

Induction

One way (and sometimes the only way) of proving that some property P holds for all integers n greater than or equal to an integer n_0 is the following two-step procedure, called **induction**:

1. **Base case:** Prove that P holds for n_0 .
2. **Inductive step:** Let $n > n_0$. Assume that P holds for all integers between n_0 and $n - 1$ (n_0 and $n - 1$ included). Prove that P holds for n .

Why does induction succeed in proving that P holds for all $n \geq n_0$? By the base case we know that P holds for n_0 . The inductive step then proves that P also holds for $n_0 + 1$. So then we know that the property holds for n_0 and $n_0 + 1$, whence the inductive step implies that it also holds for $n_0 + 2$. So then the property holds for n_0 , $n_0 + 1$ and $n_0 + 2$, whence the inductive step implies that it also holds for $n_0 + 3$, and so on. Thus for any integer $n \geq n_0$, we eventually prove that P holds for it.

There is also the following logically equivalent principle of induction:

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Exercise: Why does this induction principle succeed in proving that P holds for all $n \geq n_0$?

Usually, n_0 is either 0 or 1.

Example: Prove the identity $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ for all $n \geq 1$.

1. Base case $n = 1$: The left side of the equation is $\sum_{i=1}^1 i$ which equals 1. The right side is $\frac{1(1+1)}{2}$ which also equals 1. This verifies the base case.
2. Inductive step: We assume that the identity holds for $n - 1$. We want to prove it for n . We start with the left side of the equation (the messier of the two) and manipulate it until it resembles the right side:

$$\begin{aligned}\sum_{i=1}^n i &= \left(\sum_{i=1}^{n-1} i \right) + n \\ &= \frac{(n-1)(n-1+1)}{2} + n \quad \text{by induction assumption}\end{aligned}$$

$$\begin{aligned}
&= \frac{n^2 - n}{2} + \frac{2n}{2} \\
&= \frac{n^2 + n}{2} \\
&= \frac{n(n+1)}{2},
\end{aligned}$$

as was to be proved. QED

The following properties for $n \geq 1$ may be proved by induction.

1. There are exactly n people at a gathering. Everybody shakes everybody else's hands exactly once. How many handshakes were there?
2. $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$
4. the sum of the first n odd integers is n^2 .
5. $\sum_{i=1}^n (2i-1) = n^2$.
6. $\sum_{i=1}^n (3i^2 - i) = n^2(n+1)$.
7. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$
8. Let $A_n = 1^2 + 2^2 + 3^2 + \cdots + (2n-1)^2$. Find a formula for A_n and prove it.
9. Let $A_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$. Find a formula for A_n and prove it.
10. $\frac{dx^n}{dx} = nx^{n-1}$.
11. $1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$ for $x \neq 1$.
12. $7^n + 2$ is a multiple of 3.
13. $3^{n-2} < (n+1)!$
14. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$
15. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$
16. Let $a_1 = 2$, and for $n \geq 2$, $a_n = 3a_{n-1}$. Formulate and prove a theorem giving a_n in terms of n (no dependence on other a_i).
17. Let A and B be $k \times k$ matrices, with A invertible. Then $(ABA^{-1})^n = AB^nA^{-1}$.
18. $\sum_{i=0}^n \binom{n}{i} = 2^n$ (Notation $\binom{n}{i}$ is read "n choose i" and it denotes the number of ways to choose subsets of i elements from a collection of n elements. It happens to be $\frac{n!}{i!(n-i)!}$. To prove this by induction, you need to know that $\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$.)

19. Prove that every integer greater than 2 is a finite product of primes.
20. 8 divides $5^n + 2 \cdot 3^{n-1} + 1$.
21. $1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$.
22. $2^{n-1} \leq n!$
23. ($n \geq 2$?) $\prod_{k=2}^n \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}$.

What is wrong with the following “proofs by induction”?

1. I will prove that $5^n + 1$ is a multiple of 4.

Assume that this is true for $n - 1$. Then we can write $5^{n-1} + 1 = 4m$ for some integer m . Multiply this equation through by 5 to get that

$$5^n + 5 = 20m,$$

whence $5^n + 1 = 4(5m - 1)$. As $5m - 1$ is an integer, this proves that $5^n + 1$ is a multiple of 4. QED

2. I will prove that all horses are of the same color. This is the same as saying that for any integer $n \geq 1$ and any set of n horses, all the horses belonging to the set have the same color.

If $n = 1$, of course this only horse is the same color as itself, so the base case is proved. Now let $n > 1$. If we remove one horse from this set, the remaining $n - 1$ horses in the set are all of the same color by the induction assumption. Now bring that one horse back into the set and remove another horse. Then again all of these horses are of the same color, so the horse that was removed first is the same color as all the rest of them. QED