Exercises to go with epsilon-delta proofs and Section 1.8:

1 Fill in the blanks of the following proof that \( \lim_{x \to 2} (x^2 - 3x) = -2 \). Explain why none of the inequalities can be changed into equalities.

Let \( \epsilon > 0 \). Set \( \delta = \epsilon \). Let \( x \) satisfy \( 0 < |x - 2| < \delta \). Then

\[
| (x^2 - 3x) - (-2) | = |x^2 - 3x + 2 |
= |x - 1| |x - 2| \quad \text{(because \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)}
\leq (|x - 2| + 1) |x - 2| \quad \text{(because \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)}
< (3 + 1) |x - 2| \quad \text{(because \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)}
< 4 \delta \quad \text{(because \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)}
\leq 4 \frac{\epsilon}{4} \quad \text{(because \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)}
= \epsilon.

2 Determine the following limits and prove them with epsilon-delta proofs.

i) \( \lim_{x \to 1} (x^3 - 4) \).

ii) \( \lim_{x \to 2} \frac{1}{x} \).

iii) \( \lim_{x \to 3} \frac{x^3 - 4}{x^3 + 2} \).

iv) \( \lim_{x \to 4} \sqrt{x + 5} \).

v) \( \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \).

vi) \( \lim_{x \to 3} \frac{x - 3}{x^2 - 9} \).

3 Let \( b \in \mathbb{R} \) and \( f, g : \mathbb{R} \to \mathbb{R} \) with

\[
f(x) = \begin{cases} 
  x^3 - 4x^2, & \text{if } x \neq 5; \\
  b, & \text{if } x = 5,
\end{cases}
\quad g(x) = \begin{cases} 
  x^3 - 4x^2, & \text{if } x = 5; \\
  b, & \text{if } x \neq 5.
\end{cases}
\]

Prove that the limit of \( f(x) \) as \( x \) approaches 5 is independent of \( b \), but that the limit of \( g(x) \) as \( x \) approaches 5 depends on \( b \).

4 Prove that \( \lim_{x \to a} (mx + l) = ma + l \), where \( m \) and \( l \) are constants.

5 Let \( f : \mathbb{R} \to \mathbb{R} \) be given by \( f(x) = \frac{x}{|x|} \). Prove that \( \lim_{x \to 0^+} f(x) = 1 \) and that \( \lim_{x \to 0^-} f(x) = -1 \).