Multi-graded Hilbert functions, mixed multiplicities

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Multiplicities of ideals are useful invariants, which in good rings determine the ideal up to integral closure. Mixed multiplicities are a collection of invariants of several ideals, generalizing multiplicities, and capturing some information on the interactions among ideals. Teissier and Risler [Tei73] were the first to develop mixed multiplicities, in connection with Milnor numbers of isolated hypersurface singularities: the sequences of Milnor numbers obtained by intersecting with general i-planes arise as mixed multiplicities of the ideal generated by the partial derivatives of the defining power series with the ideal corresponding to the point (see Theorem 2.5). Rees connected mixed multiplicities to joint reductions (see Theorem 3.1).

This paper is meant to be an introduction to the topic of multi-graded Hilbert functions, mixed multiplicities, and joint reductions. There is much that is omitted, and a partial list of known results is given at the end. Familiarity with ordinary multiplicities and reductions is assumed.

Throughout, R is a Noetherian ring, s is a positive integer, $I_1, \ldots, I_s$ are ideals in R, and M is a finitely generated R-module. We will denote s-tuples of non-negative integers as $(n_1, \ldots, n_s)$ or as $\underline{n}$. All comparisons among s-tuples are componentwise. We will say that $\underline{n}$ is *sufficiently large* if there exists an s-tuple $\underline{e}$ such that $\underline{n} \geq \underline{e}$; of course, "sufficent largeness" depends on the context. By $\underline{I}^{\underline{n}}$ we denote $I_1^{n_1} \cdots I_s^{n_s}$.

1. Preliminaries

The techniques used to handle several ideals at the same time are similar to the techniques for handling single ideals. We need a multi-ideal form of the Artin–Rees Lemma, and existence of special, "sufficiently general", elements with respect to the given ideals.

Theorem 1.1: *(Generalization of the Artin–Rees Lemma) Assume that M and N are R-modules contained in a larger R-module T. Then there exists $\underline{c}$ such that for all $\underline{n} \geq \underline{c}$,*

$$I_1^{n_1} \cdots I_s^{n_s} M \cap N = I_1^{n_1 - c_1} \cdots I_s^{n_s - c_s} (I_1^{c_1} \cdots I_s^{c_s} M \cap N).$$

Proof: Let $X_1, \ldots, X_s$ be variables over R, and let A be $R[I_1X_1, \ldots, I_sX_s]$, the so-called the multi-ideal Rees ring, namely the subring of $R[X_1, \ldots, X_s]$ generated by elements a_iX_i, as a_i varies over elements of I_i. Let G be the A-submodule $T[I_1X_1, \ldots, I_sX_s]$ of $T[X_1, \ldots, X_s]$. Set $H = \oplus_{\underline{n}} \underline{I}^{\underline{n}} \underline{X}^{\underline{n}} M \cap N \subseteq G$. Then A is an $\mathbb{N}^s$-graded Noetherian ring, G

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is a finitely generated $\mathbb{N}^s$-graded A-module, and H is a graded A-submodule of G. Thus H is a finitely generated A-module. Let $h_1, \ldots, h_t$ form a homogeneous generating set of H over A. Define $\underline{c} = \max\{\deg h_i|, | i = 1, \ldots, t\}$.

The theorem is of course true if $\underline{n} = \underline{c}$. Now let $\underline{n} \geq \underline{c}$, with $n_i > c_i$ for at least one i. Fix one such i. Let $m \in I_1^{n_1} \cdots I_s^{n_s} M \cap N$. Then $m X_1^{n_1} \cdots X_s^{n_s}$ is a homogeneous element of H, and so it can be expressed as $\sum_j m_j h_j$, where each m_j is multi-homogeneous in A, and where for each $j = 1, \ldots, t$, $\deg m_j + \deg h_j = \underline{n}$. As $n_i > c_i$, for each $j = 1, \ldots, t$ we may write $m_j = \sum_k a_{kj} b_{kj}$, with each a_{kj} homogeneous of degree $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the ith place), and each b_{kj} homogeneous of degree $\deg m_j - e_i$. Hence

$$m = \sum_{k,j} a_{kj} b_{kj} h_j \in I_i \underline{I}^{\deg m_j - e_i} h_j \subseteq I_i(\underline{I}^{\deg m_j - e_i + \deg h_j} M \cap N) = I_i(\underline{I}^{\underline{n} - e_i} M \cap N),$$

which is in $I_i \underline{I}^{\underline{n} - \underline{c} - e_i}(I_1^{c_1} \cdots I_s^{c_s} M \cap N)$ by induction on $|\underline{n}|$. This proves that $\underline{I}^{\underline{n}} M \cap N \subseteq \underline{I}^{\underline{n} - \underline{c}}(\underline{I}^{\underline{c}} M \cap N)$. The other inclusion holds trivially. ∎

Lemma 1.2: *Assume that R is local with maximal ideal $\mathfrak{m}$, that the residue field $R/\mathfrak{m}$ of R is infinite, and that I_1 is not contained in prime ideals $P_1, \ldots, P_r$. Then there exist an integer $c > 0$ and a finite union V of proper $R/\mathfrak{m}$-vector subspaces of $I_1/\mathfrak{m}I_1$ such that for each $a \in I_1$ whose image in $I_1/\mathfrak{m}I_1$ is not in V, the following holds:*
(1) a is not in $\cup_i P_i$,
(2) and for all $i \geq 1$, $n_1 \geq c + i$, and $n_2, \ldots, n_s$,

$$(I_1^{n_1} \cdots I_s^{n_s} M :_M a^i) \cap I_1^c I_2^{n_2} \cdots I_s^{n_s} M = I_1^{n_1 - i} I_2^{n_2} \cdots I_s^{n_s} M.$$

(Such a is "sufficiently general".)

Proof: The proof includes the case $r = 0$.

If I_1 is nilpotent, then necessarily $r = 0$, and the lemma holds for all $a \in I_1$ and for any c such that $I_1^c = 0$.

So we may assume that I_1 is not nilpotent. Then $I_1/I_1^2 \neq 0$. Let A be the Noetherian $\mathbb{N}$-graded ring

$$A = \bigoplus_{\underline{n} \geq \underline{0}} \frac{I_1^{n_1} \cdots I_s^{n_s}}{I_1^{n_1+1} I_2^{n_2} \cdots I_s^{n_s}},$$

where I_1 has weight 1, and $I_2, \ldots, I_s$ have weight 0. Then

$$G = \bigoplus_{\underline{n} \geq \underline{0}} \frac{I_1^{n_1} \cdots I_s^{n_s} M}{I_1^{n_1+1} I_2^{n_2} \cdots I_s^{n_s} M},$$

is a finitely generated $\mathbb{N}$-graded A-module. For each prime ideal in A that is associated to G but does not contain A_1, its intersection with I_1/I_1^2 is necessarily proper, hence by Nakayama's Lemma, the image of this intersection in $I_1/\mathfrak{m}I_1$ is a proper subspace. Let

$W_1, \ldots, W_t$ be all the subspaces of $I_1/\mathfrak{m}I_1$ obtained in this way. Similarly, the images $\overline{P}_i$ of the P_i in $I_1/\mathfrak{m}I_1$ are proper subspaces. Define V to be the union of the W_i and $\overline{P}_i$ in $I_1/\mathfrak{m}I_1$. Since $I_1/\mathfrak{m}I_1$ is a finite-dimensional vector space over the infinite field $R/\mathfrak{m}$, V is a proper subset. We will prove that any element $a \in I_1$ with $a + \mathfrak{m}I_1 \notin V$ satisfies the lemma.

Since G is a Noetherian module, we can decompose its zero submodule irredundantly into primary components $0 = \cap_i M_i$. Each ideal $\sqrt{M_j :_A G}$ is an associated prime ideal of G. If A is not an element of $\sqrt{M_j :_A G}$, then since M_j is a primary module, $M_j :_G a^i = M_j$ for all $i \geq 1$. If, however, a lies in of $\sqrt{M_j :_A G}$, then by the definition of V, $\sqrt{M_j :_A G}$ contains A_1, and hence also $\oplus_{c \geq 1} A_c$. In particular, there exists an integer c such that A_c lies in all such $M_j :_A G$. Hence $\oplus I_1^c I_2^{n_2} \cdots I_s^{n_s} M / I_1^{c+1} I_2^{n_2} \cdots I_s^{n_s} M$ lies in all corresponding M_j. Therefore

$$\begin{aligned}
&(0 :_G a^i) \cap \frac{I_1^c I_2^{n_2} \cdots I_s^{n_s} M}{I_1^{c+1} I_2^{n_2} \cdots I_s^{n_s} M} \\
&\quad = \bigcap_{a \notin \sqrt{M_j :_A G}} \left(M_j :_G a^i\right) \cap \bigcap_{a \in \sqrt{M_j :_A G}} \left(M_j :_G a^i\right) \cap \frac{I_1^c I_2^{n_2} \cdots I_s^{n_s} M}{I_1^{c+1} I_2^{n_2} \cdots I_s^{n_s} M} \\
&\quad \subseteq \bigcap_{a \notin \sqrt{M_j :_A G}} M_j \cap \bigcap_{a \in \sqrt{M_j :_A G}} \left(M_j :_G a^i\right) \cap \bigcap_{a \in \sqrt{M_j :_A G}} M_j \\
&\quad \subseteq \bigcap_j M_j = 0.
\end{aligned}$$

In other words, the lemma holds. ∎

2. Hilbert-Samuel polynomials

From now on, R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, and $I_1, \ldots, I_s$ are $\mathfrak{m}$-primary ideals.

It is elementary to prove that for any $a \in I_1$,

$$0 \to \frac{\underline{I}^{\underline{n}} M :_M a}{I_1^{n_1 - 1} I_2^{n_2} \cdots I_s^{n_s} M} \to \frac{M}{I_1^{n_1 - 1} I_2^{n_2} \cdots I_s^{n_s} M} \xrightarrow{\ a\ } \frac{M}{\underline{I}^{\underline{n}} M} \to \frac{M}{\underline{I}^{\underline{n}} M + aM} \to 0$$

is a short exact sequence. Thus

$$\lambda\left(\frac{M}{\underline{I}^{\underline{n}} M}\right) - \lambda\left(\frac{M}{I_1^{n_1 - 1} I_2^{n_2} \cdots I_s^{n_s} M}\right) = \lambda\left(\frac{M}{\underline{I}^{\underline{n}} M + aM}\right) - \lambda\left(\frac{\underline{I}^{\underline{n}} M :_M a}{I_1^{n_1 - 1} I_2^{n_2} \cdots I_s^{n_s} M}\right). \quad (*)$$

For special $a \in I_1$, stronger results are obtained:

Lemma 2.1: *Let $a \in I_1$. Assume that a is not contained in any prime ideal minimal over $ann(M)$, and that there exists an integer c such that for all $n_1 \geq c$ and all sufficiently large $n_2, \ldots, n_s$,*

$$(I_1^{n_1} \cdots I_s^{n_s} M :_M a) \cap I_1^c I_2^{n_2} \cdots I_s^{n_s} M = I_1^{n_1 - 1} I_2^{n_2} \cdots I_s^{n_s} M.$$

Then for all sufficiently large $\underline{n}$, $\frac{\underline{I}^{\underline{n}}M :_M a}{I_1^{n_1-1}I_2^{n_2}\cdots I_s^{n_s}M} \cong 0 :_M a$, *and so*

$$\lambda\left(\frac{M}{\underline{I}^{\underline{n}}M}\right) - \lambda\left(\frac{M}{I_1^{n_1-1}I_2^{n_2}\cdots I_s^{n_s}M}\right) = \lambda\left(\frac{M}{\underline{I}^{\underline{n}}M + aM}\right) - \lambda\left(0 :_M a\right).$$

Proof: Necessarily $\dim M > 0$. By the generalization of the Artin–Rees Lemma (Theorem 1.1), there exists $\underline{e} \in \mathbb{N}^s$ such that for all $\underline{n} \geq \underline{e}$,

$$\underline{I}^{\underline{n}}M \cap aM \subseteq a\underline{I}^{\underline{n}-\underline{e}}M.$$

Thus $\underline{I}^{\underline{n}}M :_M a \subseteq \underline{I}^{\underline{n}-\underline{e}}M + (0 :_M a)$. As all ideals are $\mathfrak{m}$-primary, for all n_1 sufficiently large, $I_1^{n_1-e_1}M \subseteq I_1^c I_2^{e_2}\cdots I_s^{e_s}M$, so that

$$\underline{I}^{\underline{n}}M :_M a \subseteq \underline{I}^{\underline{n}-\underline{e}}M + (0 :_M a) \subseteq I_1^c I_2^{n_2}\cdots I_s^{n_s}M + (0 :_M a).$$

Hence if $\underline{n}$ is sufficiently large, by the assumption on c,

$$\begin{aligned}\frac{\underline{I}^{\underline{n}}M :_M a}{I_1^{n_1-1}I_2^{n_2}\cdots I_s^{n_s}M} &\cong \frac{(\underline{I}^{\underline{n}}M :_M a) + I_1^c I_2^{n_2}\cdots I_s^{n_s}M}{I_1^c I_2^{n_2}\cdots I_s^{n_s}M}\\ &= \frac{(0 :_M a) + I_1^c I_2^{n_2}\cdots I_s^{n_s}M}{I_1^c I_2^{n_2}\cdots I_s^{n_s}M}\\ &\cong \frac{0 :_M a}{(0 :_M a) \cap I_1^c I_2^{n_2}\cdots I_s^{n_s}M}.\end{aligned}$$

But

$$\begin{aligned}(0 :_M a) \cap I_1^c I_2^{n_2}\cdots I_s^{n_s}M &\subseteq \cap_{n_1 \geq 1}(\underline{I}^{\underline{n}}M :_M a) \cap I_1^c I_2^{n_2}\cdots I_s^{n_s}M\\ &\subseteq \cap_{n_1\geq 1} I_1^{n_1-1}I_2^{n_2}\cdots I_s^{n_s}M = 0,\end{aligned}$$

so that $\frac{\underline{I}^{\underline{n}}M :_M a}{I_1^{n_1-1}I_2^{n_2}\cdots I_s^{n_s}M} \cong (0 :_M a)$. The rest follows from Equation (*). ∎

Theorem 2.2: *For all positive integers* $n_1, \ldots, n_s$, $M/I_1^{n_1}\cdots I_s^{n_s}M$ *has finite length. Moreover, there exists a polynomial* $p \in \mathbb{Q}[X_1, \ldots, X_s]$ *such that for all sufficiently large* $(n_1, \ldots, n_s)$,

$$\lambda\left(\frac{M}{I_1^{n_1}\cdots I_s^{n_s}M}\right) = p(n_1, \ldots, n_s).$$

The degree of the polynomial is $\dim M$ *(where the degree of the constant zero polynomial is zero).*

Proof: The case $s = 1$ was proved by Samuel in [Sa51]. In general, clearly $M/I_1^{n_1}\cdots I_s^{n_s}M$ has finite length. The proof of the rest proceeds by induction on $\dim M$. If $\dim M = 0$, then for all $\underline{n} = (n_1, \ldots, n_s)$ with $|\underline{n}|$ sufficiently large, $I_1^{n_1}\cdots I_s^{n_s}M = 0$. Thus for all $\underline{n}$ sufficiently large, $\frac{M}{I_1^{n_1}\cdots I_s^{n_s}M} = M$ has finite length, so p is a constant polynomial.

Now let $\dim M > 0$. First a technicality. Let X be a variable over R, and $R' = R[X]_{\mathfrak{m}R[X]}$. Then R' is a faithfully flat extension of R, $\dim R' = \dim R$, and the residue field of R' is infinite. For any finitely generated R-module N, $\dim N = \dim (N \otimes_R R')$. In particular, if N has finite length, then $N \otimes_R R'$ has finite length. Furthermore, in that case, $\lambda_R(N) = \lambda_{R'}(N \otimes_R R')$. Thus it suffices to prove that there exists a polynomial $p \in \mathbb{Q}[X_1, \ldots, X_s]$ such that for all sufficiently large $(n_1, \ldots, n_s)$,

$$\lambda\left(\frac{M \otimes_R R'}{\underline{I}^{\underline{n}} M \otimes_R R'}\right) = \lambda\left(\frac{M}{\underline{I}^{\underline{n}} M} \otimes_R R'\right) = p(n_1, \ldots, n_s).$$

But $\underline{I}^{\underline{n}} M \otimes_R R' = (\underline{I}^{\underline{n}} \otimes_R R')(M \otimes_R R') = \underline{I}^{\underline{n}} R'(M \otimes_R R')$. Thus by replacing R by R', M by $M \otimes_R R'$, and I_i by $I_i R'$, without loss of generality R has an infinite residue field. Let $P_1, \ldots, P_r$ be the minimal prime ideals over $\operatorname{ann} M$. As I_1 is $\mathfrak{m}$-primary and $\dim M > 0$, necessarily I_1 is not contained in any P_i. Thus by Lemma 1.2, there exist $c \in \mathbb{N}$ and $a \in I_1 \setminus \cup_i P_i$ such that for all $n_1 > c$, and all $n_2, \ldots, n_s \in \mathbb{N}$,

$$(I_1^{n_1} \cdots I_s^{n_s} M :_M a) \cap I_1^c I_2^{n_2} \cdots I_s^{n_s} M = I_1^{n_1 - 1} I_2^{n_2} \cdots I_s^{n_s} M.$$

Then by Lemma 2.1, for all sufficiently large $\underline{n}$,

$$\lambda\left(\frac{M}{\underline{I}^{\underline{n}} M}\right) - \lambda\left(\frac{M}{I_1^{n_1-1} I_2^{n_2} \cdots I_s^{n_s} M}\right) = \lambda\left(\frac{M}{\underline{I}^{\underline{n}} M + aM}\right) - \lambda\left(0 :_M a\right).$$

As by the choice of a, $\dim (M/aM) < \dim M$, by induction there exists a polynomial $q \in \mathbb{Q}[X_1, \ldots, X_s]$ of degree $\dim (M/aM) = \dim M - 1$ such that for all $\underline{n}$ sufficiently large, $\lambda(\frac{M}{\underline{I}^{\underline{n}} M + aM}) = q(\underline{n})$. Then the right side of the equation is a polynomial of degree exactly $\dim M - 1$. The polynomial p can be built in the standard way by recursion: p has degree exactly one more than the degree of q, namely p has degree $\dim M$. ∎

The homogeneous part of degree $d = \dim M$ of the polynomial $p \in \mathbb{Q}[X_1, \ldots, X_s]$ as in Theorem 2.2 can be written as

$$\sum_{d_1 + \cdots + d_s = d} \frac{1}{d_1! \cdots d_s!} e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) X_1^{d_1} \cdots X_s^{d_2},$$

where $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$ denotes a rational number. This number is called the *mixed multiplicity* of M of type $(d_1, \ldots, d_s)$ with respect to $I_1, \ldots, I_s$. If $s = 1$, this is the usual multiplicity of M with respect to I_1.

We prove next that $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$ is actually an integer:

Theorem 2.3: *Use the set-up as above. Then $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$ is an integer. If $\dim M = 0$, then $e(I_1^{[0]}, \ldots, I_s^{[0]}; M) = \lambda(M)$. If $\dim M > 0$, $R/\mathfrak{m}$ is an infinite field, and $d_i > 0$, then there exists $a_i \in I_i$ such that $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$ equals*

$$e(I_1^{[d_1]}, \ldots, I_{i-1}^{[d_{i-1}]}, I_i^{[d_i - 1]}, I_{i+1}^{[d_{i+1}]}, \ldots, I_s^{[d_s]}; M/a_i M).$$

In particular, there exist d_1 elements of I_1, d_2 elements of I_2, etc., d_s elements of I_s, labelled $a_1, \ldots, a_d$, such that

$$e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = \lambda\left(\frac{M}{(a_1, \ldots, a_d)M}\right) - \lambda\left(\frac{(a_1, \ldots, a_{d-1})M :_M a_d}{(a_1, \ldots, a_{d-1})M}\right).$$

Proof: Clearly $d = \dim M$. The theorem is trivial if $d = 0$. Thus we assume that $d > 0$.

Let X be an indeterminate over R, and $R' = R[X]_{\mathfrak{m}R[X]}$. As for all $\underline{n}$, $\lambda\left(\frac{M}{\underline{I}^{\underline{n}}M}\right) = \lambda\left(\frac{M \otimes_R R'}{\underline{I}^{\underline{n}}M \otimes_R R'}\right)$, it follows that $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = e(I_1^{[d_1]}R', \ldots, I_s^{[d_s]}R'; M \otimes_R R')$. Thus we may assume that R has an infinite residue field. By possibly first permuting the I_i, without loss of generality $d_1 > 0$, and then choose $a_1 \in I_1$ as in Lemma 1.2, such that a_1 avoids each prime ideal minimal over $\operatorname{ann} M$. Then by Lemma 2.1, for all sufficiently large $\underline{n}$,

$$\lambda\left(\frac{M}{\underline{I}^{\underline{n}}M}\right) - \lambda\left(\frac{M}{I_1^{n_1-1}I_2^{n_2}\cdots I_s^{n_s}M}\right) = \lambda\left(\frac{M}{\underline{I}^{\underline{n}}M + a_1M}\right) - \lambda\left(0 :_M a_1\right),$$

and the two sides are polynomials of degree $d-1$. If $d = 1$, then we read off that $e(I_1^{[1]}, I_2^{[0]}, \ldots, I_s^{[0]}; M) = e(I_1^{[0]}, I_2^{[0]}, \ldots, I_s^{[0]}; M/a_1M) - \lambda(0 :_M a_1) = \lambda(M/a_1M) - \lambda(0 :_M a_1)$. Whenever $d > 1$, the homogeneous parts of the polynomials of degree $d-1$ give that

$$\sum_{d_1+\cdots+d_s=d, d_1>0} \frac{1}{d_1!\cdots d_s!} e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) d_1 n_1^{d_1-1} n_2^{d_2} \cdots n_s^{d_2}$$
$$= \sum_{d_1+\cdots+d_s=d-1} \frac{1}{d_1!\cdots d_s!} e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M/a_1M) n_1^{d_1} \cdots n_s^{d_2},$$

for all sufficiently large $\underline{n}$, so that $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = e(I_1^{[d_1-1]}, I_2^{[d_2]}, \ldots, I_s^{[d_s]}; M/a_1M)$. Then by repeating the construction of a_1, there exist $d_1 - 1$ elements in I_1, d_2 elements in I_2, etc., d_s elements in I_s, labelled consecutively as $a_2, \ldots, a_d$, such that

$$e(I_1^{[d_1-1]}, I_2^{[d_2]}, \ldots, I_s^{[d_s]}; M/a_1M) = \lambda\left(\frac{M}{(a_1, \ldots, a_d)M}\right) - \lambda\left(\frac{(a_1, \ldots, a_{d-1})M :_M a_d}{(a_1, \ldots, a_{d-1})M}\right).$$

Now combine the last two displays. ∎

It is not immediately clear that all these integers $e(I_1^{[d_1]}, I_2^{[d_2]}, \ldots, I_s^{[d_s]}; M)$ are positive.

Lemma 2.4: *Let $d = \dim M > 0$, and $a_1, \ldots, a_d \in \mathfrak{m}$ such that $J = (a_1, \ldots, a_d) + \operatorname{ann} M$ is $\mathfrak{m}$-primary. For all $i = 1, \ldots, s$, set $M_i = M/(a_1, \ldots, a_{i-1})M$. If for all i, either a_i is a non-zerodivisor on M_i or $(0 :_{M_i} a_i) \cap \mathfrak{m}^n M_i = 0$ for n sufficiently large, then the multiplicity of M with respect to J equals $\lambda\left(\frac{M}{(a_1,\ldots,a_d)M}\right) - \lambda\left(\frac{(a_1,\ldots,a_{d-1})M :_M a_d}{(a_1,\ldots,a_{d-1})M}\right)$.*

Proof: The choice of a_i guarantees that for all n sufficiently large, $(0 :_{M_i} a_i) \cap J^n M_i = 0$.

By Equation (*),

$$\lambda(M/J^nM) - \lambda(M/J^{n-1}M) = \lambda(M/(J^nM + a_1M) - \lambda((J^nM :_M a_1)/J^{n-1}M).$$

There exists a polynomial $p \in \mathbb{Q}[X]$ of degree d such that for all sufficiently large n, $p(n) = \lambda(M/J^nM)$. Also, there exists a polynomial $q \in \mathbb{Q}[X]$ of degree $d-1$ such that for all sufficiently large n, $q(n) = \lambda(M/(J^nM + a_1M))$. Set $r(X) = q(X) - p(X) + p(X-1)$. By the display above, r is a polynomial in $\mathbb{Q}[X]$ of degree at most $d-1$ such that for sufficiently large n, $\lambda((J^nM :_M a_1)/J^{n-1}M) = r(n)$.

If $d = 1$, then $p(X) = eX + f$ for some rational numbers (actually integers) e and f. The leading coefficient, e, is the multiplicity of M with respect to J. Hence the display says that for n large enough, $e = p(n) - p(n-1) = \lambda(M/a_1M) - \lambda((a_1^nM :_M a_1)/a_1^{n-1}M)$. But for possibly even larger n,

$$\frac{a_1^nM :_M a_1}{a_1^{n-1}M} = \frac{a_1^{n-1}M + (0 :_M a_1)}{a_1^{n-1}M} \cong \frac{0 :_M a_1}{(0 :_M a_1) \cap a_1^{n-1}M} = 0 :_M a_1,$$

which proves the case $d = 1$.

Now let $d > 1$, and set $I = (a_2, \ldots, a_d)$. By the Artin–Rees Lemma, there exists an integer c such that for all $n \geq c$, $I^nM :_M a_1 \subseteq (0 :_M a_1) + I^{n-c}M$. Then

$$\lambda\left(\frac{J^nM :_M a_1}{J^{n-1}M}\right) = \lambda\left(\frac{J^{n-1}M + (I^nM :_M a_1)}{J^{n-1}M}\right) \leq \lambda\left(\frac{J^{n-1}M + (0 :_M a_1) + I^{n-c}M}{J^{n-1}M}\right)$$
$$\leq \lambda\left(\frac{J^{n-1}M + (0 :_M a_1)}{J^{n-1}M}\right) + \lambda\left(\frac{J^{n-1}M + I^{n-c}M}{J^{n-1}M}\right).$$

As $\frac{J^{n-1}M+I^{n-c}M}{J^{n-1}M}$ is a module over $R/(J^{c-1}+\operatorname{ann} M)$, its length is bounded by $\mu(M)\mu(I^{n-c})$ times the length of $R/(J^{c-1} + \operatorname{ann} M)$, which is a polynomial of degree at most $d-2$. By the assumption on a_1, $\frac{J^{n-1}M+(0:_M a_1)}{J^{n-1}M}$ is isomorphic to $(0 :_M a_1)$ for large n. Thus $r(X)$ is a polynomial of degree at most $d-2$, and in Equation (*), the reading of the polynomial in degree $d-1$ yields that the multiplicity of M with respect to J equals the multiplicity of M/a_1M with respect to I. Then by induction on d, the multiplicity of J on M equals $\lambda\left(\frac{M}{(a_1,\ldots,a_d)M}\right) - \lambda\left(\frac{(a_1,\ldots,a_{d-1})M:_M a_d}{(a_1,\ldots,a_{d-1})M}\right)$. ∎

Now we combine Theorem 2.3 and Lemma 2.4:

Theorem 2.5: *(Teissier-Risler [Tei73], Rees [Re84]) With the set-up as above, the number $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$ is a positive integer. Whenever $R/\mathfrak{m}$ is an infinite field, this number equals the mutliplicity of an ideal J in R, where J is generated by d_1 sufficiently general elements of I_1, d_2 sufficiently general elements of I_2, etc., d_s sufficiently general elements of I_s.*

Proof: Set $d = \dim M$. Then $d = d_1 + \cdots + d_s$. The theorem is trivial if $d = 0$. Thus we assume that $d > 0$.

Let X be an indeterminate over R, and $S = R[X]_{\mathfrak{m}R[X]}$. Then for any R-module N of finite length, NS is an S-module of the same length, so in particular $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) =$

$e((I_1S)^{[d_1]}, \ldots, (I_sS)^{[d_s]}; M \otimes_R S)$. The ring S has an infinite residue field. By Theorem 2.3, there exist d_1 (sufficiently general) elements in I_1S, etc., d_s elements in I_sS, call them $a_1, \ldots, a_d$, such that $e((I_1S)^{[d_1]}, \ldots, (I_sS)^{[d_s]}; M \otimes_R S)$ equals $\lambda\left(\frac{MS}{(a_1,\ldots,a_d)MS}\right) - \lambda\left(\frac{(a_1,\ldots,a_{d-1})MS :_{MS} a_d}{(a_1,\ldots,a_{d-1})MS}\right)$. Set $M_i = MS/(a_1, \ldots, a_{i-1})MS$. The construction in the proof of Theorem 2.3 requires that each a_i avoid the minimal primes over $\operatorname{ann}(M_i)$ and satisfy the property that for some c all $n_i \ge \underline{c}$ and all $n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_s$,

$$(I_1^{n_1} \cdots I_s^{n_s} M_i :_{M_i} a_i) \cap I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^c I_{i+1}^{n_{i+1}} \cdots I_s^{n_s} M_i = I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^{n_i - 1} I_{i+1}^{n_{i+1}} \cdots I_s^{n_s} M_i.$$

By the generalized Artin–Rees Lemma (Theorem 1.1), for all $\underline{n}$ sufficiently large, the module $I_1^{n_1} \cdots I_s^{n_s} M_i :_{M_i} a_i$ is contained in $I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^c I_{i+1}^{n_{i+1}} \cdots I_s^{n_s} M_i$. Thus for all sufficiently large $\underline{n}$,

$$(0 :_{M_i} a_i) \cap \underline{I}^{\underline{n}} M_i = 0.$$

Thus $a_1, \ldots, a_d$ satisfy the conditions of Lemma 2.4, which proves the theorem. ∎

An important ingredient in the proof above is the passage from R to a faithfully flat extension $S = R[X]_{\mathfrak{m}R[X]}$ with an infinite residue field that preserves lengths.

This technique is also the main tool in the proofs of the following (and the proofs are left to the reader):

(1) It is clear that whenever $I_1, \ldots, I_s, J_1, \ldots, J_s$ are $\mathfrak{m}$-primary ideals such that for all $i = 1, \ldots, s$, $J_i \subseteq I_i$, then $\lambda(M/\underline{J}^{\underline{n}}M) \ge \lambda(M/\underline{I}^{\underline{n}}M)$. But even more: whenever $d_1 + \cdots + d_s = \dim M$, then $e(J_1^{[d_1]}, \ldots, J_s^{[d_s]}; M) \ge e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$. (Hint: It is enough to prove this in the case $J_2 = I_2$, …, $J_s = I_s$.)

(2) For any positive integers $l_1, \ldots, l_s$,

$$e((I_1^{l_1})^{[d_1]}, \ldots, (I_s^{l_s})^{[d_s]}; M) = l_1^{d_1} \cdots l_s^{d_s} e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M).$$

(3) (Associativity formula for mixed multiplicities)

$$e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = \sum_P \lambda_{R_P}(M_P) e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M/PM),$$

where P varies over the prime ideals in R containing $\operatorname{ann}(M)$ for which $\dim(R/P) = \dim M$.

(4) Mixed multiplicities behave well on short exact sequences. Explicitly, if $0 \to N \to M \to K \to 0$ is a short exact sequence of R-modules, then

(i) If $\dim M = \dim N = \dim K$,

$$e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; N) + e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; K);$$

(ii) If dim $M = $ dim $N >$ dim K, $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; N)$;

(iii) If dim $M = $ dim $K >$ dim N, $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; K)$.

(5) Let $(R, \mathfrak{m}) \subseteq (R', \mathfrak{m}')$ be a module-finite extension of local domains of dimension d. Then

$$e((I_1R')^{[d_1]}, \ldots, (I_sR')^{[d_s]}; R') = \frac{e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; R)\mathrm{rk}_R(S)}{[R'/\mathfrak{m}' : R/\mathfrak{m}]}.$$

3. Joint reductions

There is interaction between mixed multiplicities also via the concept of joint reductions. Joint reductions were first defined by Rees in [Re84]. We set-up some notation. Let d be a positive integer. Then a d-tuple $(a_1, \ldots, a_d)$ of elements of R is said to be a *joint reduction* of the d-tuple $(J_1, \ldots, J_d)$ with respect to M, if for each $i = 1, \ldots, d$, $a_i \in J_i$, and $J = \sum_{i=1}^d a_i J_1 \cdots J_{i-1} J_{i+1} \cdots J_d$ is a reduction of the ideal $I = J_1 \cdots J_s$ with respect to M. In other words, there exists an integer l such that $JI^lM = I^{l+1}M$. If each J_i is one of the I_j, and if each I_j appears d_j times, then we say that $(a_1, \ldots, a_d)$ is a joint reduction of $I_1, \ldots, I_s$ of type $(d_1, \ldots, d_s)$ with respect to M.

The reader can rework the proof of Theorem 2.5 to see that when the residue field is infinite, there exists a joint reduction $(a_1, \ldots, a_d)$ of $I_1, \ldots, I_s$ of type $(d_1, \ldots, d_s)$ with respect to M such that $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$ equals the multiplicity of M with respect to $(a_1, \ldots, a_d)$. The case $d = 1$ is the case of ordinary reductions, and the rest is proved by induction on d.

Theorem 3.1: *Let $d = \dim M$. If $(a_1, \ldots, a_d)$ is a joint reduction of $I_1, \ldots, I_s$ of type $(d_1, \ldots, d_s)$ with respect to M, then $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$ equals the multiplicity of M with respect to $(a_1, \ldots, a_d)$.*

Proof: Necessarily $(a_1, \ldots, a_d) + \mathrm{ann}\, M$ is $\mathfrak{m}$-primary.

The theorem is clear if $d = 0$. Now assume that $d = 1$. Then by assumption there exists an integer l such that $a_1(I_1 \cdots I_s)^l M = (I_1 \cdots I_s)^{l+1} M$. Thus a_1R is a reduction of I_1 with respect to $M' = (I_2 \cdots I_s)^l M$. There exists an integer c such that for all $n \geq c$, $I_1^n \in (a_1^{n-c}R + \mathrm{ann}\,(M'))$. Then for all $n \geq 1$, $\lambda(M'/a_1^n M') \geq \lambda(M'/I_1^n M') \geq \lambda(M'/a_1^{n-c}M')$. Each of the three quantities eventually equals a polynomial in n of degree 1, the first and the third polynomials have the same leading coefficient, namely the multiplicity of a_1R on M, so necessarily the middle polynomial has the same leading coefficient. But $\lambda(M'/I_1^n M') = \lambda(M/I_1^n M') - \lambda(M/M') = \lambda(M/\underline{I}^{\underline{n}} M) - \lambda(M/M')$, $\lambda(M/M')$ is a constant, so the leading coefficient of the polynomial $\lambda(M/\underline{I}^{\underline{n}} M)$ equals the leading coefficient of the three polynomials above, which proves the case $d = 1$.

Now let $d > 1$. By the associativity formula for multiplicities and mixed multiplicities, without loss of generality $\mathrm{ann}\, M = 0$ is a prime ideal, and $M = R$. For simplicity of notation we may assume that $d_1 > 0$. A step of Lemma 2.4 proves that the multiplicity of $(a_1, \ldots, a_d)$ on R is the multiplicity of $(a_2, \ldots, a_d)$ on R/a_1R, and that the mixed multiplicity $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; R)$ equals $e(I_1^{[d_1-1]}, I_2^{[d_2]}, \ldots, I_s^{[d_s]}; R/a_1R)$.

By passage to R/a_1R, $(a_2, \ldots, a_d)$ is a joint reduction of $I_2, \ldots, I_s$ of type $(d_1 - 1, d_2, \ldots, d_s)$ with respect to $I_1^c(R/a_1R)$ for some integer c. By induction on dimension, $e(I_1^{[d_1-1]}, I_2^{[d_2]}, \ldots, I_s^{[d_s]}; I_1^c(R/a_1R))$ equals the multiplicity of $I_1^c(R/a_1R)$ with respect to $(a_2, \ldots, a_d)$. But the short exact sequence $0 \to \frac{I_1^c + a_1R}{a_1R} \to \frac{R}{a_1R} \to \frac{R}{I_1^c} \to 0$ shows that $\dim R/a_1R \dim (I_1^c(R/a_1R)) = \dim R - 1 > \dim R/I_1^c = 0$, so that by Property (4), $e(I_1^{[d_1-1]}, I_2^{[d_2]}, \ldots, I_s^{[d_s]}; I_1^c(R/a_1R)) = e(I_1^{[d_1-1]}, I_2^{[d_2]}, \ldots, I_s^{[d_s]}; R/a_1R)$, and the multiplicity of $I_1^c(R/a_1R)$ with respect to $(a_2, \ldots, a_d)$ is the same as the multiplicity of $I_1^c(R/a_1R)$ with respect to $(a_2, \ldots, a_d)$. Now combine all the equalities. □

The converse holds sometimes. Namely, let $a_1, \ldots, a_d$ be d_1 elements of I_1, etc., d_s elements of I_s, such that $e(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; R)$ equals the multiplicity of R with respect to $(a_1, \ldots, a_d)$. Then if R is formally equidimensional, $(a_1, \ldots, a_d)$ is a joint reduction of the d-tuple $(I_1, \ldots, I_1, I_2, \ldots, I_2, \ldots, I_s, \ldots, I_s)$ with respect to R, where each I_i is listed d_i times. (The proof of this is in [Sw93], and it requires techniques from [B69].)

4. Other topics

We finish with listing a few other topics related to mixed multiplicities which an interested reader may wish to read. Teissier [Tei78], Rees and Sharp [ReSh78] proved various Minkowski-type inequalities. Katz and Verma [KaVe89] generalized mixed multiplicities to ideals not all of which are zero-dimensional. Verma [Ve88, Ve92] connected mixed multiplicities to multiplicities of multi-graded Rees algebras. Cohen–Macaulayness of multi-ideal Rees algebras was studied by Verma [Ve88], [Ve90], [Ve91]; Tang [Ta99]; Herrmann, Hyry, Ribbe [HHR93]; and Herrmann, Hyry, Ribbe, Tang [HHRT97]. Trung [Tr01] generalized mixed multiplicities to bigraded algebras, using filter-regular sequences. Trung and Verma [TrVe06] interpreted mixed multiplicities of monomial ideals in terms of mixed volumes of polytopes.

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