also, $(S/\tilde{I}) \otimes_S (S/J) = R/I$. Thus it suffices to prove that whenever z is a regular element on a ring R and on a module M , then an R -free resolution of M tensored with $R/(z)$ gives an $(R/(z))$ -free resolution of M/zM . In other words, it suffices to prove that $\text{Tor}_i^R(M, (R/(z))) = 0$ for $i > 0$. This Tor can be computed in another way as well: $0 \rightarrow R \rightarrow R \rightarrow R/(z) \rightarrow 0$ is an R -free resolution of $R/(z)$, and we can tensor it with M to get the sequence $0 \rightarrow M \rightarrow M \rightarrow M/zM \rightarrow 0$ that is even exact. This proves the proposition. \square

Jahan [1] proved that when R is the polynomial ring in variables x_i over a field, a monomial prime filtration of S/\tilde{I} , when reduced modulo all the $Y_{ij} - x_i$, yields a monomial prime filtration of R/I , with the prime ideal quotients reducing to prime ideal quotients.

This holds despite the fact that a primary decomposition of \tilde{I} does not reduce to a primary decomposition of I . Here is an example of that: let $I = (x_1^2, x_1x_2)$. Then $\tilde{I} = (x_1Y_{i1}, x_1x_2)$. The primary decomposition of $\tilde{I} = (x_1) \cap (Y_{i2}, x_2)$, and the reduction of the components on the right is $(x_1), (x_1, x_2)$, whose intersection is not I .

Here is another positive result on polarizations:

Lemma 2.3 *Let I and J be monomial ideals in a permutable regular sequence. Then the polarization of $I \cap J$ is $\tilde{I} \cap \tilde{J}$.*

Proof. By Lemma 1.2, $I \cap J$ is generated by the least common multiples of g and h , where g varies over the monomial generators of I and h varies over the monomial generators of J . But the polarization of the least common multiple of any two monomials is the least common multiple of the polarizations of the two monomials, which is then a generator of $\tilde{I} \cap \tilde{J}$. \square

References

1. A. S. Jahan, Prime filtrations of monomial ideals and polarizations, [arXiv:math.AC/0605119](https://arxiv.org/abs/math/0605119)
2. I. Kaplansky, R -sequences and homological dimension, *Nagoya Math. J.* **20** (1962), 195–199.