

Every numerical semigroup is one over d of infinitely many symmetric numerical semigroups

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Abstract. For every numerical semigroup S and every positive integer $d > 1$ there exist infinitely many symmetric numerical semigroups \overline{S} such that $S = \{n \in \mathbb{Z} : dn \in \overline{S}\}$. If d is odd, there exist infinitely many pseudo-symmetric numerical semigroups \overline{S} such that $S = \{n \in \mathbb{Z} : dn \in \overline{S}\}$.

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This note was motivated by the recent results of Rosales and García-Sánchez [4, 5] that for every numerical semigroup S there exist infinitely many symmetric numerical semigroups \overline{S} such that $S = \{n \in \mathbb{Z} : 2n \in \overline{S}\}$. The main result in this note is to prove that 2 is not a special integer, bigger positive integers work as well. See Theorem 0.5. The Rosales–García-Sánchez construction for $d = 2$ gives all the possible \overline{S} , whereas the construction below does not.

Throughout, S stands for a numerical semigroup, $F(S)$ stands for its Frobenius number, $\text{PF}(S)$ for the set of all pseudo-Frobenius numbers (i.e., all $n \in \mathbb{Z} \setminus S$ such that $n + (S \setminus \{0\}) \subseteq S$), and d for a positive integer strictly bigger than 1. The notation dS stands for the set $\{ds : s \in S\}$ and $\frac{S}{d}$ stands for $\{n \in \mathbb{N} : dn \in S\}$.

The goal is to construct infinitely many (symmetric and pseudo-symmetric) numerical semigroups T such that $S = \frac{T}{d}$.

The ring-theoretic consequence, by a result of Kunz [1], is that for any affine domain of the form $A = k[t^{a_1}, \dots, t^{a_m}]$, with a_1, \dots, a_m positive integers generating a numerical semigroup and with k a field, there exist infinitely many (Gorenstein) affine extension domains R of the same form such that any equation $X^d - a$ with $a \in A$ has a solution in A if and only if it has a solution in R . Such rings are called d -closed.

It is clear that if $S = \frac{T}{a}$ and $T = \frac{U}{b}$, then $S = \frac{U}{ab}$. Thus it suffices to prove that for every S and for every positive prime integer d there exist infinitely many (symmetric) numerical semigroups T for which $S = \frac{T}{d}$. The proofs below, however, will not assume that d is a prime.

Definition 0.1. Let d be a positive integer. A numerical semigroup S is said to be **d -symmetric** if for all integers $n \in \mathbb{Z}$, whenever d divides n , either n or $F(S) - n$ is in S .

Observe that a symmetric numerical semigroup is d -symmetric for all d , that a 1-symmetric numerical semigroup S is symmetric, and that a 2-symmetric numerical semigroup S is symmetric if $F(S)$ is an odd integer.

Proposition 0.2. *Let $S \subseteq T$ be numerical semigroups such that $F(S) = F(T)$. If S is d -symmetric, so is T , and $\frac{S}{d} = \frac{T}{d}$.*

Proof. Let $m \in \mathbb{Z}$ be a multiple of d . If $m \notin T$, then $m \notin S$, so by the d -symmetric assumption on S , $F(T) - m = F(S) - m \in S \subseteq T$. Thus T is d -symmetric. It remains to prove that $\frac{T}{d} \subseteq \frac{S}{d}$. Let $m \in \frac{T}{d}$. Suppose that $m \notin \frac{S}{d}$. Then $dm \in T \setminus S$. Since S is d -symmetric, $F(S) - dm \in S \subseteq T$, whence $F(T) = F(S) = (F(S) - dm) + dm \in T$, which is a contradiction. \square

By Theorem 1 and Proposition 2 in [3, Rosales and Branco], every numerical semigroup S with odd $F(S)$ can be embedded in a symmetric numerical semigroup T such that $F(S) = F(T)$. There are only finitely many choices for such T , but they are in general not unique. For example, let $S = \langle 12, 16, 21, 22, 23 \rangle$. Then S is 4-symmetric, $F(S) = 41$, and the numbers n for which $n, F(S) - n$ are not in S are 10, 11, 14, 15, 26, 27, 30, 31. If one adds 10 to S , then $2 \cdot 10 + 21 = 41$ would be in the numerical semigroup, so the Frobenius number would not be preserved. Thus any symmetric numerical semigroup T containing S with $F(S) = F(T)$ needs to contain 31. However, there are symmetric (and 4-symmetric) T that contain 11 and there are those that contain 30. All the possible symmetric T containing S with $F(S) = F(T)$ are as follows: $\langle 11, 12, 16, 21, 22, 23, 26, 31 \rangle$, $\langle 12, 14, 16, 21, 22, 23, 31 \rangle$, $\langle 12, 15, 16, 21, 22, 23, 31 \rangle$, $\langle 12, 16, 21, 22, 23, 26, 27, 30, 31 \rangle$.

Proposition 0.3. *Let S be a numerical semigroup and d, t and e positive integers. Let $g_1, \dots, g_t \in \mathbb{N} \setminus S$, with $g_1 = F(S)$. Let h_1, \dots, h_e be integers satisfying the following properties:*

- (1) *For all distinct $i, j \in \{1, \dots, e\}$, d does not divide $h_i - h_j$, and does not divide h_i .*
- (2) $h_1 = \min\{h_1, \dots, h_e\}$.
- (3) *For all $i = 1, \dots, e$, $h_i - dF(S) > \frac{1}{2}h_1$.*
- (4) $\gcd(d, h_1, \dots, h_e) = 1$.

Set $T = dS + \langle h_i - dg_j : i = 1, \dots, e; j = 1, \dots, t \rangle + \langle h_1 + 1, h_1 + 2, \dots, 2h_1 + 1 \rangle$. Then T is a numerical semigroup, $F(T) = h_1$, and $S = \frac{T}{d}$.

If $\text{PF}(S) \subseteq \{g_1, \dots, g_t\}$, then T is d -symmetric.

Proof. The set $\langle h_1 + 1, \dots, 2h_1 + 1 \rangle$ is contained in T , and thus T is a numerical semigroup with $F(T) \leq h_1$. Suppose that $h_1 \in T$. Then

$$h_1 = ds + \sum_{i,j} a_{ij}(h_i - dg_j)$$

for some $s \in S$ and some non-negative integers a_{ij} . Since d does not divide h_1 , at least one a_{ij} is non-zero. By condition (3), at most one a_{ij} is non-zero, and so it is necessarily 1. Then $h_1 - h_i = ds - dg_j$, so that by condition (1), $i = 1$ and $s = g_j \in S$, which is a contradiction. This proves that $h_1 \notin T$, whence $h_1 = F(T)$.

By (3), $h_1 > 2dF(S) > dF(S)$.

Clearly $dS \subseteq T$, so $S \subseteq \frac{T}{d}$. Let $n \in \mathbb{Z}$ such that $dn \in T$. We want to prove that $n \in S$. If $dn \geq h_1$, by the previous paragraph $n > F(S)$, whence $n \in S$. Now suppose that $dn < h_1$. Since $dn \in T$, write

$$dn = ds + \sum_{i,j} a_{ij}(h_i - dg_j),$$

for some $s \in S$ and some non-negative integers a_{ij} . As above, either $dn = ds + h_i - dg_j$ for some i, j , or $dn = ds$. The former case is impossible as h_i is not a multiple of d , so necessarily $dn = ds$ and $n = s \in S$. This proves that $S = \frac{T}{d}$.

It remains to prove that T is d -symmetric if $\text{PF}(S) \subseteq \{g_1, \dots, g_t\}$. Let $n \in \mathbb{Z}$ with $n = dm$ for some $m \in \mathbb{Z}$. If $n \notin T$, then $m \notin S$, and by [2, Proposition 12] there exists $g_i \in \text{PF}(S)$ such that $g_i - m \in S$. Then $h_1 - n = h_1 - dm = (h_1 - dg_i) + d(g_i - m) \in T$. Thus T is d -symmetric. \square

Corollary 0.4. (Rosales–García-Sánchez [5]) *Every numerical semigroup is one half of infinitely many symmetric numerical semigroups.*

Proof. Let $\text{PF}(S) = \{g_1, \dots, g_t\}$ and let h_1 be an arbitrary odd integer bigger than $4F(S)$. Then by Proposition 0.3, there exists a 2-symmetric numerical semigroup T such that $\frac{T}{2} = S$ and such that $F(T) = h_1$. We already observed that a 2-symmetric numerical semigroup with an odd Frobenius number is symmetric. Since there are infinitely many choices for h_1 , we are done. \square

In general, the construction in the proof of Proposition 0.3 does not necessarily give a symmetric numerical group T . Say $S = \langle 3, 4 \rangle$ and $d = 4$. Then $F(S) = 5$, $\text{PF}(S) = \{5\}$. The maximal possible e is $d - 1 = 3$, so if we take $h_1 = 41$, $h_2 = 42$, $h_3 = 43$, the hypotheses of the theorem are satisfied, and the construction gives $T = \langle 12, 16, 21, 22, 23 \rangle$. By the theorem, $F(T) = 41$ and $S = \frac{T}{4}$, but T is not symmetric as neither 10 nor 31 are in T . One can find a symmetric numerical semigroup U such that $S = \frac{U}{4}$ by using the Rosales–García-Sánchez construction twice (with $d = 2$), or one can apply the following main theorem of this paper:

Theorem 0.5. *Let S be a numerical semigroup and let d be a positive integer greater than or equal to 2. Then there exist infinitely many symmetric numerical semigroups T such that $S = \frac{T}{d}$.*

Proof. By choosing large odd integers h_1 that are relatively prime to d , applying Proposition 0.3 with $e = 1$ and $\{g_1, \dots, g_t\} = \text{PF}(S)$ gives a d -symmetric numerical semigroup T such that $S = \frac{T}{d}$ and $F(T) = h_1$. But then by Theorem 1 in [3] there exists a symmetric numerical semigroup U containing T such that $F(U) = F(T)$. By Proposition 0.2, $\frac{T}{d} = \frac{U}{d}$. Thus there exists a symmetric numerical semigroup U such that $S = \frac{U}{d}$ and $F(U) = h_1$. Since there are infinitely many choices of h_1 , we are done. \square

Recall that a numerical semigroup S is **pseudo-symmetric** if $F(S)$ is even and if for all $n \in \mathbb{Z} \setminus \{F(S)/2\}$, either n or $F(S) - n$ is in S . The following is a modification of the main theorem for pseudo-symmetric semigroups:

Theorem 0.6. *Let S be a numerical semigroup and let d be an odd integer greater than or equal to 3. Then there exist infinitely many pseudo-symmetric numerical semigroups T such that $S = \frac{T}{d}$.*

Proof. By choosing large even integers h_1 that are relatively prime to d , applying Proposition 0.3 with $e = 1$ and $\{g_1, \dots, g_t\} = \text{PF}(S)$ gives a d -symmetric numerical semigroup T such that $S = \frac{T}{d}$ and $F(T) = h_1$. Similar to Theorem 1 in [3], there exists a pseudo-symmetric numerical semigroup U containing T such that $F(U) = F(T)$, say $U = T \cup \{n \in \mathbb{N} \mid h_1/2 < n < h_1, h_1 - n \notin T\}$. By Proposition 0.2, $\frac{T}{d} = \frac{U}{d}$. Thus U is a pseudo-symmetric numerical semigroup such that $S = \frac{U}{d}$ and $F(U) = h_1$. Since there are infinitely many choices of h_1 , we are done. \square

References

- [1] E. Kunz, The value semigroup of a one-dimensional Gorenstein ring, *Proc. Amer. Math. Soc.* **25** (1970), 748–751.
- [2] J. C. Rosales and M. B. Branco, Numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups, *J. Pure Appl. Algebra* **171**(2–3) (2002), 303–314.
- [3] J. C. Rosales and M. B. Branco, Irreducible numerical semigroups, *Pacific J. Math.* **209** (2003), 131–143.
- [4] J. C. Rosales and P. A. García-Sánchez, Every numerical semigroup is one half of a symmetric numerical semigroup, *Proc. Amer. Math. Soc.* **136** (2008), 475–477.
- [5] J. C. Rosales and P. A. García-Sánchez, Every numerical semigroup is one half of infinitely many symmetric numerical semigroups, *Comm. in Algebra* **36** (2008), 2910–2916.

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