On the embedded primes of the Mayr-Meyer ideals

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This paper investigates the doubly exponential ideal membership property of the Mayr-Meyer ideals. These ideals were first defined by Mayr and Meyer in [MM], where their doubly exponential behavior was first observed, and subsequently these ideals were further analyzed by Bayer and Stillman [BS], Demazure [D], Koh [K].

The analysis in this paper, as well as in [S1, S2], is from the point of view of the structure of the associated primes. The motivation came from a question raised by Bayer, Huneke and Stillman of whether the doubly exponential behavior is due to the number of minimal and associated primes, or to the nature of one of them. The complete answer for the case of the Mayr-Meyer ideals with the fewest possible number of variables (the case $n = 1$) is given in [S1]. For all other cases, it was proved in [S2] that the doubly exponential behavior is due to the embedded primes. [S2] also computed all minimal components, the minimal primes, their heights, and the intersection of all minimal components.

This paper provides partial answers about the embedded primes. In the analysis a new family of ideals emerges which also has the doubly exponential ideal membership property. The advantage of this family is that the determination of the associated primes of the ideals in the family can be found recursively, see Section 7.

The main tool used below for finding the associated primes of the Mayr-Meyer ideals

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are various short exact sequences, and the fact that the associated primes of the middle module in a short exact sequence is contained in the union of the associated primes of the two other modules. Theorem 8.1 gives a set of prime ideals obtained in this way, which therefore contains all associated primes of the Mayr-Meyer ideals. However, not all primes in this set need to be associated to the middle module. Removing the redundant prime ideals is a much harder process, and is not completed here. Most of Sections 3 and 4 is taken up by removing (some) redundancies.

The Mayr-Meyer ideals $J = J(n, d)$ depend on two parameters, n and d, where the number of variables in the ring is $O(n)$ and the degree of the given generators of the ideal is $O(d)$. (See definitions in the first section.) Both n and d are positive integers. Throughout this paper it is assumed that $n \geq 2$.

For given n, d, the construction described above yields a set $S(n, d)$ of prime ideals which contain the set of associated primes of $J(n, d)$. The set $S(n, d)$ is made explicit in Theorem 8.1. The cardinality of $S(n, d)$ is $160n - 270 + 31d + n(n-1) + \delta_{n=2}(d^2 - d) + \delta_{n>2}(d^3 - d)(n-1) + 31(d^{2^1} + \cdots + d^{2^{n-3}}) + (n-1)d^{2^1} + (n-2)d^{2^2} + \cdots + 3d^{2^{n-3}} + 18d^{2^{n-2}}$. This number is doubly exponential in n. Sections 2, 3, and 4 find $31 + 15d + \delta_{n=2}(d^2 - d) + \delta_{n>2}(d^3 - d)(n-1)$ prime ideals which are indeed embedded primes of $J(n, d)$, showing that the number of embedded primes of $J(n, d)$ grows with n and d. Sections 3 and 4 also prove that there exist no embedded primes of certain kinds. It is not proved whether $J(n, d)$ in fact has a doubly exponential number of embedded primes. Of the primes in $S(n, d)$, the largest height is achieved by the prime ideals $Q_{23,n-2,n,1,\alpha}$ and Q_{24}, whose heights are 2 less than the dimension of the ring. However, I do not know if these ideals are associated.

The generators of the Mayr-Meyer ideals in levels 1 through $n-1$ have similar forms, so that there is hope that the associated primes of the Mayr-Meyer ideals could be arrived at via recursion. I was unable to reduce the search for the associated primes of $J(n, d)$ to that of finding the associated primes of $J(n-1, d)$. However, Section 5 modifies the problem of finding the associated primes of $J(n, d)$ to that of finding the associated primes of an ideal $K(n, d)$ to which recursion can be applied. The recursive procedure is carried out in Section 7.

Many questions remain about the embedded primes of $J(n, d)$. Some are listed at the end.

Originally I attempted to find the embedded components, not just the embedded primes, but that became unwieldy. See `http://math.nmsu.edu/~iswanson` for these and other computations with the Mayr-Meyer ideals which are not included here.

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1. Notation

The definition below of the Mayr-Meyer ideals is taken from [S2]: it is somewhat different from the original definition by Mayr and Meyer in [MM], but equivalent to the original one from the point of view of primary decompositions. See [S2] for complete justification. Namely, for any fixed integers $n, d \geq 2$, let $R = k[s, f, b_{ri}, c_{ri} | r = 0, \ldots, n-1; i = 1, \ldots, 4]$ be a polynomial ring in $8n+2$ variables over a field k, and let the Mayr-Meyer ideal $J(n,d)$ be the ideal in R generated by the following polynomials h_{ri}: first the four level 0 generators:

$$h_{0i} = c_{0i}\left(s - f b_{0i}^d\right), i = 1, 2, 3, 4;$$

then the eight level 1 generators:

$$\begin{aligned}
h_{13} &= f c_{01} - s c_{02}, \\
h_{14} &= f c_{04} - s c_{03}, \\
h_{15} &= s\left(c_{03} - c_{02}\right), \\
h_{16} &= f\left(c_{02} b_{01} - c_{03} b_{04}\right), \\
h_{1,6+i} &= f c_{02} c_{1i}\left(b_{02} - b_{1i} b_{03}\right), i = 1, \ldots, 4,
\end{aligned}$$

the first four level r generators, $r = 2, \ldots, n$:

$$\begin{aligned}
h_{r3} &= s c_{01} c_{11} \cdots c_{r-3,1}\left(c_{r-2,4} c_{r-1,1} - c_{r-2,1} c_{r-1,2}\right), \\
h_{r4} &= s c_{01} c_{11} \cdots c_{r-3,1}\left(c_{r-2,4} c_{r-1,4} - c_{r-2,1} c_{r-1,3}\right), \\
h_{r5} &= s c_{01} c_{11} \cdots c_{r-2,1}\left(c_{r-1,3} - c_{r-1,2}\right), \\
h_{r6} &= s c_{01} c_{11} \cdots c_{r-3,1} c_{r-2,4}\left(c_{r-1,2} b_{r-1,1} - c_{r-1,3} b_{r-1,4}\right),
\end{aligned}$$

the last four level r generators, $r = 2, ..., n-1$:

$$h_{r,6+i} = s c_{01} c_{11} \cdots c_{r-3,1} c_{r-2,4} c_{r-1,2} c_{ri}\left(b_{r-1,2} - b_{ri} b_{r-1,3}\right), i = 1, \ldots, 4,$$

and the last level n generator:

$$h_{n7} = s c_{01} c_{11} \cdots c_{n-3,1} c_{n-2,4} c_{n-1,2}\left(b_{n-1,2} - b_{n-1,3}\right).$$

For notational purposes we also define the following ideals in R:

$$\begin{aligned}
E &= (s - f b_{01}^d) + \left(b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d\right), \\
F &= \left(b_{02} - b_{11}b_{03}, b_{14} - b_{11}, b_{13} - b_{11}, b_{12} - b_{11}, b_{12}^d - 1\right) \\
C_r &= (c_{r1}, c_{r2}, c_{r3}, c_{r4}), r = 0, \ldots, n-1 \\
C_n &= (0), \\
D_0 &= \left(c_{04} - c_{01}, c_{03} - c_{02}, c_{01} - c_{02}b_{01}^d\right), \\
D_r &= (c_{r4} - c_{r1}, c_{r3} - c_{r2}, c_{r2} - c_{r1}), r = 1, \ldots, n-1, \\
D_n &= (0), \\
B_0 &= B_1 = (0), \\
B_r &= (1 - b_{2i}, 1 - b_{3i}, \ldots, 1 - b_{ri} | i = 1, \ldots, 4), r = 2, \ldots, n-1. \\
B_{kr} &= (1 - b_{ki}, 1 - b_{k+1,i}, \ldots, 1 - b_{ri} | i = 1, \ldots, 4), r = 2, \ldots, n-1. \\
p_1 &= C_1 + E + D_0, \\
p_r &= C_r + E + F + B_{r-1} + D_0 + D_1 + \cdots + D_{r-1}, r \geq 2.
\end{aligned}$$

With this notation, here is the table of all minimal primes over $J(n,d)$, as computed in [S2], where α and β are dth roots of unity, and Λ varies over all subsets of $\{1,2,3,4\}$:

minimal prime	height	component of $J(n,d)$
$P_0 = (c_{01}, c_{02}, c_{03}, c_{04})$	4	$p_0 = P_0$
$P_{1\alpha\beta} = p_1 + (b_{01} - \alpha b_{02}, b_{02} - \beta b_{03})$	11	$p_{1\alpha\beta} = P_{1\alpha\beta}$
$P_{r\alpha\beta} = p_r$	$7r+4$	$p_{r\alpha\beta} = P_{r\alpha\beta}$, $2 \leq r < n$
$+(b_{01} - \alpha b_{02}, b_{02} - \beta b_{03}, \beta - b_{1i})$	$7n$	$p_{r\alpha\beta} = P_{r\alpha\beta}$, $r = n$
$P_{-1} = (s, f)$	2	$p_{-1} = P_{-1}$
$P_{-2} = (s, c_{01}, c_{02}, c_{04}, b_{03}, b_{04})$	6	$p_{-2} = (s, c_{01}, c_{02}, c_{04}, b_{03}^d, b_{04})$
$P_{-3} = (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01} - c_{03}b_{04})$	6	$p_{-3} = P_{-3}$
$P_{-4\Lambda} = (s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02})$	10	$p_{-4\Lambda} = (s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}^d)$
$+(c_{1i}, b_{1j} \| i \notin \Lambda, j \in \Lambda)$		$+(c_{1i} \| i \notin \Lambda)$
		$+(b_{1j}^d, b_{02} - b_{1j}b_{03}, b_{1j} - b_{1j'} \| j, j' \in \Lambda)$

The intersection of all components primary to the $P_{-4\Lambda}$ was computed to be

$$p_{-4} = (s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}^d) + (c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j})|i, j = 1, \ldots, 4).$$

The following summarizes the elementary facts about primary decompositions used in the paper:

Facts:

1.1: For any ideals I, I' and I'' with $I \subseteq I''$, $(I + I') \cap I'' = I + I' \cap I''$.

1.2: For any ideal I and element x, $(x) \cap I = x(I : x)$.

1.3: For any ideals I and I', and any element x, $(I + xI') : x = (I : x) + I'$.

1.4: Let $x_1, \ldots, x_n$ be variables over a ring R. Let $S = R[x_1, \ldots, x_n]$. For any $f_1 \in R$, $f_2 \in R[x_1]$, …, $f_n \in R[x_1, \ldots, x_{n-1}]$, let L be the ideal $(x_1 - f_1, \ldots, x_n - f_n)S$ in S. Then an ideal I in R is primary (respectively, prime) if and only if $IS+L$ is a primary (respectively, prime) in S. Furthermore, $\cap_i q_i = I$ is a primary decomposition of I if and only if $\cap_i(q_i S + L)$ is a primary decomposition of $IS + L$.

1.5: Let x be an element of a Noetherian ring R and I an ideal. Then there exists an integer k such that for all m, $I : x^m \subseteq I : x^k$. Then $I : x^k$ is also denoted as $I : x^\infty$. Also, $I = \left(I : x^k\right) \cap \left(I + (x^k)\right)$. Thus to find a (possibly redundant) primary decomposition of I it suffices to find primary decompositions of possibly larger ideals $I : x^k$ and $I + (x^k)$.

1.6: Let I be an ideal in a ring R. Then for any $x \in R$, $\text{Ass}\left(\frac{R}{I}\right) \subseteq \text{Ass}\left(\frac{R}{I:x}\right) \cup \text{Ass}\left(\frac{R}{I+(x)}\right)$, and every associated prime of $\frac{R}{I:x}$ is an associated prime of $\frac{R}{I}$. (Use the short exact sequence $0 \longrightarrow \frac{R}{I:x} \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{I+(x)} \longrightarrow 0$.)

1.7: Let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be variables over a field k and I an ideal in $k[\underline{x}] = k[x_1, \ldots, x_n]$ and J an ideal in $k[\underline{y}] = k[y_1, \ldots, y_m]$. Then

$$Ik[\underline{x}, \underline{y}] \cap Jk[\underline{x}, \underline{y}] = IJk[\underline{x}, \underline{y}].$$

For simpler notation it will be assumed throughout that the characteristic of k does not divide d, but most of the work goes through without that assumption. Also, $J(n, d)$ will often be abbreviated to J.

We will use the extended Kronecker delta notation δ_P as follows: whenever P is true, then $A\delta_P$ equals A, and when P is false, $A\delta_P$ has no effect on the rest of the expression. To shorten notation, whenever the range of subscripts i and j is not specified, it is understood that they vary in the set $\{1, 2, 3, 4\}$. Thus for example, (c_{1i}) stands for $(c_{11}, c_{12}, c_{13}, c_{14})$.

2. Sixteen embedded components

The Mayr-Meyer ideals do have embedded primes. The (possible) embedded primes will be denoted as $Q_{r\_}$, with r varying from 1 to 24, and the second part of the subscript depending on r.

Here is the first batch: for every subset $\Lambda \subseteq \{1,2,3,4\}$, define

$$Q_{1\Lambda} = (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01} - c_{03}b_{04}) + (c_{1i} | i \notin \Lambda) + (b_{1i} - b_{1j} | i, j \in \Lambda).$$

Of all associated primes of $J(n,d)$ found so far, these primes contain only P_{-3}. We prove below that each of these 16 prime ideals is an associated prime of J, with its embedded component being

$$q_{1\Lambda} = \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}\right) + (c_{1i} | i \notin \Lambda) + (b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j} | i, j \in \Lambda).$$

It is clear that the sixteen $Q_{1\Lambda}$ are prime and the $q_{1\Lambda}$ are primary. Note that the height of $Q_{1\emptyset}$ is 10, but if $\Lambda \neq \emptyset$, then the height of $Q_{1\Lambda}$ equals 9. Not only is the height of $Q_{1\emptyset}$ larger than those of the others, but it even contains all $Q_{1\{i\}}$, $i = 1, \ldots, 4$. By a computation similar to the one for p_{-4} in Section 2 of [S2], the intersection of all the $q_{1\Lambda}$ is

$$q_1 = \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}) | i, j = 1, \ldots, 4\right).$$

Observe that $J : (fc_{02}c_{03}(c_{02} - c_{03}))^\infty$ contains

$$\left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}(b_{02} - b_{1i}b_{03})\right).$$

This latter ideal contains J and decomposes as

$$\begin{aligned}
&= \bigcap_\Lambda \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}, b_{02} - b_{1j}b_{03} | i \notin \Lambda, j \in \Lambda\right) \\
&= (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01} - c_{03}b_{04}) \\
&\quad \cap \bigcap_\Lambda \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}, b_{02} - b_{1j}b_{03}, b_{1j} - b_{1j'} | i \notin \Lambda, j, j' \in \Lambda\right) \\
&= p_{-3} \cap q_1.
\end{aligned}$$

Since the element $fc_{02}c_{03}(c_{02} - c_{03})$ is a non-zerodivisor on these components, this proves that

$$J : (fc_{02}c_{03}(c_{02} - c_{03}))^\infty = p_{-3} \cap q_1.$$

To prove that each $Q_{1\Lambda}$ is associated to J, it now suffices to prove that none of the $Q_{1\Lambda}$-primary components of $J : (fc_{02}c_{03}(c_{02} - c_{03}))^\infty$ is redundant. So let Λ be a subset of $\{1,2,3,4\}$. First suppose that $\Lambda \neq \emptyset$. Then $J' = J : (fc_{02}c_{03}(c_{02} - c_{03}))^\infty \left(\prod_{\substack{i \in \Lambda \\ j \notin \Lambda}} c_{1i}(b_{1i} - b_{1j})\right)$ is exactly $p_{-3} \cap q_{1\Lambda} \neq p_{-3}$, so that $Q_{1\Lambda}$ is associated to J.

Finally suppose that $\Lambda = \emptyset$. Then $J' = J : (fc_{02}c_{03}(c_{02} - c_{03}))^\infty \left(\prod_{i \neq j}(b_{1i} - b_{1j})\right)$ is exactly $p_{-3} \cap q_{1\emptyset} \cap \bigcap_{i=1}^4 q_{1\{i\}}$. Since the element $b_{02}^{d-1} b_{03} \prod_i b_{1i}$ is in $p_{-3} \cap \bigcap_{i=1}^4 q_{1\{i\}}$ but not in $q_{1\emptyset}$, this proves that $Q_{1\emptyset}$ is associated to J.

Thus $J(n,d)$ has at least 16 embedded primes, which are as follows:

embedded prime	height	component of $J(n,d)$
$Q_{1\Lambda} = (s, c_{01}, c_{04}, b_{02}, b_{03})$	9, if $\Lambda \neq \emptyset$	$q_{1\Lambda} = \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d\right)$
$+(c_{02}b_{01} - c_{03}b_{04}) + (c_{1i} \vert i \notin \Lambda)$	10, if $\Lambda = \emptyset$	$+(c_{02}b_{01} - c_{03}b_{04}) + (c_{1i} \vert i \notin \Lambda)$
$+(b_{1i} - b_{1j} \vert i, j \in \Lambda)$		$+(b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j} \vert i, j \in \Lambda)$

3. $\mathbf{15(d+1)}$ more embedded primes (plus $\mathbf{d^2 - d}$ if $\mathbf{n = 2}$)

In this section we find $15(d+1)$ ($+d^2 - d$ when $n = 2$) more embedded primes of $J(n,d)$. This shows that the number of embedded primes of $J(n,d)$ grows with d. As usual, Λ ranges over all subsets of $\{1,2,3,4\}$ and α and β vary over the dth roots of unity:

$$Q_{2\Lambda\alpha} = (s, c_{01}, c_{03} - c_{02}, c_{04}, b_{01}, b_{02}, b_{03}, b_{04}) + (c_{1i} | i \notin \Lambda) + (b_{1i} - \alpha | i \in \Lambda)\,,$$
$$Q_{3\Lambda} = C_0 + (s, b_{01}, b_{02}, b_{03}, b_{04}) + (c_{1i} | i \notin \Lambda) + (b_{1i} - b_{1j} | i, j \in \Lambda)\,,$$
$$Q_{4,2\alpha\beta} = (s, c_{01}, c_{04}, c_{02} - c_{03}, b_{01}, b_{02}, b_{03}, b_{04}, b_{11} - \alpha, b_{14} - \alpha, b_{12} - \beta, b_{13} - \beta) + C_1.$$

These ideals are clearly prime ideals. Let $x = f^3 c_{21} b_{13}(b_{21} - b_{22})$ when $n > 2$ and $x = f^3$ when $n = 2$. We prove below that the $Q_{2\Lambda\alpha}$ and the $Q_{3\Lambda}$, for all non-empty subsets $\Lambda \subseteq \{1,2,3,4\}$ and all dth roots of unity α, are associated primes of J, and that when $n = 2$, also the $Q_{4,2\alpha\beta}$ are associated whenever α and β are distinct dth roots of unity. Furthermore, we prove that these $15(d+1)$ ($+d^2 - d$ when $n = 2$) primes are the only new embedded primes of J which do not contain x.

Consider the ideal

$$\begin{aligned}
\hat{J} = (c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d) + s(c_{02} - c_{03}) \\
+ c_{02}b_{02}^d(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d) \\
+ (c_{02}b_{01} - c_{03}b_{04}, c_{02}(s - fb_{02}^d), c_{02}c_{1i}(b_{02} - b_{1i}b_{03}), c_{02}b_{02}^d - c_{03}b_{03}^d) \\
+ c_{02}b_{02}^{2d}(c_{11} - c_{12}, c_{14} - c_{13}, c_{13} - c_{12}, c_{11}(b_{11} - b_{14}), c_{11}(b_{12} - b_{13}), c_{11}\delta_{n\geq 3}).
\end{aligned}$$

It is easy to see that x multiplies $\hat{J}$ into J and that $\hat{J}$ contains J. We will find all associated primes of $\hat{J}$, of which the only new ones are the $Q_{2\Lambda\alpha}$, $Q_{3\Lambda}$, and $Q_{4,2\alpha\beta}$. We will show that x is not in any of the associated primes, so that then $\hat{J} = J : x$. Thus $\operatorname{Ass}(R/\hat{J}) = \operatorname{Ass}(R/J : x)$, and every associated prime of $\hat{J}$ is also associated to J. Thus it suffices to find all associated primes of $\hat{J}$ and to show that x is not in any of them.

By Fact 1.6, $\operatorname{Ass}\left(\frac{R}{\hat{J}}\right) \subseteq \operatorname{Ass}\left(\frac{R}{\hat{J}:c_{02}}\right) \cup \operatorname{Ass}\left(\frac{R}{\hat{J}+(c_{02})}\right)$. Note that $\hat{J}+(c_{02}) = (c_{01}, c_{02}, c_{04}, c_{03}b_{03}^d, sc_{03}, c_{03}b_{04}) = p_0 \cap p_{-2}$ is an intersection of some minimal components of J, and x is a non-zerodivisor modulo each of them. Hence it suffices to find the associated primes of $\hat{J} : c_{02}$, and to show that x is not in any of them:

$$\begin{aligned}
\hat{J} : c_{02} = (&c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d) \\
&+ b_{02}^d(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d) + (c_{1i}(b_{02} - b_{1i}b_{03})) \\
&+ b_{02}^{2d}(c_{11} - c_{12}, c_{14} - c_{13}, c_{13} - c_{12}, c_{11}(b_{11} - b_{14}), c_{11}(b_{12} - b_{13}), c_{11}\delta_{n\geq 3}) \\
&+ (s(c_{02} - c_{03}), c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}) : c_{02} \\
= (&c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d) \\
&+ b_{02}^d(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d) + (c_{1i}(b_{02} - b_{1i}b_{03})) \\
&+ b_{02}^{2d}(c_{11} - c_{12}, c_{14} - c_{13}, c_{13} - c_{12}, c_{11}(b_{11} - b_{14}), c_{11}(b_{12} - b_{13}), c_{11}\delta_{n\geq 3}) \\
&+ (c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d) \\
&+ s(c_{02} - c_{03}, b_{01} - b_{04}, b_{02}^d - b_{03}^d) \\
= (&c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d) \\
&+ b_{02}^d(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d) + (c_{1i}(b_{02} - b_{1i}b_{03})) \\
&+ b_{02}^{2d}(c_{11} - c_{12}, c_{14} - c_{13}, c_{13} - c_{12}, c_{11}(b_{11} - b_{14}), c_{11}(b_{12} - b_{13}), c_{11}\delta_{n\geq 3}) \\
&+ (c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d).
\end{aligned}$$

Again by Fact 1.6, $\operatorname{Ass}\left(\frac{R}{\hat{J}:c_{02}}\right) \subseteq \operatorname{Ass}\left(\frac{R}{\hat{J}:c_{02}b_{02}^d}\right) \cup \operatorname{Ass}\left(\frac{R}{(\hat{J}:c_{02})+(b_{02}^d)}\right)$. Note that

$$(\hat{J} : c_{02}) + (b_{02}^d) = (c_{01}, c_{04}, s, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d),$$

which decomposes as

$$\begin{aligned}
&= (c_{01}, c_{04}, s, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{03}, b_{01}) \\
&\quad \cap (c_{01}, c_{04}, s, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), b_{03}^d, c_{02}b_{01} - c_{03}b_{04}) \\
&= p_{-4} \cap p_{-3} \cap q_1,
\end{aligned}$$

as in Section 2. Thus x is a non-zerodivisor on this ideal, and no new associated primes appear. Thus it suffices to find the associated primes of $\hat{J} : c_{02}b_{02}^d$:

$$\begin{aligned}
\hat{J} : c_{02}b_{02}^d = (&c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d) \\
&+ (b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d) \\
&+ b_{02}^d(c_{11} - c_{12}, c_{14} - c_{13}, c_{13} - c_{12}, c_{11}(b_{11} - b_{14}), c_{11}(b_{12} - b_{13}), c_{11}\delta_{n\geq 3}) \\
&+ (c_{1i}(b_{02} - b_{1i}b_{03}), c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d) : b_{02}^d.
\end{aligned}$$

The next two displays will compute the colon ideal in the last row. As in the computation of p_{-4} in [S2],

$$
\begin{aligned}
&(c_{1i}(b_{02}-b_{1i}b_{03}), c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d)\\
&=\bigcap_{\Lambda}\Big((c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}, b_{02}-b_{1j}b_{03} | i\notin\Lambda, j\in\Lambda)\Big)\\
&=\Big((c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d) + C_1\Big)\\
&\bigcap_{\Lambda\neq\emptyset}\Big(((c_{02}b_{1j}^d - c_{03})b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, (b_{01}-b_{04}b_{1j}^d)b_{03}^d, c_{1i}, b_{02}-b_{1j}b_{03} | i\notin\Lambda, j\in\Lambda)\Big),
\end{aligned}
$$

which coloned with b_{02}^d equals

$$
\begin{aligned}
&\Big((c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d) + C_1\Big)\\
&\bigcap_{\Lambda\neq\emptyset}\Big((c_{02}b_{1j}^d - c_{03}, b_{01}-b_{04}b_{1j}^d, c_{1i}, b_{02}-b_{1j}b_{03}, b_{1j}-b_{1j'} | i\notin\Lambda, j, j'\in\Lambda)\Big)\\
&=\big(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01}-c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02}-b_{1i}b_{03})\big)\\
&\qquad+\big(c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}(b_{01}-b_{1i}^d b_{04}), c_{1i}(c_{03}-b_{1i}^d c_{02}) | i, j\big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\hat{J} : c_{02}b_{02}^d &= (c_{01}-c_{02}b_{02}^d, c_{04}-c_{02}b_{02}^d, s - fb_{02}^d, c_{02}-c_{03}, b_{02}^d - b_{01}^d, b_{01}-b_{04}, b_{03}^d - b_{01}^d)\\
&+ b_{02}^d(c_{11}-c_{12}, c_{14}-c_{13}, c_{13}-c_{12}, c_{11}(b_{11}-b_{14}), c_{11}(b_{12}-b_{13}), c_{11}\delta_{n\geq 3})\\
&+\big(c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}b_{01}(1-b_{1i}^d), c_{02}c_{1i}(1-b_{1i}^d) | i, j\big).
\end{aligned}
$$

Now let $J' = \hat{J} : c_{02}b_{02}^{2d}$ and $J'' = (\hat{J} : c_{02}b_{02}^d) + (b_{02}^d)$. By Fact 1.6, the set of associated primes of $\hat{J} : c_{02}b_{02}^d$ is contained in the union of the sets of associated primes of J' and J''.

First we analyze J'':

$$
\begin{aligned}
J'' &= (c_{01}, c_{04}, c_{02}-c_{03}, s, b_{0i}^d, b_{01}-b_{04})\\
&\qquad+\big(c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}b_{01}(1-b_{1i}^d), c_{02}c_{1i}(1-b_{1i}^d) | i, j\big).
\end{aligned}
$$

This decomposes as follows:

$$
\begin{aligned}
J'' &= (C_0 + \big(s, b_{0i}^d, b_{01}-b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}b_{01}(1-b_{1i}^d)\big))\\
&\quad\cap\big(c_{01}, c_{04}, c_{02}-c_{03}, s, b_{0i}^d, b_{01}-b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}(1-b_{1i}^d)\big).
\end{aligned}
$$

Let q_2 denote the ideal in the second row. Clearly, q_2 decomposes as the intersection of $Q_{2\Lambda\alpha}$-primary components. The ideal in the first row decomposes as

$$
\begin{aligned}
&(C_0 + \big(s, b_{0i}^d, b_{01}-b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}(1-b_{1i}^d)\big))\\
&\cap(C_0 + \big(s, b_{0i}^d, b_{01}, b_{04}, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j})\big)).
\end{aligned}
$$

Let q_3 be the ideal in the second row above. Then q_3 is an intersection of the $Q_{3\Lambda}$-primary components. The ideal in the first row contains q_2, and is thus redundant for computing the associated primes of J''. Thus the set of associated primes of J'' is a subset of $\{Q_{2\Lambda\alpha}, Q_{3\Lambda}\}$. Clearly x is not in any $Q_{2\Lambda\alpha}$ and $Q_{3\Lambda}$.

It remains to compute a decomposition of J':

$$\begin{aligned} J' = (&c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d, c_{02} - c_{03}, b_{02}^d - b_{01}^d, b_{01} - b_{04}, b_{03}^d - b_{01}^d) \\ &+ (c_{11} - c_{12}, c_{14} - c_{13}, c_{13} - c_{12}, c_{11}(b_{11} - b_{14}), c_{11}(b_{12} - b_{13}), c_{11}\delta_{n\geq 3}) \\ &+ \left(c_{11}(b_{02} - b_{1i}b_{03}), c_{11}^2(b_{1i} - b_{1j}), c_{11}(1 - b_{1i}^d) | i, j\right). \end{aligned}$$

By coloning with and adding c_{11}^2:

$$\begin{aligned} J' = (&c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d) \\ &+ \left(c_{11} - c_{12}, c_{14} - c_{13}, c_{13} - c_{12}, \delta_{n\geq 3}, b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j}, 1 - b_{1i}^d\right)) \\ &\cap ((c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d, c_{02} - c_{03}, b_{02}^d - b_{01}^d, b_{01} - b_{04}, b_{03}^d - b_{01}^d) \\ &+ (c_{11} - c_{12}, c_{14} - c_{13}, c_{13} - c_{12}, c_{11}(b_{11} - b_{14}), c_{11}(b_{12} - b_{13}), c_{11}\delta_{n\geq 3}) \\ &+ \left(c_{11}(b_{02} - b_{1i}b_{03}), c_{11}^2, c_{11}(1 - b_{1i}^d)\right)) \\ = &\, p_2\delta_{n=2} \\ &\cap ((c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d, c_{02} - c_{03}, b_{02}^d - b_{01}^d, b_{01} - b_{04}, b_{03}^d - b_{01}^d) + C_1) \\ &\cap ((c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d, c_{02} - c_{03}, b_{02}^d - b_{01}^d, b_{01} - b_{04}, b_{03}^d - b_{01}^d) + D_1 \\ &+ \left(b_{11} - b_{14}, b_{12} - b_{13}, \delta_{n\geq 3}, b_{02} - b_{1i}b_{03}, c_{11}^2, 1 - b_{1i}^d\right)). \end{aligned}$$

By coloning with and adding b_{03} on the third component,

$$\begin{aligned} J' = &\, p_2\delta_{n=2} \cap p_1 \\ &\cap ((c_{01} - c_{02}b_{02}^d, c_{04} - c_{02}b_{02}^d, s - fb_{02}^d, c_{02} - c_{03}, b_{02}^d - b_{01}^d, b_{01} - b_{04}, b_{03}^d - b_{01}^d) + D_1 \\ &+ \left(\delta_{n\geq 3}, b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j}, c_{11}^2, 1 - b_{1i}^d\right)) \\ &\cap ((c_{01}, c_{04}, s, c_{02} - c_{03}, b_{01}^d, b_{01} - b_{04}, b_{02}, b_{03}, b_{11} - b_{14}, b_{12} - b_{13}, \delta_{n\geq 3}, c_{11}^2, 1 - b_{1i}^d) + D_1). \end{aligned}$$

The second to the last ideal above properly contains $p_2\delta_{n=2}$, and the last ideal $q_{4,2}$ is an intersection of $Q_{4,2\alpha\beta}$-primary components when $n = 2$. This proves that

$$J' = \hat{J} : c_{02}b_{02}^{2d} = J : c_{02}b_{02}^{2d}x = p_1 \cap p_2\delta_{n=2} \cap q_{4,2}\delta_{n=2},$$

and all $Q_{4,2\alpha\beta}$ with $\alpha \neq \beta$ are associated to J. As x is a non-zerodivisor modulo this ideal, this also finishes the proof that $\hat{J} = J : x$. Furthermore, this proves that the set of new embedded primes of J which do not contain x is contained in the set of associated primes of the ideal $\hat{J} : c_{02}b_{02}^d = J : c_{02}b_{02}^d x$, and that this latter set is a subset of

$$\{Q_{2\Lambda\alpha}, Q_{3\Lambda}, Q_{4,2\alpha\beta}\delta_{n=2}\}.$$

It remains to prove that the prime ideals $Q_{2\emptyset\alpha}$ are not associated to J, and that every element of

$$\{Q_{2\Lambda\alpha}, Q_{3\Lambda} | \Lambda \neq \emptyset, \alpha^d = 1\}$$

is associated to J. By construction, it suffices to show that the displayed prime ideals are associated to $J : c_{02}b_{02}^d x$ and that the $Q_{2\emptyset\alpha}$ are not associated to $J : c_{02}b_{02}^d x$.

Let Λ be a subset of $\{1, 2, 3, 4\}$. Let K be the ideal $J : c_{02}b_{02}^d x$ coloned with a power of the element

$$y = \prod_{\substack{i \in \Lambda \\ j \notin \Lambda}} c_{1i}(1 - b_{1j}^d)c_{02}.$$

Let α be a dth root of unity α, and Γ a subset of $\{1, 2, 3, 4\}$ with $\Gamma \neq \Lambda$. Note that $y \in Q_{2\Gamma\alpha} \setminus Q_{2\Lambda\alpha}$. Also, y is an element of each $Q_{4,2\alpha\beta}$, and of $Q_{3\Gamma}$, $Q_{3\Lambda}$. Also, if $\Lambda \neq \emptyset$, then $y \in p_1$, and if $\Lambda \neq \{1, 2, 3, 4\}$, then $y \in p_2$. If $\Lambda = \emptyset$, then $K = p_1$, which proves that $Q_{2\emptyset\alpha}$ is not associated. If $\Lambda \neq \emptyset$, then $K \neq p_2\delta_{n=2}$, which proves that the $Q_{2\Lambda\alpha}$ is associated to J.

Similarly, by coloning $J : c_{02}b_{02}^d x$ with $\prod_{\substack{i \in \Lambda \\ j, j' \notin \Lambda}} c_{1i}(b_{1j} - b_{1j'})(b_{1i} - b_{1j}) \prod_{j=1}^4 (1 - b_{1j}^d)$ we get that each $Q_{3\Lambda}$ is associated to J if and only if $\Lambda \neq \emptyset$. Thus

Theorem 3.1: *Set $x = fc_{21}b_{13}(b_{21} - b_{22})$ when $n > 2$ and $x = f$ when $n = 2$. Then the set of embedded primes of J which do not contain x is*

$$\{Q_{1\Lambda}, Q_{2\Lambda'\alpha}, Q_{3\Lambda'}, Q_{4,2\alpha\beta}\delta_{n=2} | \Lambda' \neq \emptyset, \alpha^d = \beta^d = 1, \alpha \neq \beta\},$$

and each of these primes is associated to J. ■

For clarity we record the new embedded primes in a table:

embedded prime ($\Lambda \neq \emptyset, \alpha^d = 1, \beta^d = 1, \alpha \neq \beta$)	height
$Q_{2\Lambda\alpha} = (s, c_{01}, c_{03} - c_{02}, c_{04}, b_{01}, b_{02}, b_{03}, b_{04}) + (c_{1i} \mid i \notin \Lambda) + (b_{1i} - \alpha \mid i \in \Lambda)$	12
$Q_{3\Lambda} = C_0 + (s, b_{01}, b_{02}, b_{03}, b_{04}) + (c_{1i} \mid i \notin \Lambda) + (b_{1i} - b_{1j} \mid i, j \in \Lambda)$	12
$Q_{4,2\alpha\beta} = (s, c_{01}, c_{04}, c_{02} - c_{03}, b_{01}, b_{02}, b_{03}, b_{04})$	16
$(b_{11} - \alpha, b_{14} - \alpha, b_{12} - \beta, b_{13} - \beta) + C_1$ if $n = 2$ only	

4. $(\mathbf{n-1})(\mathbf{d^3-d})$ more embedded primes, for $\mathbf{n>2}$

The embedded primes of J found so far do not contain $b_{2i}-b_{2j}$. Without this assumption there are many more embedded primes of J, and the number of these primes grows with n and d. In this section, $(n-1)(d^3-d)$ more embedded primes are found in the case when $n>2$. The main theorem of this section, Theorem 4.1, says that these primes are the only new ones not containing the element x, where x is

$$x = \begin{cases} f^3(c_{21}\cdots c_{r-1,1})b_{13}^{2d+1}(b_{23}\cdots b_{r-1,3})(1-b_{r1}), & \text{if } r<n, \\ f^3(c_{21}\cdots c_{r-1,1})b_{13}^{2d+1}(b_{23}\cdots b_{r-1,3}), & \text{if } r=n. \end{cases}$$

Throughout this section, $n>2$.

For each $r\in\{2,\ldots,n\}$ and α,β and γ in k such that $\alpha^d=\beta^d=\gamma^d=1$, define

$$\begin{aligned} Q_{4r\alpha\beta\gamma} &= (s, c_{01}, c_{04}, c_{02}-c_{03}, b_{01}, b_{02}, b_{03}, b_{04}) \\ &\quad + (b_{12}-b_{2i}b_{13}, b_{2i}-b_{2j})\delta_{r>2} + (b_{11}-\alpha, b_{14}-\alpha, b_{12}-\beta, b_{13}-\gamma) \\ &\quad + C_1 + D_2 + \cdots + D_{r-1} + C_r + B_{3,r-1}. \end{aligned}$$

It is proved in this section that these prime ideals are associated to J if and only if $\{\alpha,\beta,\gamma\}|>1$, i.e., if α, β and γ are not identical. We also prove that these $(n-1)(d^3-d)$ prime ideals are the only new associated prime ideals of J which do not contain the element x defined above.

For all $2\le r\le n$, with the convention that $c_{ni}=b_{ni}=1$, $C_n=(0)$, all these cases can be analyzed simultaneously. Consider the ideal

$$\begin{aligned} K = \big(&c_{01}-c_{02}b_{02}^d, c_{01}-c_{04}, s(c_{02}-c_{03}), c_{02}b_{01}-c_{03}b_{04}, c_{02}(s-fb_{02}^d), c_{02}b_{02}^d-c_{03}b_{03}^d\big) \\ &+ c_{02}(b_{02}^d, c_{13}b_{03}^d)(b_{02}^d-b_{01}^d, c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d-b_{01}^d) + (c_{02}c_{1i}(b_{02}-b_{1i}b_{03})) \\ &+ c_{02}b_{02}^{2d}(c_{1i}b_{1i}^d-c_{13}b_{13}^d) + c_{02}c_{13}b_{03}^{2d}\left(D_1+(b_{11}-b_{14}, 1-b_{1i}^d)\right) \\ &+ c_{02}c_{13}b_{03}^{2d}\left((c_{11}, b_{02}, b_{03})(1-b_{2i}) + (b_{12}-b_{2i}b_{13}, b_{2i}-b_{2j})\right)\delta_{r>2} \\ &+ \sum_{k=2}^{r-2} c_{02}b_{03}^{2d}c_{13}\left(D_k+(1-b_{k+1,i})\right) + c_{02}c_{13}b_{03}^{2d}\left(D_{r-1}+C_r\right). \end{aligned}$$

It is easy to see that K contains J and that x multiplies K into J, except possibly that x multiplies the element $c_{02}b_{03}^{2d}c_{11}c_{13}(1-b_{2i})$ into J:

$$\begin{aligned} c_{02}b_{03}^{2d}c_{11}c_{13}(1-b_{2i})x &\in (f^2c_{02}b_{03}^{2d}c_{11}c_{13}(1-b_{2i})c_{21}b_{13}^{2d+1}) + J \\ &= (f^2c_{02}b_{02}^{2d}c_{11}c_{13}(1-b_{2i})c_{21}b_{13}) + J \\ &= (sfc_{01}c_{11}^2(1-b_{2i})c_{2i}b_{13}) + J \\ &= (sfc_{04}c_{11}c_{12}(1-b_{2i})c_{2i}b_{13}) + J \\ &= (sfc_{04}c_{11}c_{12}(b_{13}-b_{12})c_{2i}) + J \end{aligned}$$

$$
\begin{aligned}
&= (sfc_{01}c_{11}(c_{13}b_{13} - c_{12}b_{12})c_{2i}) + J \\
&= (sfc_{02}c_{11}(c_{13}b_{13} - c_{12}b_{12})b_{03}^{d}c_{2i}) + J \\
&= (sfc_{02}c_{11}(c_{13} - c_{12})b_{02}b_{03}^{d-1}c_{2i}) + J \\
&= (sfc_{02}c_{11}b_{11}(c_{13} - c_{12})b_{03}^{d}c_{2i}) + J \\
&= (sfc_{01}c_{11}b_{11}(c_{13} - c_{12})c_{2i}) + J = J.
\end{aligned}
$$

The intermediate goal in this section is to find a primary decomposition of K. It turns out that x is a non-zerodivisor on K, which proves that $K = J : x$, and thus determines all associated primes of J which do not contain x.

By Fact 1.6, $\operatorname{Ass}\left(\frac{R}{K}\right) \subseteq \operatorname{Ass}\left(\frac{R}{K:c_{02}}\right) \cup \operatorname{Ass}\left(\frac{R}{K+(c_{02})}\right)$. The second set is easy: $K+(c_{02})$ equals $(c_{01}, c_{02}, c_{04}, sc_{03}, c_{03}b_{04}, c_{03}b_{03}^{d}) = p_0 \cap p_{-2}$, which is an intersection of some minimal components of J (none of which contain x), so it suffices to find the associated primes of $K : c_{02}$. By Facts 1.3 and 1.4:

$$
\begin{aligned}
K : c_{02} = \big(&c_{01} - c_{02}b_{02}^{d}, c_{01} - c_{04}, s - fb_{02}^{d}\big) \\
&+ (b_{02}^{d}, c_{13}b_{03}^{d})(b_{02}^{d} - b_{01}^{d}, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^{d} - b_{01}^{d}) + (c_{1i}(b_{02} - b_{1i}b_{03})) \\
&+ b_{02}^{2d}(c_{1i}b_{1i}^{d} - c_{13}b_{13}^{d}) + c_{13}b_{03}^{2d}\left(D_1 + (b_{11} - b_{14}, 1 - b_{1i}^{d})\right) \\
&+ c_{13}b_{03}^{2d}\left((c_{11}, b_{02}, b_{03})(1 - b_{2i}) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j})\right)\delta_{r>2} \\
&+ \sum_{k=2}^{r-2} b_{03}^{2d}c_{13}\left(D_k + (1 - b_{k+1,i})\right) + c_{13}b_{03}^{2d}\left(D_{r-1} + C_r\right) \\
&+ (s(c_{02} - c_{03}), c_{02}b_{01} - c_{03}b_{04}, c_{02}b_{02}^{d} - c_{03}b_{03}^{d}) : c_{02}.
\end{aligned}
$$

The latter colon ideal equals

$$
(c_{02}b_{02}^{d} - c_{03}b_{03}^{d}, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^{d} - b_{04}b_{02}^{d}) + s(c_{02} - c_{03}, b_{01} - b_{04}, b_{02}^{d} - b_{03}^{d}),
$$

so that

$$
\begin{aligned}
K : c_{02} = \big(&c_{01} - c_{02}b_{02}^{d}, c_{01} - c_{04}, s - fb_{02}^{d}\big) \\
&+ (b_{02}^{d}, c_{13}b_{03}^{d})(b_{02}^{d} - b_{01}^{d}, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^{d} - b_{01}^{d}) + (c_{1i}(b_{02} - b_{1i}b_{03})) \\
&+ b_{02}^{2d}(c_{1i}b_{1i}^{d} - c_{13}b_{13}^{d}) + c_{13}b_{03}^{2d}\left(D_1 + (b_{11} - b_{14}, 1 - b_{1i}^{d})\right) \\
&+ c_{13}b_{03}^{2d}\left((c_{11}, b_{02}, b_{03})(1 - b_{2i}) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j})\right)\delta_{r>2} \\
&+ \sum_{k=2}^{r-2} b_{03}^{2d}c_{13}\left(D_k + (1 - b_{k+1,i})\right) + c_{13}b_{03}^{2d}\left(D_{r-1} + C_r\right) \\
&+ (c_{02}b_{02}^{d} - c_{03}b_{03}^{d}, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^{d} - b_{04}b_{02}^{d}).
\end{aligned}
$$

Again by Fact 1.6, $\operatorname{Ass}\left(\frac{R}{K:c_{02}}\right) \subseteq \operatorname{Ass}\left(\frac{R}{K:c_{02}b_{03}^d}\right) \cup \operatorname{Ass}\left(\frac{R}{(K:c_{02})+(b_{03}^d)}\right)$. Note that

$$\begin{aligned}(K : c_{02}) + (b_{03}^d) = (c_{01} - c_{02}b_{02}^d, c_{01} - c_{04}, s - fb_{02}^d, b_{03}^d) + (c_{1i}(b_{02} - b_{1i}b_{03}))\\ + b_{02}^d(b_{01}^d, b_{02}^d, c_{02} - c_{03}, b_{01} - b_{04}) + (c_{02}b_{02}^d, c_{02}b_{01} - c_{03}b_{04}, b_{04}b_{02}^d).\end{aligned}$$

By Fact 1.6, $\operatorname{Ass}\left((K : c_{02}) + (b_{03}^d)\right) \subseteq \operatorname{Ass}\left(\frac{R}{((K:c_{02})+(b_{03}^d)):b_{02}^d}\right) \cup \operatorname{Ass}\left(\frac{R}{(K:c_{02})+(b_{03}^d,b_{02}^d)}\right)$. Then

$$\begin{aligned}&(K : c_{02}) + (b_{03}^d, b_{02}^d) = (s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{02}b_{01} - c_{03}b_{04})\\ &= (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01} - c_{03}b_{04})\\ &\quad \cap (s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{02}b_{01} - c_{03}b_{04}).\end{aligned}$$

The first component is P_{-3}, and the second component is the intersection of ideals primary to the $Q_{1\Lambda}$, as Λ varies over the subsets of $\{1, 2, 3, 4\}$. None of these prime ideals contains x.

Next, $((K : c_{02}) + (b_{03}^d)) : b_{02}^d$ equals

$$\begin{aligned}&= C_0 + (s, b_{01}, b_{02}^d, b_{03}^d, b_{04}) + (b_{03}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{02}b_{01} - c_{03}b_{04}) : b_{02}^d\\ &= C_0 + (s, b_{01}, b_{02}^d, b_{03}^d, b_{04}) + C_1,\end{aligned}$$

and again x is a non-zerodivisor modulo this ideal, and the associated prime of this ideal ($Q_{3\emptyset}$) is not associated to J by Theorem 3.1.

This finishes the analysis of the associated primes of $(K : c_{02}) + (b_{03}^d)$. It remains to analyze $K : c_{02}b_{03}^d$. This colon ideal is

$$\begin{aligned}K : c_{02}b_{03}^d &= \left(c_{01} - c_{02}b_{02}^d, c_{01} - c_{04}, s - fb_{02}^d\right)\\ &+ c_{13}(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d)\\ &+ b_{02}^d(c_{1i}b_{1i}^d - c_{13}b_{13}^d) + c_{13}b_{03}^d\left(D_1 + (b_{11} - b_{14}, 1 - b_{1i}^d)\right)\\ &+ c_{13}b_{03}^d\left((c_{11}, b_{02}, b_{03})(1 - b_{2i}) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j})\right)\delta_{r>2}\\ &+ \sum_{k=2}^{r-2} b_{03}^d c_{13}\left(D_k + (1 - b_{k+1,i})\right) + c_{13}b_{03}^d\left(D_{r-1} + C_r\right) + \left(L : b_{03}^d\right),\end{aligned}$$

where

$$\begin{aligned}L = (c_{1i}(b_{02} - b_{1i}b_{03})) + b_{02}^d(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d)\\ + (c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d).\end{aligned}$$

The next two pages will compute $L : b_{03}^d$. First of all, coloning with b_{02}^d gives:

$$\begin{aligned}L : b_{02}^d = (b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d)\\ + \left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d\right) : b_{02}^d,\end{aligned}$$

which by a computation on page 8 equals

$$\begin{aligned} L : b_{02}^d = (&b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d) \\ &+ (c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), b_{01}c_{1i}(1 - b_{1i}^d), c_{02}c_{1i}(1 - b_{1i}^d)), \end{aligned}$$

so that $L : b_{02}^d b_{01}$ equals

$$(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(1 - b_{1i}^d)).$$

Note that neither b_{01} nor b_{02} is a zero-divisor modulo $L : b_{02}^d b_{01}$, so that by Fact 1.5,

$$\begin{aligned} L &= (L : b_{02}^d b_{01}) \cap (L + (b_{02}^d b_{01})) \\ &= \Big(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(1 - b_{1i}^d)\Big) \\ &\quad \cap \Big((c_{1i}(b_{02} - b_{1i}b_{03})) + b_{02}^d(b_{01}, b_{02}^d, b_{03}^d, c_{02} - c_{03}, b_{04}) \\ &\qquad + (c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d)\Big) \\ &= \Big(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(1 - b_{1i}^d)\Big) \\ &\quad \cap \Big(C_1 + b_{02}^d(b_{01}, b_{02}^d, b_{03}^d, c_{02} - c_{03}, b_{04}) + (c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d)\Big) \\ &\quad \bigcap_{\Lambda \neq \emptyset} \Big((c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d) + (c_{1i} | i \notin \Lambda) \\ &\qquad + (b_{02} - b_{1i}b_{03}, b_{1i}^d b_{03}^d(b_{01}, b_{03}^d, c_{02} - c_{03}, b_{04}), b_{03}^d(c_{02}b_{1i}^d - c_{03}) | i \in \Lambda)\Big). \end{aligned}$$

This is still part of the effort to compute $L : b_{03}^d$. Coloning the second component above with b_{03}^d equals

$$C_1 + (b_{01}, b_{02}^d) + \Big(b_{02}^d(b_{01}, b_{02}^d, c_{02} - c_{03}, b_{04}) + (c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04})\Big) : b_{03}^d.$$

But

$$\begin{aligned} &b_{02}^d(b_{01}, b_{02}^d, c_{02} - c_{03}, b_{04}) + (c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}) \\ &\quad = \Big(b_{02}^{2d}, b_{01}, c_{02} - c_{03}, b_{04}, c_{02}b_{02}^d - c_{03}b_{03}^d\Big) \cap \Big(b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}\Big), \end{aligned}$$

so that

$$\begin{aligned} &C_1 + (b_{01}, b_{02}^d) + \Big(b_{02}^d(b_{01}, b_{02}^d, c_{02} - c_{03}, b_{04}) + (c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04})\Big) : b_{03}^d = \\ &= C_1 + (b_{01}, b_{02}^d) + \Big(b_{02}^{2d}, b_{01}, c_{02} - c_{03}, b_{04}, c_{02}b_{02}^d, c_{02}b_{03}^d\Big) \cap \Big(b_{02}^d, c_{03}, c_{02}b_{01}\Big) \\ &= C_1 + (b_{01}, b_{02}^d, c_{03}b_{03}^d, (c_{02} - c_{03})c_{03}, b_{04}c_{03}), \end{aligned}$$

so that finally $L : b_{03}^d$ equals

$$\Big(b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(1 - b_{1i}^d)\Big)$$

$$\cap \Big(C_1 + (b_{01}, b_{02}^d, c_{03}b_{03}^d, (c_{02}-c_{03})c_{03}, b_{04}c_{03})\Big)$$
$$\bigcap_{\Lambda\neq\emptyset}\Big((b_{01}, c_{03}b_{04}) + (c_{1i}|i \notin \Lambda)$$
$$+ (b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j}, b_{1i}^d(b_{03}^d, c_{02}-c_{03}, b_{04}), c_{02}b_{1i}^d - c_{03}|i,j\in\Lambda)\Big)$$
$$= \Big(b_{02}^d - b_{01}^d, c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d - b_{01}^d, c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}(1-b_{1i}^d)\Big)$$
$$\cap \Big((b_{01}, b_{02}^d, c_{03}b_{03}^d, (c_{02}-c_{03})c_{03}, b_{04}c_{03})$$
$$+ (c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}b_{1i}^d(c_{02}-c_{03}, b_{04}), c_{1i}(c_{02}b_{1i}^d - c_{03})\Big)$$
$$= (b_{02}^d - b_{01}^d, c_{03}(b_{03}^d - b_{01}^d), (c_{02}-c_{03})c_{03}, b_{04}c_{03} - b_{01}c_{02})$$
$$+ (c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}b_{1i}^d(c_{02}-c_{03}, b_{04}-b_{01}), c_{1i}(c_{02}b_{1i}^d - c_{03}))$$
$$+ b_{01}(c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d - b_{01}^d, c_{1i}(1-b_{1i}^d)).$$

Thus finally

$$K : c_{02}b_{03}^d = \big(c_{01} - c_{02}b_{02}^d, c_{01}-c_{04}, s - fb_{02}^d\big) + c_{13}(c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d - b_{01}^d)$$
$$+ b_{02}^d(c_{1i}b_{1i}^d - c_{13}b_{13}^d) + c_{13}b_{03}^d\left(D_1 + (b_{11}-b_{14}, 1-b_{1i}^d)\right)$$
$$+ c_{13}b_{03}^d\left((c_{11}, b_{02}, b_{03})(1-b_{2i}) + (b_{12} - b_{2i}b_{13}, b_{2i}-b_{2j})\right)\delta_{r>2}$$
$$+ \sum_{k=2}^{r-2} b_{03}^d c_{13}\left(D_k + (1-b_{k+1,i})\right) + c_{13}b_{03}^d\left(D_{r-1} + C_r\right)$$
$$+ (b_{02}^d - b_{01}^d, c_{03}(b_{03}^d - b_{01}^d), (c_{02}-c_{03})c_{03}, b_{04}c_{03} - b_{01}c_{02})$$
$$+ (c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}b_{1i}^d(c_{02}-c_{03}, b_{04}-b_{01}), c_{1i}(c_{02}b_{1i}^d - c_{03}))$$
$$+ b_{01}(c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d - b_{01}^d, c_{1i}(1-b_{1i}^d)).$$

By Fact 1.6, $\operatorname{Ass}\left(\frac{R}{K:c_{02}b_{03}^d}\right) \subseteq \operatorname{Ass}\left(\frac{R}{K:c_{02}b_{03}^d c_{13}}\right) \cup \operatorname{Ass}\left(\frac{R}{(K:c_{02}b_{03}^d)+(c_{13})}\right)$. Note that

$$(K : c_{02}b_{03}^d) + (c_{13}) = (c_{01} - c_{02}b_{02}^d, c_{01}-c_{04}, s - fb_{02}^d, c_{13})$$
$$+ b_{02}^d(c_{1i}b_{1i}^d) + (b_{02}^d - b_{01}^d, c_{03}(b_{03}^d - b_{01}^d), (c_{02}-c_{03})c_{03}, b_{04}c_{03} - b_{01}c_{02})$$
$$+ (c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i}-b_{1j}), c_{1i}b_{1i}^d(c_{02}-c_{03}, b_{04}-b_{01}), c_{1i}(c_{02}b_{1i}^d - c_{03}))$$
$$+ b_{01}(c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d - b_{01}^d, c_{1i}(1-b_{1i}^d)).$$

No b_{13}, c_{2i} or b_{2i} appear in a minimal generating set of this ideal, so that by Theorem 3.1, $(K : c_{02}b_{03}^d)+(c_{13})$ gives no new embedded primes of J. Furthermore, x is a non-zerodivisor on all of these. Thus it remains to analyze the associated primes of $K : c_{02}b_{03}^d c_{13}$:

$$K : c_{02}b_{03}^d c_{13} = \big(c_{01} - c_{02}b_{02}^d, c_{01}-c_{04}, s - fb_{02}^d, c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d - b_{01}^d\big)$$
$$+ b_{03}^d\left(D_1 + (b_{11}-b_{14}, 1-b_{1i}^d)\right)$$

$$
\begin{aligned}
&+ b_{03}^d \left((c_{11}, b_{02}, b_{03})(1-b_{2i}) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j})\right) \delta_{r>2} \\
&+ \sum_{k=2}^{r-2} b_{03}^d \left(D_k + (1 - b_{k+1,i})\right) + b_{03}^d \left(D_{r-1} + C_r\right) \\
&+ (b_{02} - b_{13}b_{03}, c_{1i}(b_{1i} - b_{13}), b_{13}^d(c_{02} - c_{03}, b_{04} - b_{01}), c_{02}b_{13}^d - c_{03}, b_{01}(1 - b_{13}^d)) \\
&+ (b_{02}^d(c_{13}b_{13}^d - c_{1i}b_{1i}^d | i \neq 3) + L') : c_{13},
\end{aligned}
$$

where

$$
\begin{aligned}
L' = &(b_{02}^d - b_{01}^d, c_{03}(b_{03}^d - b_{01}^d), (c_{02} - c_{03})c_{03}, b_{04}c_{03} - b_{01}c_{02}, c_{1i}c_{1j}(b_{1i} - b_{1j}) | i, j \neq 3) \\
&+ (c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d(c_{02} - c_{03}, b_{04} - b_{01}), c_{1i}(c_{02}b_{1i}^d - c_{03}) | i \neq 3) \\
&+ b_{01}(c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d, c_{1i}(1 - b_{1i}^d) | i \neq 3). \\
= &(b_{02}^d - b_{01}^d, c_{03}(b_{03}^d - b_{01}^d), (c_{02} - c_{03})c_{03}, (b_{04} - b_{01})c_{03}, c_{1i}c_{1j}(b_{1i} - b_{1j}) | i, j \neq 3) \\
&+ (c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d(c_{02} - c_{03}, b_{04} - b_{01}), c_{1i}c_{03}(b_{1i}^d - 1) | i \neq 3) \\
&+ b_{01}(c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d, c_{1i}(1 - b_{1i}^d) | i \neq 3).
\end{aligned}
$$

Clearly $(b_{02}^d(c_{13}b_{13}^d - c_{1i}b_{1i}^d | i \neq 3) + L') : c_{13}$ contains

$$
\begin{aligned}
L'' = &\, b_{02}^d(c_{13}b_{13}^d - c_{1i}b_{1i}^d) + b_{02}^d b_{13}^d(c_{1i} - c_{1j}, b_{02} - b_{1i}b_{03}, 1 - b_{1i}^d, c_{1i}(b_{1i} - b_{13}) | i, j \neq 3) \\
&+ (b_{02}^d - b_{01}^d, c_{03}(b_{03}^d - b_{01}^d), (c_{02} - c_{03})c_{03}, (b_{04} - b_{01})c_{03}) \\
&+ (c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}b_{1i}^d(c_{02} - c_{03}, b_{04} - b_{01}), c_{1i}c_{03}(b_{1i}^d - 1) | i, j \neq 3) \\
&+ b_{01}(c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d, c_{1i}(1 - b_{1i}^d) | i \neq 3).
\end{aligned}
$$

It turns out that $L'' = L' : c_{13}$, as the proof below shows.

Let $y \in (b_{02}^d(c_{13}b_{13}^d - c_{1i}b_{1i}^d) | i \neq 3) + L') : c_{13}$. Write

$$
yc_{13} = \sum_{i \neq 3} y_i b_{02}^d(c_{13}b_{13}^d - c_{1i}b_{1i}^d) + l,
$$

for some y_i in the ring and $l \in L'$. Then $y_1 b_{02}^d c_{11} b_{11}^d \in L' + (c_{12}, c_{13}, c_{14})$, so that without loss of generality

$$
y_1 \in (c_{12}, c_{14}, b_{02}^d - b_{01}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d, b_{02} - b_{11}b_{03}, b_{11}^d - 1).
$$

Thus $y_1 b_{02}^d(c_{13}b_{13}^d - c_{11}b_{11}^d)$ is contained in

$$
\begin{aligned}
&b_{02}^d(c_{13}b_{13}^d - c_{11}b_{11}^d)\left(c_{1j}, c_{02} - c_{03}, b_{01} - b_{04}, b_{03}^d - b_{01}^d, b_{02} - b_{11}b_{03}, b_{11}^d - 1, | j \neq 1, 3\right) \\
&\subseteq L' + b_{02}^d(c_{13}b_{13}^d - c_{11}b_{11}^d)\left(c_{1j}, b_{02} - b_{11}b_{03}, b_{11}^d - 1 | j \neq 1, 3\right) \\
&\subseteq L' + b_{02}^d(c_{13}b_{13}^d - c_{11}b_{11}^d)(c_{1j} | j \neq 1, 3) + b_{02}^d c_{13} b_{13}^d(b_{02} - b_{11}b_{03}, b_{11}^d - 1) \\
&\subseteq L' + b_{02}^d c_{11}(c_{13}b_{13}^d - c_{1j}b_{1j}^d)) + b_{02}^d c_{13} b_{13}^d(c_{1j} - c_{11}, b_{02} - b_{11}b_{03}, b_{11}^d - 1 | j \neq 1, 3).
\end{aligned}
$$

Thus for some $y' \in b_{02}^d c_{13} b_{13}^d(c_{1j} - c_{11}, b_{02} - b_{11}b_{03}, b_{11}^d - 1 | j \neq 1, 3) \subseteq L''$ and some y_2', y_4'

in the ring,

$$(y-y')c_{13}-\sum_{i=2,4} y'_i b_{02}^d(c_{13}b_{13}^d - c_{1i}b_{1i}^d) \in L'.$$

Then $y'_2 b_{02}^d c_{12} b_{12}^d$ is in $L' + (c_{13}, c_{14})$, so that

$$\begin{aligned} y'_2 c_{12} b_{12}^d \in{} & (c_{13}, c_{14}, b_{02}^d - b_{01}^d, c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d - b_{01}^d) \\ & + (c_{1i}(1-b_{1i}^d), c_{1i}(b_{02}-b_{1i}b_{03}), c_{11}c_{12}(b_{11}-b_{12}) | i = 1,2), \end{aligned}$$

whence

$$y'_2 \in (c_{13}, c_{14}, c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d - b_{01}^d, 1-b_{12}^d, b_{02}-b_{12}b_{03}, c_{11}(b_{11}-b_{12})).$$

By reasoning similar to the one for y_1, there exists $y'' \in L''$ and y''_4 in the ring such that $(y-y'-y'')c_{13} - y''_4 b_{02}^d(c_{13}b_{13}^d - c_{14}b_{14}^d) \in L'$. Then $y''_4 b_{02}^d c_{14} b_{14}^d \in L' + (c_{13})$, and again one can conclude that $y''_4 \in L'' + (c_{13})$. Thus y is an element of L' modulo L'', so that $L'' = L' : c_{13}$. Thus finally

$$\begin{aligned} K : c_{02}b_{03}^d c_{13} = {} & \left(c_{01} - c_{02}b_{02}^d, c_{01}-c_{04}, s - f b_{02}^d, c_{02}-c_{03}, b_{01}-b_{04}, b_{03}^d - b_{01}^d\right) \\ & + b_{03}^d \left(D_1 + (b_{11}-b_{14}, 1-b_{1i}^d)\right) \\ & + b_{03}^d \left((c_{11}, b_{02}, b_{03})(1-b_{2i}) + (b_{12}-b_{2i}b_{13}, b_{2i}-b_{2j})\right) \delta_{r>2} \\ & + \sum_{k=2}^{r-2} b_{03}^d \left(D_k + (1-b_{k+1,i})\right) + b_{03}^d \left(D_{r-1} + C_r\right) \\ & + (b_{02}-b_{13}b_{03}, c_{1i}(b_{1i}-b_{13}), b_{02}^d b_{13}^d (b_{1i}-b_{13})b_{03}, b_{02}^d - b_{01}^d) \\ & + (c_{02}, b_{01})(1-b_{13}^d, c_{1i}c_{02}(1-b_{1i}^d) | i \neq 3). \end{aligned}$$

By Fact 1.6, $\mathrm{Ass}\left(\frac{R}{K:c_{02}b_{03}^d c_{13}}\right) \subseteq \mathrm{Ass}\left(\frac{R}{K:c_{02}b_{03}^{2d} c_{13}}\right) \cup \mathrm{Ass}\left(\frac{R}{(K:c_{02}b_{03}^d c_{13}) + (b_{03}^d)}\right)$. The latter ideal equals and decomposes as:

$$\begin{aligned} (K : c_{02}b_{03}^d c_{13}) + (b_{03}^d) = {} & (s, c_{01}, c_{04}, c_{02}-c_{03}, b_{01}-b_{04}, b_{01}^d, b_{02}-b_{13}b_{03}, b_{03}^d) \\ & + (c_{1i}(b_{1i}-b_{13}), c_{02}(b_{13}^d - 1), b_{01}(1-b_{13}^d), c_{02}c_{1i}(b_{1i}^d - 1)), b_{01}(c_{1i}(1-b_{1i}^d)) \\ = {} & \Big(C_0 + (s, b_{01}, b_{04}, b_{02}-b_{13}b_{03}, b_{03}^d, c_{1i}(b_{1i}-b_{13}))\Big) \\ & \cap \Big((s, c_{01}, c_{04}, c_{02}-c_{03}, b_{01}-b_{04}, b_{01}^d, b_{02}-b_{13}b_{03}, b_{03}^d) \\ & + (c_{1i}(b_{1i}-b_{13}), b_{13}^d - 1, c_{1i}(b_{1i}^d - 1))\Big), \end{aligned}$$

which is an intersection of $Q_{3\Lambda}$- and $Q_{2\Lambda\alpha}$-primary components, where Λ varies over all subsets of $\{1,2,3,4\}$ for which $3 \in \Lambda$. These do not give any new embedded primes of J, and furthermore none of these primes contains x.

It remains to analyze the associated primes of $K : c_{02}b_{03}^{2d}c_{13}$:

$$\begin{aligned}
K : c_{02}b_{03}^{2d}c_{13} &= \big(c_{01} - c_{02}b_{02}^d, c_{01} - c_{04}, s - fb_{02}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{02} - b_{13}b_{03}\big) \\
&\quad + D_1 + (b_{11} - b_{14}, 1 - b_{1i}^d) + ((c_{11}, b_{02}, b_{03})(1 - b_{2i}) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j}))\,\delta_{r>2} \\
&\quad + \sum_{k=2}^{r-2} (D_k + (1 - b_{k+1,i})) + D_{r-1} + C_r + (b_{13}^d(b_{1i} - b_{13})b_{03}) \\
&\quad + \big((b_{03}^d - b_{01}^d, c_{1i}(b_{1i} - b_{13})) + (c_{03}, b_{01})(c_{1i}(b_{1i}^d - 1), 1 - b_{13}^d)\big) : b_{03}^d \\
&= \big(c_{01} - c_{02}b_{02}^d, c_{01} - c_{04}, s - fb_{02}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{02} - b_{13}b_{03}\big) \\
&\quad + D_1 + (b_{11} - b_{14}, 1 - b_{1i}^d) + ((c_{11}, b_{02}, b_{03})(1 - b_{2i}) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j}))\,\delta_{r>2} \\
&\quad + \sum_{k=2}^{r-2} (D_k + (1 - b_{k+1,i})) + D_{r-1} + C_r + (b_{1i} - b_{13})(b_{03}, c_{1i}) + \big(b_{03}^d - b_{01}^d\big)\,.
\end{aligned}$$

Note that this decomposes as

$$\begin{aligned}
&\Big((c_{01} - c_{02}b_{02}^d, c_{01} - c_{04}, s - fb_{02}^d, c_{02} - c_{03}, b_{01} - b_{04}, b_{02} - b_{13}b_{03}) + D_1 \\
&\quad + (1 - b_{1i}^d, (1 - b_{2i})\delta_{r>2}, b_{13} - b_{1i}, b_{03}^d - b_{01}^d) + \sum_{k=2}^{r-2} (D_k + (1 - b_{k+1,i})) + D_{r-1} + C_r\Big) \\
&\cap \Big((s, c_{01}, c_{04}, c_{02} - c_{03}, b_{01} - b_{04}, b_{02}, b_{03}, b_{01}^d) + C_1 + (b_{11} - b_{14}, 1 - b_{1i}^d) \\
&\quad + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j})\delta_{r>2} + \sum_{k=2}^{r-2} (D_k + (1 - b_{k+1,i})) + D_{r-1} + C_r\Big).
\end{aligned}$$

(The key to this decomposition is the fact that (c_{11}, b_{03}) intersected with the first component is contained in $K : c_{02}b_{03}^{2d}c_{13}$.) The first component above is p_r, so all of its associated primes are minimal over J. It is easy to read off the associated primes of the last component as well. First note that none of these primes contain x, which finishes the proof that x is a non-zerodivisor modulo K. Thus as $J \subseteq K$ and $xK \subseteq J$, it follows that K equals $J : x$.

It remains to determine the associated prime ideals of the last component of $K : c_{02}b_{03}^{2d}c_{13}$ in the display above. The last component is the intersection of $Q_{4r\alpha\beta\gamma}$-primary components, as α, β, and γ vary over all dth roots of unity. Note that $Q_{4,r\alpha\alpha\alpha}$-component contains p_r and is thus redundant in the decomposition. But coloning with $b_{13} - b_{1i}$ for various i shows that the remaining prime ideals are indeed associated to K and thus to J.

This proves

Theorem 4.1: *Let $n > 2$. For $r \in \{2, \ldots, n-2\}$, set $x = f(c_{21} \cdots c_{r-1,1})$ $(b_{13} \cdots b_{r-1,3})$ $c_{r+1,1}(1 - b_{r1})$, and for $r = n-1, n$, set $x = f(c_{21} \cdots c_{r-1,1})(b_{13} \cdots b_{r-1,3})$. Then the set* *of embedded primes of J not containing x is contained in*

$$\{Q_{1\Lambda}, Q_{2\Lambda'\alpha}, Q_{3\Lambda'} | \Lambda, \Lambda' \subseteq \{1,2,3,4\}, |\Lambda'| > 0\}$$
$$\cup \{Q_{4,2\alpha\beta}\delta_{n=2}, Q_{4r\alpha\beta\gamma}\delta_{n>2} | r = 2, \ldots, n; \alpha^d = \beta^d = \gamma^d = 1, |\{\alpha,\beta,\gamma\}| > 1\},$$

and each listed prime ideal is associated to J.

These new associated primes are also recorded in a table:

embedded prime ($\alpha^d = \beta^d = \gamma^d = 1$, α, β, γ not all equal) $n > 2$, $r = 2, \ldots, n$	height
$Q_{4r\alpha\beta\gamma} = (s, c_{01}, c_{03} - c_{02}, c_{04}, b_{01}, b_{02}, b_{03}, b_{04})$ $+(b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j})\delta_{r>2} + (b_{11} - \alpha, b_{14} - \alpha, b_{12} - \beta, b_{13} - \gamma)$ $+C_1 + D_2 + \cdots + D_{r-1} + C_r + B_{3,r-1}$	$7r + 2 + 4\delta_{r<n}$

5. Reduction to $(\mathbf{J}(\mathbf{n}, \mathbf{d}) : \mathbf{s}\mathbf{c}_{\mathbf{02}}) + (\mathbf{c}_{\mathbf{02}}, \mathbf{f})$

In this section the finding of the embedded primes of J gets reduced to that of finding the associated primes of certain ideals on which recursion can be applied. The main methods are repeated applications of Facts 1.5 and 1.6. For example, the set of associated primes of J is contained in $\mathrm{Ass}\left(\frac{R}{J+(s)}\right) \cup \mathrm{Ass}\left(\frac{R}{J:s}\right)$.

To start off, the decomposition of $J + (s)$ is easy:

$$\begin{aligned}
J + (s) &= (s) + f\left(c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03})\right) \\
&= (s, f) \cap \left(s, c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03})\right) \\
&= p_{-1} \cap \left(s, c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03}), c_{02}\right) \\
&\qquad \cap \left(\left(s, c_{01}, c_{04}, c_{02}b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03})\right) : c_{02}\right) \\
&= p_{-1} \cap \left(s, c_{01}, c_{04}, c_{02}, c_{03}b_{03}^d, c_{03}b_{04}\right) \\
&\qquad \cap \left(s, c_{01}, c_{04}, b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d, c_{1i}(b_{02} - b_{1i}b_{03})\right) \\
&= p_{-1} \cap (s, c_{01}, c_{04}, c_{02}, c_{03}) \cap \left(s, c_{01}, c_{04}, c_{02}, b_{03}^d, b_{04}\right) \\
&\qquad \cap \left(\left(s, c_{01}, c_{04}, b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d, c_{1i}(b_{02} - b_{1i}b_{03})\right) : b_{03}^d\right) \\
&\qquad \cap \left(s, c_{01}, c_{04}, b_{02}^d, c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d, c_{1i}(b_{02} - b_{1i}b_{03}), b_{03}^d\right) \\
&= p_{-1} \cap (s, c_{01}, c_{04}, c_{02}, c_{03}) \cap p_{-2} \\
&\qquad \cap \left(s, c_{01}, c_{04}, b_{02}^d, c_{03}, b_{01}, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}b_{1i}^d\right) \\
&\qquad \cap \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}(b_{02} - b_{1i}b_{03})\right)
\end{aligned}$$

$$
\begin{aligned}
&= p_{-1} \cap (s, c_{01}, c_{04}, c_{02}, c_{03}) \cap p_{-2} \cap p_{-4} \\
&\qquad \cap \left(s, c_{01}, c_{04}, b_{02}^d, b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j})\right) \\
&\qquad \cap (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01} - c_{03}b_{04}) \\
&= p_{-1} \cap (s, c_{01}, c_{04}, c_{02}, c_{03}) \cap p_{-2} \cap p_{-4} \cap q_1 \cap p_{-3}.
\end{aligned}
$$

Recall that p_{-1}, p_{-2} and p_{-3} are minimal components of J, that p_{-4} and q_1 are the intersections of 16 components of J each, but that $(s, c_{01}, c_{04}, c_{02}, c_{03})$ is not associated to J as it is not in the list in Theorem 3.1 and not on the list of minimal primes on page 4.

This proves (by Fact 1.6):

Theorem 5.1: *The set of embedded primes of J is contained in $\{Q_{1\Lambda} | \Lambda\} \cup Ass\left(\frac{R}{J:s}\right)$.* ∎

The next task is to compute $J : s$ and to analyze its associated primes. Any associated prime of $J : s$ is also associated to J. Computing $J : s$ is straightforward (next Theorem), but analyzing its associated primes takes many steps and the rest of this paper.

Theorem 5.2: *Let J_2 be the ideal in R generated by all the h_{rj}/s, $r \geq 2$. (Note that all these h_{rj} are multiples of s.) Then $J : s$ equals*

$$
\begin{aligned}
J : s = &\left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d\right) + J_2 + c_{02}(fb_{01}^d - s) \\
&+ (fc_{02}, c_{02}^2)\left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) \\
&+ c_{02}\left(b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03})\right) \\
&+ c_{02}\left(c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{02}c_{1i}(1 - b_{1i}^d)\right),
\end{aligned}
$$

where the indices i and j vary from 1 to 4.

Proof: First observe that

$$
J = s\left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d\right) + sJ_2 + fK + (fc_{01} - sc_{02}),
$$

where $K = \left(c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d, c_{01} - c_{04}, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03})\right)$. Thus $J : s = \left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d\right) + J_2 + (fK + (fc_{01} - sc_{02})) : s$. Let $x \in (fK + (fc_{01} - sc_{02})) : s$. Write $xs = kf + a(fc_{01} - sc_{02})$ for some $k \in K$ and $a \in R$. By adding to x a multiple of $fc_{01} - sc_{02}$ and an element of fK, and correspondingly changing a and k, without loss of generality no s appears in a, and as $fK \cap (s) = sfK$, without loss of generality also no s appears in k. From $xs = kf + a(fc_{01} - sc_{02})$ it follows that

$$
a \in (K + (s)) : fc_{01} = (s) + (K : c_{01}),
$$

and as no s appears in a and the generators of K, actually $a \in K : c_{01}$. By Fact 1.4, $K : c_{01} = K : c_{02}b_{02}^d = (c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d) + (K' : c_{02}b_{02}^d)$, where

$$
K' = \left(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03})\right).
$$

Then

$$K' : c_{02} = \left(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}\right) : c_{02} + (c_{1i}(b_{02} - b_{1i}b_{03}))$$
$$= \left(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03})\right),$$

and by the same proof as on page 8,

$$K' : c_{02}b_{02}^d = \left(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03})\right)$$
$$+ \left(c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(c_{03} - b_{1i}^d c_{02})\right).$$

Thus

$$a \in K : c_{01} = \left(c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d\right)$$
$$+ \left(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03})\right)$$
$$+ \left(c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(c_{03} - b_{1i}^d c_{02})\right).$$

Recall that $x \in K : s$ and $sx = kf + a(fc_{01} - sc_{02})$ for some $k \in K$ and $a \in K : c_{01}$. Thus $s(x + ac_{02}) = f(k + ac_{01})$, and as no s appears in a and in k, $x + ac_{02} = 0$, so that $x \in c_{02}(K : c_{01})$. Thus

$$\begin{aligned}
J : s \subseteq\ & \left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d\right) + J_2 + (fc_{01} - sc_{02}) + fK \\
& + c_{02}\left(c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d\right) \\
& + c_{02}\left(c_{02}b_{02}^d - c_{03}b_{03}^d, c_{02}b_{01} - c_{03}b_{04}, b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03})\right) \\
& + c_{02}\left(c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{1i}(c_{03} - b_{1i}^d c_{02})\right) \\
=\ & \left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d\right) + J_2 + c_{02}(fb_{01}^d - s) \\
& + f\left(c_{01} - c_{02}b_{02}^d, c_{04} - c_{03}b_{03}^d, c_{01} - c_{04}, c_{02}b_{01} - c_{03}b_{04}, c_{02}c_{1i}(b_{02} - b_{1i}b_{03})\right) \\
& + c_{02}^2\left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{02}^d - b_{03}^d, b_{01} - b_{04}, c_{1i}(1 - b_{1i}^d)\right) \\
& + c_{02}\left(b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04})\right) \\
=\ & \left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d\right) + J_2 + c_{02}(fb_{01}^d - s) \\
& + fc_{02}\left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) \\
& + c_{02}^2\left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}, c_{1i}(1 - b_{1i}^d)\right) \\
& + c_{02}\left(b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04})\right).
\end{aligned}$$

It is easy to verify that the other inclusion also holds, which proves the theorem. ■

Incidentally, this also shows:

Proposition 5.3: *The Mayr-Meyer ideal $J(n, d)$ is not a radical ideal: the element $sc_{02}(b_{01} - b_{04})$ is in $\sqrt{J}$ but not in J.* ■

This was already proved in [S2] with the assumption that $d \geq 2$, without giving an element of the radical which is not in the ideal.

Furthermore, it is easy to see the following:

Corollary 5.4: *Let a be one of the listed generators of $J(n,d) : s$. Then $s \cdot a$ can be written as a linear combination of the generators of $J(n,d)$ with coefficients of degree at most $2d+1$. Also, $c_{02}b_{01}^d c_{11} \cdots c_{n-2,1}(c_{n-1,1} - c_{n-1,4})$ lies in $J(n,d) : s$.* ■

Let J_2' be the ideal obtained from J_2 after rewriting each c_{01} as $c_{02}b_{01}^d$, c_{03} as c_{02}, and c_{04} as $c_{02}b_{04}^d$. Note that J_2' is a multiple of c_{02} and that the theorem above also holds with J_2' in place of J_2.

Observe that $(J : s) + (c_{02}) = C_0 = p_0$, a minimal prime ideal over J. Thus by Fact 1.6:

Theorem 5.5: *The set of embedded primes of J equals $\{Q_{1\Lambda}|\Lambda\} \cup Ass\left(\frac{R}{J:sc_{02}}\right)$, which is contained in*

$$\{Q_{1\Lambda}|\Lambda\} \cup Ass\left(\frac{R}{J : sc_{02}^2}\right) \cup Ass\left(\frac{R}{(J : sc_{02}) + (c_{02})}\right).$$

Note that

$$Ass\left(\frac{R}{(J : sc_{02}) + (c_{02})}\right) \subseteq Ass\left(\frac{R}{((J : sc_{02}) + (c_{02})) : f}\right) \cup Ass\left(\frac{R}{(J : sc_{02}) + (c_{02}, f)}\right).$$

Here are all the ideals appearing in this theorem:

$$\begin{aligned}
J : sc_{02} &= \left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d\right) + J_2'/c_{02} + (fb_{01}^d - s) \\
&\quad + (f, c_{02})\left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) + \left(b_{01}b_{03}^d - b_{04}b_{02}^d\right) \\
&\quad + \left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04}), c_{02}c_{1i}(1 - b_{1i}^d)\right), \\
J : sc_{02}^2 &= \left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d\right) + J_2'/c_{02} + (fb_{01}^d - s) \\
&\quad + \left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(1 - b_{1i}^d)\right), \\
(J : sc_{02}) + (c_{02}) &= C_0 + J_2'/c_{02} + (fb_{01}^d - s) + f\left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) \\
&\quad + \left(b_{01}b_{03}^d - b_{04}b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04})\right), \\
((J : sc_{02}) + (c_{02})) : f &= C_0 + J_2'/c_{02} + \left(fb_{01}^d - s, b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) \\
&\quad + \left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04})\right), \\
(J : sc_{02}) + (c_{02}, f) &= C_0 + J_2'/c_{02} + \left(s, f, b_{01}b_{03}^d - b_{04}b_{02}^d\right) \\
&\quad + \left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04})\right).
\end{aligned}$$

Observe that $(J : sc_{02}^2) + (b_{01}^d)$ equals

$$\left(c_{01}, c_{03} - c_{02}, c_{04}, s, b_{0i}^d, b_{01} - b_{04}, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(1 - b_{1i}^d)\right),$$

and $(((J : sc_{02}) + (c_{02})) : f) + (b_{01}^d)$ equals

$$C_0 + \left(s, b_{0i}^d, b_{01} - b_{04}\right) + \left(c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(b_{01} - b_{1i}^d b_{04})\right).$$

Clearly the associated primes of these two ideals do not contain x, where $x = fc_{21}b_{13}(b_{21} - b_{22})$ when $n > 2$ and $x = f$ when $n = 2$. Thus by Theorem 3.1, these ideals do not contribute anything new to the set of embedded primes of J.

Thus by another application of Fact 1.6,

Theorem 5.6: *The set of embedded primes of J is contained in*

$$\{Q_{1\Lambda}, Q_{2\Lambda'\alpha}, Q_{3\Lambda'} | \Lambda, \Lambda' \subseteq \{1,2,3,4\}, |\Lambda'| > 0, \alpha^d = 1\}$$
$$\cup\, Ass\left(\frac{R}{J : sc_{02}^2 b_{01}^d}\right) \cup Ass\left(\frac{R}{((J : sc_{02}) + (c_{02})) : fb_{01}^d}\right) \cup Ass\left(\frac{R}{(J : sc_{02}) + (c_{02}, f)}\right). \quad \blacksquare$$

We now determine the embedded primes of J that arise from the associated prime ideals of $J : sc_{02}^2 b_{01}^d$ and $((J : sc_{02}) + (c_{02})) : fb_{01}^d$.

Define J_2'' to be the ideal

$$J_2'' = (h_{rj} | r \geq 2) \text{ with setting } s = c_{01} = c_{04} = 1.$$

This is the same as taking the ideal J_2, rewriting each c_{01} and c_{04} as $c_{02}b_{01}^d$ (whence each element is divisible by $c_{02}b_{01}^d$), and then dividing that ideal by $c_{02}b_{01}^d$. Recall that J_2' is the ideal obtained from J_2 by rewriting each c_{01} as $c_{02}b_{01}^d$ and c_{04} as $c_{02}b_{04}^d$. Then

$$J_2'' = D_1 + \sum_{r=1}^{n-1} c_{11} \cdots c_{r1} \Big(D_{r+1} + (b_{r1} - b_{r4}, c_{r+1,1}(b_{r2} - b_{r+1,i}b_{r3})) \Big),$$

using the convention that $c_{ni} = 1 = b_{ni}$ and $D_n = (0)$, and

$$J_2'/c_{02} + (b_{01} - b_{04}) = J_2'' b_{01}^d + (b_{01} - b_{04}).$$

Thus

$$\begin{aligned}
J : sc_{02}^2 b_{01}^d &= J_2'' + \left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d, s - fb_{01}^d\right) \\
&\quad + \left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}(1 - b_{1i}^d)\right) : b_{01}^d \\
&= J_2'' + \left(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d, s - fb_{01}^d\right) \\
&\quad + \left(b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right) + c_{11}\left(b_{02} - b_{1i}b_{03}, c_{11}(b_{1i} - b_{1j}), 1 - b_{1i}^d\right),
\end{aligned}$$

and

$$\begin{aligned}
((J : sc_{02}) + (c_{02})) : fb_{01}^d &= C_0 + J_2'' + (s - fb_{01}^d, b_{01} - b_{04}) \\
&\quad + \left(b_{01}^d - b_{02}^d, b_{01}^d - b_{03}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}c_{1j}(b_{1i} - b_{1j}), c_{1i}b_{01}(1 - b_{1i}^d)\right) : b_{01}^d \\
&= C_0 + J_2'' + \left(s - fb_{01}^d, b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}\right)
\end{aligned}$$

$$+ c_{11}\left(b_{02} - b_{1i}b_{03}, c_{11}(b_{1i} - b_{1j}), 1 - b_{1i}^d\right).$$

Let L be either of the two ideals above. Then L is of the form

$$L_0 + J_2'' + (s - fb_{01}^d, b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}) + c_{11}(b_{02} - b_{1i}b_{03}, c_{11}(b_{1i} - b_{1j}), 1 - b_{1i}^d),$$

where L_0 is either C_0 or $(c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d)$.

By Fact 1.6, $\mathrm{Ass}(R/L) \subseteq \mathrm{Ass}(R/(L : c_{11})) \cup \mathrm{Ass}(R/(L + (c_{11})))$. It will be proved that the only embedded prime of J in this larger union set $\mathrm{Ass}(R/(L : c_{11})) \cup \mathrm{Ass}(R/(L + (c_{11})))$ are the $Q_{4r\alpha\beta\gamma}$ or the $Q_{4,2\alpha\beta}$.

First of all,

$$L + (c_{11}) = L_0 + C_1 + (s - fb_{01}^d, b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}) = L_0 + p_1,$$

which equals the intersection of minimal components $p_{1\alpha\beta}$ if $L_0 = (c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d)$, and is not associated to J by Theorem 3.1 if $L_0 = C_0$.

Thus it remains to find the associated primes of $L : c_{11}$ in order to find the associated primes of L which are also associated to J. For this first note that $J_2'' = D_1 + c_{11}J_2'''$ for some (obvious) ideal J_2''' in R. Thus

$$\begin{aligned} L : c_{11} = L_0 + D_1 + J_2''' \\ + (s - fb_{01}^d, b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}, b_{02} - b_{1i}b_{03}, c_{11}(b_{1i} - b_{1j}), 1 - b_{1i}^d). \end{aligned}$$

Note that $L : c_{11}b_{03}$ equals

$$L_0 + D_1 + J_2''' + (s - fb_{01}^d, b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}, b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j}, 1 - b_{1i}^d),$$

which decomposes:

$$\begin{aligned} &= \bigcap_{r=2}^{n}\Big(L_0 + D_1 + \cdots + D_{r-1} + C_r + B_{r-1} \\ &\qquad + (s - fb_{01}^d, b_{01}^d - b_{02}^d, b_{04}^d - b_{03}^d, b_{01} - b_{04}, b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j}, 1 - b_{1i}^d)\Big) \\ &= \bigcap_{r=2}^{n}\Big(L_0 + p_r\Big). \end{aligned}$$

As before, when $L_0 = (c_{01} - c_{02}b_{01}^d, c_{03} - c_{02}, c_{04} - c_{02}b_{04}^d)$, the above is just the intersection of some minimal components of J, and when $L_0 = C_0$, the associated primes are of the form $C_0 + P_{r\alpha\beta}$, $r \geq 2$, whence are not associated to J by Theorems 3.1 and 4.1.

Thus it remains to find the associated primes of

$$(L : c_{11}) + (b_{03}) = L_0 + D_1 + J_2''' + (s, b_{01}^d, b_{01} - b_{04}, b_{02}, b_{03}, c_{11}(b_{1i} - b_{1j}), 1 - b_{1i}^d),$$

which similarly decomposes (first add c_{11} and colon with c_{11}) as

$$= \bigcap_{r=2}^{n} \Big(L_0 + C_1 + D_2 + \cdots D_{r-1} + C_r + B_{3,r-1} + (s, b_{01}^d, b_{01} - b_{04}, b_{02}, b_{03}, 1 - b_{1i}^d) + (b_{11} - b_{14}) + (b_{12} - b_{2i} b_{13}, b_{2i} - b_{21}) \delta_{r>2} + (b_{12} - b_{13}) \delta_{n=2} \Big)$$

$$\bigcap_{r=2}^{n} \Big(L_0 + D_1 + \cdots + D_{r-1} + C_r + B_{r-1} + (s, b_{01}^d, b_{01} - b_{04}, b_{02}, b_{03}, b_{1i} - b_{1j}, 1 - b_{1i}^d) \Big),$$

from which it is easy to read off the associated primes. By Theorems 3.1 and 4.1, only the $Q_{4r\alpha\beta\gamma}$ or the $Q_{4,2\alpha\beta}$ among these are embedded primes of J.

This proves the following:

Theorem 5.7: *The set of embedded primes of J is contained in*

$$\{Q_{1\Lambda}, Q_{2\Lambda'\alpha}, Q_{3\Lambda'} | \Lambda, \Lambda' \subseteq \{1,2,3,4\}, |\Lambda'| > 0\} \cup \{Q_{4,2\alpha\beta} \delta_{n=2} | \alpha \neq \beta\} \cup \{Q_{4r\alpha\beta\gamma} \delta_{n>2} | r = 2, \ldots, n; |\{\alpha, \beta, \gamma\}| > 1\} \cup Ass\left(\frac{R}{(J : sc_{02}) + (c_{02}, f)} \right),$$

where α, β and γ vary over dth roots of unity. The listed $31 + 15d + (d^2 - d)\delta_{n=2} + (n-1)(d^3 - d)\delta_{n>2}$ prime ideals are indeed associated to $J = J(n,d)$. ■

6. More ideals with the doubly exponential ideal membership property

In this section we define a new family of ideals with the doubly exponential ideal membership property. This family is intricately connected to $(J(n,d) : sc_{02}) + (c_{02}, f)$, and is thus by the previous section a key family in finding the associated primes of the Mayr-Meyer ideals. The advantage of the new family is that a recursive procedure can be used to find its associated primes.

For any integers $n \geq 2, d$, let $K(n,d)$ be the ideal generated by the following:

$g_{01} = b_{01} b_{03}^d - b_{04} b_{02}^d$,

$g_{1i} = c_{1i} \left(b_{02} - b_{1i} b_{03} \right), i = 1, \ldots, 4$,

$g_{1,4+i} = c_{1i} \left(b_{01} - b_{1i}^d b_{04} \right), i = 1, \ldots, 4$,

$g_{1ij} = c_{1i} c_{1j} \left(b_{1i} - b_{1j} \right), 1 \leq i < j \leq 4$,

$g_{21} = b_{04}^d c_{11} - b_{01}^d c_{12}$,

$g_{22} = b_{04}^d c_{14} - b_{01}^d c_{13}$,

$g_{23} = b_{01}^d (c_{12} - c_{13})$,

$g_{24} = b_{04}^d (c_{12} b_{11} - c_{13} b_{14})$,

$$g_{2,4+i} = b_{04}^d c_{12}c_{2i}\left(b_{12} - b_{2i}b_{13}\right), i = 1, \ldots, 4, \text{ when } n > 2,$$
$$g_{25} = b_{04}^d c_{12}c_{2i}\left(b_{12} - b_{13}\right), \text{ when } n = 2,$$
$$g_{r1} = b_{01}^d c_{11}\cdots c_{r-3,1}\left(c_{r-2,4}c_{r-1,1} - c_{r-2,1}c_{r-1,2}\right), r = 2, \ldots, n,$$
$$g_{r2} = b_{01}^d c_{11}\cdots c_{r-3,1}\left(c_{r-2,4}c_{r-1,4} - c_{r-2,1}c_{r-1,3}\right), r = 2, \ldots, n,$$
$$g_{r3} = b_{01}^d c_{11}\cdots c_{r-2,1}\left(c_{r-1,3} - c_{r-1,2}\right), r = 2, \ldots, n,$$
$$g_{r4} = b_{01}^d c_{11}\cdots c_{r-3,1}c_{r-2,4}\left(c_{r-1,2}b_{r-1,1} - c_{r-1,3}b_{r-1,4}\right), r = 2, \ldots, n,$$
$$g_{r,4+i} = b_{01}^d c_{11}\cdots c_{r-3,1}c_{r-2,4}c_{r-1,2}c_{ri}\left(b_{r-1,2} - b_{ri}b_{r-1,3}\right), i = 1, \ldots, 4, r = 2, \ldots, n-1,$$
$$g_{n5} = b_{01}^d c_{11}\cdots c_{n-3,1}c_{n-2,4}c_{n-1,2}\left(b_{n-1,2} - b_{n-1,3}\right).$$

Note that the maximal degree of a generator of $K(n,d)$ is $d+n+1$.

By calculations in the previous section, $(J : sc_{02}) + (c_{02}, f) = K(n,d) + C_0 + (s,f)$. Note that the generators of $K(n,d)$ do not involve any of the variables $s, f, c_{01}, c_{02}, c_{03}, c_{04}$, so we may think of $K(n,d)$ as living in a polynomial ring over k with fewer variables as well.

Theorem 6.1: *Set $x = b_{01}^d c_{11}\cdots c_{n-2,1}(c_{n-1,1} - c_{n-1,4})$. With notation as above, the element x of R lies in $K(n,d)$, has degree $d+n-1$, but when written as an R-linear combination of the given generators of $K(n,d)$, the degree of at least one coefficient is doubly exponential in n.*

In other words, the family $K(n,d)$ satisfies the doubly exponential ideal membership property.

Proof: We use that $(J(n,d) : sc_{02}) + (c_{02}, f) = K(n,d) + (s, f, c_{01}, c_{02}, c_{03}, c_{04})$. As x and the generators of $K(n,d)$ do not involve any variables $s, f, c_{01}, c_{02}, c_{03}, c_{04}$, then x is in $K(n,d)$ if and only if x is in

$$K(n,d) + (s, f, c_{01}, c_{02}, c_{03}, c_{04}) = (J(n,d) : sc_{02}) + (c_{02}, f),$$

and then analogously this holds if and only if x is in $J(n,d) : sc_{02}$. But this is indeed the case by Corollary 5.4. Thus we can write $x = \sum_{ri} A_{ri}g_{ri} + \sum_{ij} A_{1ij}g_{1ij}$, for some $A_{ri}, A_{1ij} \in R$. Let $N(n,d)$ be the maximum degree of the A_{ri}, A_{1ij} with $i \geq 1$. Multiplying through by sc_{02} gives

$$sc_{02}x = b_{01}^d sc_{02}c_{11}\cdots c_{n-2,1}(c_{n-1,1} - c_{n-1,4}) = \sum_{r;i\geq 1} a_{ri}sc_{02}g_{ri} + \sum_{ij} a_{1ij}sc_{02}g_{1ij}.$$

By Corollary 5.4, each of $s(c_{01} - b_{02}^d c_{02})$, $sc_{02}g_{ri}$ and $sc_{02}g_{1ij}$ can be rewritten as a linear combination of the generators of $J(n,d)$ with coefficients of those generators having degrees at most $2d+1$. In particular,

$$\begin{aligned} sc_{01}c_{11}\cdots c_{n-2,1}(c_{n-1,1}-c_{n-1,4}) &= s(c_{01}-b_{01}^d c_{02})c_{11}\cdots c_{n-2,1}(c_{n-1,1}-c_{n-1,4}) \\ &\quad + b_{01}^d sc_{02}c_{11}\cdots c_{n-2,1}(c_{n-1,1}-c_{n-1,4}) \\ &= s(c_{01}-b_{01}^d c_{02})c_{11}\cdots c_{n-2,1}(c_{n-1,1}-c_{n-1,4}) + \sum_{r;i\geq 1} a_{ri}sc_{02}g_{ri} + \sum_{ij} a_{1ij}sc_{02}g_{1ij} \end{aligned}$$

can be written as a linear combination of the generators of the Mayr-Meyer ideal whose coefficients have degrees at most $2d+1+\max\{n, \deg a_{ri}, \deg a_{1ij}\} \leq 2d+1+\max\{n, N(n,d)\}$. By the work of Mayr and Meyer, this maximum is in fact doubly exponential in n, so that also $N(n,d)$ is doubly exponential in n. ∎

A remark is in order: just as $J(n,d)$ is a shortened version of the original Mayr-Meyer ideals $J_l(n,d)$, similarly there exists a lengthened version $K_l(n,d)$ of $K(n,d)$: $K_l(n,d) + (s, f, c_{01}, c_{02}, c_{03}, c_{04}) = (J_l(n,d) : sc_{02}) + (f, c_{02})$. The difference is that the lengthened ideals live in a ring with more variables, have more generators, but of smaller degree. However, by Fact 1.4, the primary decompositions of the two are equivalent. It is easy to verify that $s_n - f_n \in K_l(n,d)$, that its degree is 1, but that with correspondingly chosen generators of $K_l(n,d)$, any expression of $s_n - f_n$ as a linear combination of those generators will use coefficients whose degrees are doubly exponential in n.

7. Associated primes of $K(n,d)$

For simplicity of notation we will assume in this section that k is an algebraically closed field whose characteristic is relatively prime to d.

Sometimes we will write the ideal $K(n,d)$ also as $K(n,d; c_{r'i}, b_{ri} | r' = 1, \ldots, n-1; r = 0, \ldots, n; i = 1, \ldots, 4)$, to emphasize the defining variables.

Let M be the ideal generated by $g_{01}, g_{1i'}, g_{1ij}$, $1 \leq i < j \leq 4$, $i' = 1, \ldots, 8$. Namely, M is generated by generators of $K(n,d)$ of level 0 and 1. Let N be the ideal generated by the generators of $K(n,d)$ of level 2, namely by the g_{2j}. Finally, let L be the ideal generated by all higher level generators, namely by the g_{rj} with $r \geq 3$. Thus $K(n,d) = M + N + L$.

These new ideals M, N and L will sometimes be specified with the variables and degrees $(n, d; c_{r'i}, b_{ri} | r = 0, \ldots, n; r' = 1, \ldots, n-1; i = 1, \ldots, 4)$ attached to them.

Then define M_1, N_1 and L_1 as the corresponding level ideals of

$$K(n-1, d^2) = K(n-1, d^2; c_{r'i}, b_{ri} | r = 1, \ldots, n-1; r' = 2, \ldots, n-1; i = 1, \ldots, 4).$$

Then (up to a renaming of variables), $K(n-1, d^2)$ equals $M_1 + N_1 + L_1$.

We will prove that finding the set of associated primes of $K(n,d)$ reduces to finding the set of associated primes of $(c_{11}, c_{12}, c_{13}, c_{14},\ b_{01}, b_{02}, b_{03}, b_{04}) + K(n-1, d^2)$, and by

Fact 1.4, this reduces to finding the set of associated primes of $K(n-1,d^2)$.

Thus the associated prime ideals of $K(n,d)$ follow a recursion pattern.

By Fact 1.6, $\mathrm{Ass}\left(\frac{R}{K(n,d)}\right) \subseteq \mathrm{Ass}\left(\frac{R}{K(n,d)+(b_{04}^d)}\right) \cup \mathrm{Ass}\left(\frac{R}{K(n,d):b_{04}^d}\right)$. But $L \subseteq C_1 b_{01}^d \subseteq C_1 b_{04}^d + M$, so that

$$\begin{aligned}
K(n,d)+(b_{04}^d) &= N + M + (b_{04}^d) \\
&= (b_{04}^d, b_{01}b_{03}^d - b_{04}b_{02}^d) + c_{1i}(b_{02}-b_{1i}b_{03}, c_{1j}(b_{1i}-b_{1j}), b_{01}-b_{1i}^d b_{04}) \\
&= \bigcap_{\Lambda}\big((b_{04}^d, b_{01}b_{03}^d - b_{04}b_{02}^d) + (c_{1i}|i\notin\Lambda) \\
&\qquad + (b_{02}-b_{1i}b_{03}, b_{1i}-b_{1j}, b_{01}-b_{1i}^d b_{04}|i,j\in\Lambda)\big) \\
&= \bigcap_{\Lambda\neq\emptyset}\big((b_{04}^d) + (c_{1i}|i\notin\Lambda) + (b_{02}-b_{1i}b_{03}, b_{1i}-b_{1j}, b_{01}-b_{1i}^d b_{04}|i,j\in\Lambda)\big) \\
&\qquad \cap \big(C_1 + (b_{01}^d, b_{04}^d, b_{01}b_{03}^d - b_{04}b_{02}^d)\big) \cap \big(C_1 + (b_{04}^d, b_{03}^{d^2}, b_{01}b_{03}^d - b_{04}b_{02}^d)\big).
\end{aligned}$$

Thus the associated primes of $K(n,d)+(b_{04}^d)$ are $C_1+(b_{03},b_{04})$ and

$$G_{5\Lambda} = (b_{01},b_{04}) + (c_{1i}|i\notin\Lambda) + (b_{02}-b_{1i}b_{03}, b_{1i}-b_{1j}|i,j\in\Lambda)\,,$$

where Λ varies over all subsets of $\{1,2,3,4\}$. However, the former prime ideal is not associated to $K(n,d)$ as $K(n,d) : \bigcup_{i=1}^4 (b_{02}-b_{1i}b_{03}) = C_1 + (b_{01}b_{03}^d - b_{04}b_{02}^d)$. Also, $G_{5\emptyset}$ is not associated for the same reason. On the other hand, $G_{5\Lambda}$ is associated to $K(n,d)$ when $\Lambda\neq\emptyset$ as it is minimal over it. Thus

Proposition 7.1: *The set of associated primes of* $K(n,d)$ *is contained in* $\{G_{5\Lambda'}\} \cup$ *Ass* $\left(\frac{R}{K(n,d):b_{04}^d}\right)$, *as* Λ' *varies over non-empty subsets of* $\{1,2,3,4\}$. ∎

Thus the next step is to calculate $K(n,d) : b_{04}^d$. First let L' (respectively N') be the ideal obtained from L (respectively N) by rewriting each $c_{1i}b_{01}^d$ as $c_{1i}b_{1i}^{d^2}b_{04}^d$. Then both L' and N' are multiples of b_{04}^d. As $c_{1i}(b_{01}-b_{1i}^d b_{04}) \in M$, it follows that $L'+N'+M = K(n,d)$. Thus

$$\begin{aligned}
K(n,d) : b_{04}^d &= L'/b_{04}^d + N'/b_{04}^d + (M : b_{04}^d) \\
&= L'/b_{04}^d + (c_{11} - b_{12}^{d^2}c_{12}, c_{14}-c_{11}, b_{13}^{d^2}c_{13} - b_{12}^{d^2}c_{12}) \\
&\quad + (c_{12}b_{11} - c_{13}b_{14}, c_{12}c_{2i}(b_{12}-b_{2i}b_{13}), b_{01}b_{03}^d - b_{04}b_{02}^d) \\
&\quad + c_{1i}(b_{02}-b_{1i}b_{03}, c_{1j}(b_{1i}-b_{1j}), b_{01}-b_{1i}^d b_{04}).
\end{aligned}$$

By Fact 1.6, the associated primes of $K(n,d) : b_{04}^d$ are in the union of the associated primes of the two ideals obtained from $K(n,d) : b_{04}^d$ by respectively adding and coloning with c_{12}. Note that $(K(n,d) : b_{04}^d) + (c_{12})$ equals

$$= (c_{11}, c_{12}, c_{14}, b_{01}b_{03}^d - b_{04}b_{02}^d) + c_{13}(b_{13}^{d^2}, b_{14}, b_{02}-b_{13}b_{03}, b_{01}-b_{13}^d b_{04})$$

$$= (C_1 + (b_{01}b_{03}^d - b_{04}b_{02}^d)) \cap (c_{11}, c_{12}, c_{14}, b_{13}^{d^2}, b_{14}, b_{02} - b_{13}b_{03}, b_{01} - b_{13}^d b_{04}),$$

whose associated prime ideals are

$$G_6 = C_1 + (b_{01}b_{03}^d - b_{04}b_{02}^d),$$
$$G_7 = (c_{11}, c_{12}, c_{14}, b_{01}, b_{02}, b_{13}, b_{14}).$$

Note that G_6 is minimal over $K(n,d)$ and it is straightforward to verify that G_7 is associated to $K(n,d)$. Hence

Proposition 7.2: *Ass* $\left(\frac{R}{K(n,d)}\right) \subseteq \{G_{5\Lambda'}, G_j | j = 6, 7\} \cup$ *Ass* $\left(\frac{R}{K(n,d) : b_{04}^d c_{12}}\right)$*, where* Λ' *varies over all non-empty subsets of* $\{1,2,3,4\}$*.* ■

Let L'' be the ideal obtained from L'/b_{04}^d by rewriting each c_{11} and c_{14} as $b_{12}^{d^2}c_{12}$. Note that in the displayed $K(n,d) : b_{04}^d$ above, the summand L'/b_{04}^d may be replaced by L'', and that $L'' = (L_1 + N_1)c_{12}b_{12}^{d^2}$. Thus by Fact 1.3,

$$\begin{aligned}
K(n,d) : b_{04}^d c_{12} &= (L_1 + N_1)b_{12}^{d^2} + (c_{11} - b_{12}^{d^2}c_{12}, c_{14} - c_{11}, c_{2i}(b_{12} - b_{2i}b_{13}))\\
&+ (b_{02} - b_{12}b_{03}, c_{1i}(b_{12} - b_{1i}), b_{01} - b_{12}^d b_{04})\\
&+ b_{12}^{d^2}(b_{02} - b_{11}b_{03}, b_{02} - b_{14}b_{03}, b_{01} - b_{11}^d b_{04}, b_{01} - b_{14}^d b_{04})\\
&+ \big((b_{01}b_{03}^d - b_{04}b_{02}^d, b_{13}^{d^2}c_{13} - b_{12}^{d^2}c_{12}, c_{12}b_{11} - c_{13}b_{14}) + c_{13}(b_{02} - b_{13}b_{03}, b_{01} - b_{13}^d b_{04})\big) : c_{12}.
\end{aligned}$$

The ideal in the last two rows, before taking the colon with c_{12}, decomposes as (by coloning with and adding c_{13}):

$$\begin{aligned}
&\big(b_{13}^{d^2}c_{13} - b_{12}^{d^2}c_{12}, c_{12}b_{11} - c_{13}b_{14}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{02} - b_{13}b_{03}, b_{01} - b_{13}^d b_{04}\big)\\
&\quad \cap (c_{13}, b_{12}^{d^2}c_{12}, c_{12}b_{11}, b_{01}b_{03}^d - b_{04}b_{02}^d),
\end{aligned}$$

which coloned with c_{12} equals

$$\begin{aligned}
&\big(b_{13}^{d^2}c_{13} - b_{12}^{d^2}c_{12}, c_{12}b_{11} - c_{13}b_{14}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{02} - b_{13}b_{03}, b_{01} - b_{13}^d b_{04}\big)\\
&\quad \cap (c_{13}, b_{12}^{d^2}, b_{11}, b_{01}b_{03}^d - b_{04}b_{02}^d)\\
=&(b_{13}^{d^2}c_{13} - b_{12}^{d^2}c_{12}, c_{12}b_{11} - c_{13}b_{14}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{01}b_{03}^d - b_{04}b_{02}^d)\\
&\quad + (b_{02} - b_{13}b_{03}, b_{01} - b_{13}^d b_{04}) \cdot (c_{13}, b_{12}^{d^2}, b_{11}).
\end{aligned}$$

It follows that

$$\begin{aligned}
K(n,d) : b_{04}^d c_{12} &= (L_1 + N_1)b_{12}^{d^2} + (c_{11} - b_{12}^{d^2}c_{12}, c_{14} - c_{11}, c_{2i}(b_{12} - b_{2i}b_{13}))\\
&+ (b_{02} - b_{12}b_{03}, c_{1i}(b_{12} - b_{1i}), b_{01} - b_{12}^d b_{04})\\
&+ b_{12}^{d^2}(b_{02} - b_{11}b_{03}, b_{02} - b_{14}b_{03}, b_{01} - b_{11}^d b_{04}, b_{01} - b_{14}^d b_{04})\\
&+ (b_{13}^{d^2}c_{13} - b_{12}^{d^2}c_{12}, c_{12}b_{11} - c_{13}b_{14}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{01}b_{03}^d - b_{04}b_{02}^d)
\end{aligned}$$

$$
\begin{aligned}
&+ (b_{02} - b_{13}b_{03}, b_{01} - b_{13}^d b_{04}) \cdot (c_{13}, b_{12}^{d^2}, b_{11}) \\
&= (L_1 + N_1) b_{12}^{d^2} + (c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, c_{2i}(b_{12} - b_{2i}b_{13})) \\
&\quad + (b_{02} - b_{12}b_{03}, c_{12}b_{12}^{d^2}(b_{12} - b_{11}), c_{13}(b_{12} - b_{13}), c_{12}b_{12}^{d^2}(b_{12} - b_{14}), b_{01} - b_{12}^d b_{04}) \\
&\quad + b_{12}^{d^2}((b_{12} - b_{1i})b_{03}, (b_{12}^d - b_{1i}^d)b_{04}, c_{13} - c_{12}) \\
&\quad + (c_{12}b_{11} - c_{13}b_{14}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}) + b_{11}((b_{12} - b_{13})b_{03}, (b_{12}^d - b_{13}^d)b_{04}).
\end{aligned}
$$

When $n = 2$, this ideal is much simpler than when $n > 2$. We first analyze the cases $n \geq 3$. By Fact 1.6, the set of associated primes of V is contained in $\operatorname{Ass}\left(\frac{R}{(K(n,d):b_{04}^d c_{12}) + (b_{12}^{d^2})}\right) \cup \operatorname{Ass}\left(\frac{R}{K(n,d):b_{04}^d c_{12} b_{12}^{d^2}}\right)$. Note that

$$
\begin{aligned}
(K(n,d) : b_{04}^d c_{12}) + (b_{12}^{d^2}) &= (c_{11}, c_{14}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}, b_{12}^{d^2}, c_{13}(b_{12} - b_{13})) \\
&\quad + (c_{12}b_{11} - c_{13}b_{14}, c_{2i}(b_{12} - b_{2i}b_{13})) + b_{11}(b_{13}^{d^2}, (b_{12} - b_{13})b_{03}, (b_{12}^d - b_{13}^d)b_{04}).
\end{aligned}
$$

To decompose this, by Fact 1.4, for notational simplicity it suffices to decompose

$$V' = (b_{12}^{d^2}, c_{13}(b_{12}-b_{13}), c_{12}b_{11}-c_{13}b_{14}, c_{2i}(b_{12}-b_{2i}b_{13}))+b_{11}(b_{13}^{d^2}, (b_{12}-b_{13})b_{03}, (b_{12}^d-b_{13}^d)b_{04}).$$

Then by first coloning with and adding c_{13}, then repeating with $b_{03}b_{11}$:

$$
\begin{aligned}
V' &= (b_{12}^{d^2}, c_{2i}b_{12}(1 - b_{2i}), b_{12} - b_{13}, c_{12}b_{11} - c_{13}b_{14}) \\
&\quad \cap \big((b_{12}^{d^2}, c_{2i}(b_{12} - b_{2i}b_{13}), c_{13}) + b_{11}(c_{12}, b_{13}^{d^2}, (b_{12} - b_{13})b_{03}, (b_{12}^d - b_{13}^d)b_{04})\big) \\
&= (b_{12}^{d^2}, c_{2i}b_{12}(1 - b_{2i}), b_{12} - b_{13}, c_{12}b_{11} - c_{13}b_{14}) \\
&\quad \cap (b_{12}^{d^2}, c_{2i}b_{12}(1 - b_{2i}), c_{13}, c_{12}, b_{12} - b_{13}) \\
&\quad \cap \big((b_{12}^{d^2}, c_{2i}(b_{12} - b_{2i}b_{13}), c_{13}) + b_{11}(c_{12}, b_{13}^{d^2}, b_{03}, (b_{12}^d - b_{13}^d)b_{04})\big).
\end{aligned}
$$

As the second component contains the first one, the second one is redundant. By decomposing the remaining two components of V', V' further decomposes as:

$$
\begin{aligned}
V' &= (b_{12}^{d^2}, c_{2i}(1 - b_{2i}), b_{12} - b_{13}, c_{12}b_{11} - c_{13}b_{14}) \\
&\quad \cap (b_{12}, b_{13}, c_{12}b_{11} - c_{13}b_{14}) \\
&\quad \cap (b_{12}^{d^2}, c_{2i}(b_{12} - b_{2i}b_{13}), c_{13}, b_{11}) \\
&\quad \cap \Big((b_{12}^{d^2}, c_{2i}(b_{12} - b_{2i}b_{13}), c_{12}, c_{13}, b_{13}^{d^2}, b_{03}, (b_{12}^d - b_{13}^d)b_{04})\Big).
\end{aligned}
$$

Thus V' is the intersection of the four ideals above. Let the ith ideal be V_i. The ideal V_1 decomposes into primary ideals as follows:

$$V_1 = \bigcap_{\Lambda} \Big((b_{12}^{d^2}, b_{12} - b_{13}, c_{12}b_{11} - c_{13}b_{14}) + (c_{2i} | i \notin \Lambda) + (1 - b_{2i} | i \in \Lambda)\Big),$$

V_2 is a prime ideal, V_3 decomposes as

$$
\begin{aligned}
V_3 &= \bigcap_\Lambda \Big((b_{12}^{d^2}, c_{13}, b_{11}) + (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i}b_{13} | i \in \Lambda) \Big) \\
&= \bigcap_\Lambda \Big((b_{12}^{d^2}, c_{13}, b_{11}) + (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i}^{d^2}, b_{2i} - b_{2i} | i, j \in \Lambda) \Big) \\
&\quad \bigcap_{\Lambda \neq \emptyset} \Big((b_{12}^{d^2}, b_{13}^{d^2}, c_{13}, b_{11}) + (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i}b_{13} | i \in \Lambda) \Big) \\
&= \bigcap_\Lambda \Big((b_{12}^{d^2}, c_{13}, b_{11}) + (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i}^{d^2}, b_{2i} - b_{2j} | i, j \in \Lambda) \Big) \\
&\quad \bigcap_{\Lambda \neq \emptyset} \Big((b_{12}^{d^2}, b_{13}^{d^2}, c_{13}, b_{11}) + (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j} | i, j \in \Lambda) \Big) \\
&\quad \cap (b_{12}, b_{13}, c_{13}, b_{11}),
\end{aligned}
$$

which are all primary ideals. The last primary factor of V_3 properly contains V_2 and is thus redundant in a primary decomposition of V'. Finally,

$$
\begin{aligned}
V_4 = &\cap (b_{12}^{d^2}, c_{2i}(b_{12} - b_{2i}b_{13}), c_{12}, c_{13}, b_{13}^{d^2}, b_{03}, b_{12}^d - b_{13}^d) \\
&\cap (b_{12}^{d^2}, c_{2i}(b_{12} - b_{2i}b_{13}), c_{12}, c_{13}, b_{13}^{d^2}, b_{03}, b_{04}) \\
= &\bigcap_\Lambda \Big((b_{12}^{d^2}, c_{12}, c_{13}, b_{03}, b_{12}^d - b_{13}^d) + (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j}, 1 - b_{2i}^d | i, j \in \Lambda) \Big) \\
&\bigcap_{\Lambda \neq \emptyset} \Big((b_{12}^d, c_{12}, c_{13}, b_{13}^d, b_{03}) + (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j} | i, j \in \Lambda) \Big) \\
&\cap \Big(b_{12}, b_{13}, c_{12}, c_{13}, b_{03} \Big) \\
&\bigcap_\Lambda \Big((b_{12}^{d^2}, c_{12}, c_{13}, b_{13}^{d^2}, b_{03}, b_{04}) + (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j} | i, j \in \Lambda) \Big) \\
&\cap (b_{12}, b_{13}, c_{12}, c_{13}, b_{03}, b_{04}).
\end{aligned}
$$

The ideals in the last and the third to the last rows above contain V_2, and are thus redundant in a primary decomposition of V. Now the (possibly redundant) associated primes of V' can be easily read off, and by adding $(c_{11}, c_{14}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04})$ one gets the associated primes of $(K(n,d) : b_{04}^d c_{12}) + (b_{12}^{d^2})$ (see page 31):

$$
\begin{aligned}
G_{8\Lambda} &= (c_{11}, c_{14}, b_{01}, b_{02}, b_{12}, b_{13}, c_{12}b_{11} - c_{13}b_{14}) + (c_{2i} | i \notin \Lambda) + (1 - b_{2i} | i \in \Lambda), \\
G_9 &= (c_{11}, c_{14}, b_{01}, b_{02}, b_{12}, b_{13}, c_{12}b_{11} - c_{13}b_{14}), \\
G_{10\Lambda} &= (c_{11}, c_{13}, c_{14}, b_{01}, b_{02}, b_{11}, b_{12}) + (c_{2i} | i \notin \Lambda) + (b_{2i} | i \in \Lambda), \\
G_{11\Lambda'} &= (c_{11}, c_{13}, c_{14}, b_{01}, b_{02}, b_{11}, b_{12}, b_{13}) + (c_{2i} | i \notin \Lambda') + (b_{2i} - b_{2j} | i, j \in \Lambda'), \\
G_{12\Lambda\alpha} &= C_1 + (b_{01}, b_{02}, b_{03}, b_{12}, b_{13}) + (c_{2i} | i \notin \Lambda) + (b_{2i} - \alpha | i \in \Lambda), \alpha^d = 1,
\end{aligned}
$$

$$G_{13\Lambda'} = C_1 + (b_{01}, b_{02}, b_{03}, b_{12}, b_{13}) + (c_{2i} | i \notin \Lambda') + (b_{2i} - b_{2j} | i, j \in \Lambda'),$$
$$G_{14\Lambda} = C_1 + (b_{01}, b_{02}, b_{03}, b_{04}, b_{12}, b_{13}) + (c_{2i} | i \notin \Lambda) + (b_{2i} - b_{2j} | i, j \in \Lambda),$$

where Λ varies over all subsets of $\{1, 2, 3, 4\}$ and Λ' varies over all non-empty subsets of $\{1, 2, 3, 4\}$. By Proposition 7.2 it follows that

Proposition 7.3: *Whenever* $n \geq 3$,

$$Ass\left(\frac{R}{K(n,d)}\right) \subseteq \{G_{i\Lambda}, G_j, G_{k\Lambda'} | i = 8, 10, 14; j = 6, 7, 9; k = 5, 11, 13\}$$
$$\cup \{G_{12\Lambda\alpha}\} \cup Ass\left(\frac{R}{K(n,d) : b_{04}^d c_{12} b_{12}^{d^2}}\right),$$

where α varies over the dth roots of unity and Λ over all subsets of $\{1, 2, 3, 4\}$ and Λ' varies over all non-empty subsets of $\{1, 2, 3, 4\}$. ∎

Next,

$$\begin{aligned}
K(n,d) : b_{04}^d c_{12} b_{12}^{d^2} &= L_1 + N_1 + (c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, b_{02} - b_{12} b_{03}, b_{01} - b_{12}^d b_{04}) \\
&+ (c_{12}(b_{12} - b_{11}), c_{12}(b_{12} - b_{14}), (b_{12} - b_{1i}) b_{03}), (b_{12}^d - b_{1i}^d) b_{04}, c_{13} - c_{12}) \\
&+ \Big((c_{2i}(b_{12} - b_{2i} b_{13}), c_{13}(b_{12} - b_{13}), c_{12} b_{11} - c_{13} b_{14}, b_{11} b_{13}^{d^2} - b_{14} b_{12}^{d^2}) \\
&\quad + b_{11}((b_{12} - b_{13}) b_{03}, (b_{12}^d - b_{13}^d) b_{04})\Big) : b_{12}^{d^2}.
\end{aligned}$$

The ideal $\hat{V}$ in the last two rows, before coloning with $b_{12}^{d^2}$, (partially) decomposes as (first coloning with and adding c_{13}):

$$\begin{aligned}
\hat{V} &= (c_{2i} b_{12}(1 - b_{2i}), b_{12} - b_{13}, c_{12} b_{11} - c_{13} b_{14}, b_{12}^{d^2}(b_{11} - b_{14})) \\
&\quad \cap \big(c_{2i}(b_{12} - b_{2i} b_{13}), c_{13}, c_{12} b_{11}, b_{11} b_{13}^{d^2} - b_{14} b_{12}^{d^2}, (b_{12} - b_{13}) b_{03} b_{11}, (b_{12}^d - b_{13}^d) b_{04} b_{11})\big).
\end{aligned}$$

The second component decomposes further into:

$$\begin{aligned}
\bigcap_{\Lambda} \Big(&(c_{13}, c_{12} b_{11}, b_{11} b_{13}^{d^2} - b_{14} b_{12}^{d^2}, (b_{12} - b_{13}) b_{03} b_{11}, (b_{12}^d - b_{13}^d) b_{04} b_{11}) \\
&+ (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i} b_{13} | i \in \Lambda)\Big).
\end{aligned}$$

For $\Lambda \neq \emptyset$, the corresponding component of $\hat{V}$ simplifies to

$$\begin{aligned}
&(c_{13}, c_{12} b_{11}) + (c_{2i} | i \notin \Lambda) \\
&\quad + (b_{12} - b_{2i} b_{13}, b_{13}^{d^2}(b_{11} - b_{14} b_{2i}^{d^2}), b_{13}(b_{2i} - 1) b_{03} b_{11}, b_{13}^d (b_{2i}^d - 1) b_{04} b_{11} | i \in \Lambda\Big),
\end{aligned}$$

and for $\Lambda = \emptyset$ it equals to and (partially) decomposes as

$$\begin{aligned}
&C_2 + (c_{13}, c_{12} b_{11}, b_{11} b_{13}^{d^2} - b_{14} b_{12}^{d^2}, (b_{12} - b_{13}) b_{03} b_{11}, (b_{12}^d - b_{13}^d) b_{04} b_{11}) \\
&= \big(C_2 + (c_{13}, b_{11}, b_{14} b_{12}^{d^2})\big)
\end{aligned}$$

$$
\begin{aligned}
&\cap \big(C_2 + (c_{13}, c_{12}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, (b_{12} - b_{13})b_{03}b_{11}, (b_{12}^d - b_{13}^d)b_{04}b_{11})\big)\\
= &\big(C_2 + (c_{13}, b_{11}, b_{14}b_{12}^{d^2})\big)\\
&\cap \big(C_2 + (c_{13}, c_{12}, (b_{11} - b_{14})b_{12}^{d^2}, b_{12} - b_{13})\big)\\
&\cap \big(C_2 + (c_{13}, c_{12}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}b_{11}, (b_{12}^d - b_{13}^d)b_{04}b_{11}\big).
\end{aligned}
$$

The last component in the last display above decomposes as:

$$
\begin{aligned}
&\big(C_2 + (c_{13}, c_{12}, (b_{11} - b_{14})b_{12}^{d^2}, b_{03}, b_{12}^d - b_{13}^d\big)\\
&\quad \cap \big(C_2 + (c_{13}, c_{12}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}, b_{04})\big)\\
&\quad \cap \big(C_2 + (c_{13}, c_{12}, b_{14}b_{12}^{d^2}, b_{11})\big).
\end{aligned}
$$

Thus from the combined decomposition above of $\hat{V}$, $\hat{V} : b_{12}^{d^2}$ equals

$$
\begin{aligned}
\hat{V} : b_{12}^{d^2} = &(c_{2i}(1 - b_{2i}), b_{12} - b_{13}, c_{12}b_{11} - c_{13}b_{14}, b_{11} - b_{14})\\
&\cap \bigcap_{\Lambda \neq \emptyset} \big((c_{13}, c_{12}b_{14}) + (c_{2i} | i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j} | i, j \in \Lambda)\\
&\quad + (b_{11} - b_{14}b_{2i}^{d^2}, (b_{2i} - 1)b_{03}b_{14}, (b_{2i}^d - 1)b_{04}b_{14} | i \in \Lambda)\big)\\
&\cap \big(C_2 + (c_{13}, b_{11}, b_{14})\big)\\
&\cap \big(C_2 + (c_{13}, c_{12}, b_{11} - b_{14}, b_{12} - b_{13})\big)\\
&\cap \big(C_2 + (c_{13}, c_{12}, b_{11} - b_{14}, b_{03}, b_{12}^d - b_{13}^d\big)\\
&\cap \big(C_2 + (c_{13}, c_{12}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}, b_{04})\big)\\
&\cap \big(C_2 + (c_{13}, c_{12}, b_{14}, b_{11})\big).
\end{aligned}
$$

Next we compute this intersection. The intersection of the last two ideals is

$$
C_2 + (c_{13}, c_{12}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}b_{11}, b_{04}b_{11}, b_{03}b_{14}, b_{04}b_{14}),
$$

and the intersection of the two immediately preceding is

$$
C_2 + (c_{13}, c_{12}, b_{11} - b_{14}, b_{03}(b_{12} - b_{13}), b_{12}^d - b_{13}^d\big).
$$

Thus the intersection of the last four components in $\hat{V} : b_{12}^{d^2}$ is

$$
\begin{aligned}
&C_2 + (c_{13}, c_{12}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}(b_{11} - b_{14}), b_{04}(b_{11} - b_{14})\\
&\quad + (b_{03}b_{11}(b_{12} - b_{13}), b_{04}b_{14}(b_{12}^d - b_{13}^d)),
\end{aligned}
$$

and then the intersection of the last five components in $\hat{V} : b_{12}^{d^2}$ is

$$
\begin{aligned}
&C_2 + (c_{13}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}(b_{11} - b_{14}), b_{04}(b_{11} - b_{14})\\
&\quad + (b_{03}b_{11}(b_{12} - b_{13}), b_{04}b_{14}(b_{12}^d - b_{13}^d)) + c_{12}(b_{11}, b_{14}).
\end{aligned}
$$

Now this ideal, together with the second through the sixteenth components of $\hat{V} : b_{12}^{d^2}$

above are all of the form

$$(c_{13}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}(b_{11} - b_{14}), b_{04}(b_{11} - b_{14}), b_{03}b_{11}(b_{12} - b_{13}), b_{04}b_{14}(b_{12}^d - b_{13}^d))$$
$$+ c_{12}(b_{11}, b_{14}) + (c_{2i} | i \notin \Lambda)$$
$$+ (b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j}(b_{11} - b_{14}b_{2i}^{d^2}, (b_{2i} - 1)b_{03}b_{14}, (b_{2i}^d - 1)b_{04}b_{14} | i, j \in \Lambda)\big),$$

as Λ varies over all sixteen subsets of $\{1, 2, 3, 4\}$. The intersection of all these sixteen ideals is

$$(c_{13}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}(b_{11} - b_{14}), b_{04}(b_{11} - b_{14}), b_{03}b_{11}(b_{12} - b_{13}), b_{04}b_{14}(b_{12}^d - b_{13}^d))$$
$$+ c_{12}(b_{11}, b_{14}) + c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i} - b_{2j}), b_{11} - b_{14}b_{2i}^{d^2}), (b_{2i} - 1)b_{03}b_{14}, (b_{2i}^d - 1)b_{04}b_{14})).$$

Thus finally $\hat{V} : b_{12}^{d^2}$ equals (intersection of above with the first factor of $\hat{V} : b_{12}^{d^2}$):

$$(b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}(b_{11} - b_{14}), b_{04}(b_{11} - b_{14}), b_{03}b_{11}(b_{12} - b_{13}), b_{04}b_{14}(b_{12}^d - b_{13}^d))$$
$$+ c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i} - b_{2j}), b_{11} - b_{14}b_{2i}^{d^2}, (b_{2i} - 1)b_{03}b_{14}, (b_{2i}^d - 1)b_{04}b_{14}))$$
$$+ (c_{12}(b_{11} - b_{14}, c_{12}b_{11} - c_{13}b_{14}) + c_{13}(c_{2i}(1 - b_{2i}), b_{12} - b_{13}, b_{11} - b_{14}).$$

Thus $K(n, d) : b_{04}^d c_{12} b_{12}^{d^2}$ equals (refer to page 33):

$$L_1 + N_1 + (c_{11} - b_{12}^{d^2}c_{12}, c_{14} - c_{11}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04})$$
$$+ (c_{12}(b_{12} - b_{11}), c_{12}(b_{12} - b_{14}), (b_{12} - b_{1i})b_{03}, (b_{12}^d - b_{1i}^d)b_{04}, c_{13} - c_{12})$$
$$+ (b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, b_{03}(b_{11} - b_{14}), b_{04}(b_{11} - b_{14})$$
$$+ (b_{03}b_{11}(b_{12} - b_{13}), b_{04}b_{14}(b_{12}^d - b_{13}^d))$$
$$+ c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i} - b_{2j}), b_{11} - b_{14}b_{2i}^{d^2}, (b_{2i} - 1)b_{03}b_{14}, (b_{2i}^d - 1)b_{04}b_{14}))$$
$$+ (c_{12}(b_{11} - b_{14}), c_{12}b_{11} - c_{13}b_{14}) + c_{13}(c_{2i}(1 - b_{2i}), b_{12} - b_{13}, b_{11} - b_{14})$$
$$= L_1 + N_1 + (c_{11} - b_{12}^{d^2}c_{12}, c_{14} - c_{11}, c_{13} - c_{12}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04})$$
$$+ (c_{12}(b_{12} - b_{1i}), (b_{12} - b_{1i})b_{03}, (b_{12}^d - b_{1i}^d)b_{04}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, (b_{11} - b_{14})b_{04})$$
$$+ c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i} - b_{2j}), b_{11} - b_{2i}^{d^2}b_{14}, c_{12}(1 - b_{2i})).$$

By Fact 1.6, any associated prime of $K(n, d) : b_{04}^d c_{12} b_{12}^{d^2}$ is associated either to $(K(n, d) : b_{04}^d c_{12} b_{12}^{d^2}) + (c_{12})$ or to $K(n, d) : b_{04}^d c_{12}^2 b_{12}^{d^2}$. These ideals are as follows:

$$S = K(n, d) : b_{04}^d c_{12}^2 b_{12}^{d^2} = L_1 + N_1$$
$$+ (c_{11} - b_{12}^{d^2}c_{12}, c_{14} - c_{11}, c_{13} - c_{12}, b_{02} - b_{12}b_{03}, b_{12} - b_{1i}, b_{01} - b_{12}^d b_{04}, c_{2i}(1 - b_{2i})),$$

and

$$(K(n, d) : b_{04}^d c_{12} b_{12}^{d^2}) + (c_{12}) = L_1 + N_1 + C_1$$
$$+ (b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}, (b_{12} - b_{1i})b_{03}, (b_{12}^d - b_{1i}^d)b_{04})$$

$$+ c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i} - b_{2j}), b_{11} - b_{2i}^{d^2} b_{14}) + (b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, (b_{11} - b_{14})b_{04}).$$

This last ideal decomposes (coloning with and adding b_{03}):

$$= \Big(L_1 + N_1 + C_1 + (b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}, b_{12} - b_{1i}) + c_{2i}(b_{12}(1 - b_{2i}), c_{2j}(b_{2i} - b_{2j}))\Big)$$
$$\cap \Big(L_1 + N_1 + C_1 + (b_{01} - b_{12}^d b_{04}, b_{02}, b_{03}, (b_{12}^d - b_{1i}^d)b_{04})$$
$$+ c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i} - b_{2j}), b_{11} - b_{2i}^{d^2} b_{14}) + (b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}, (b_{11} - b_{14})b_{04})\Big).$$

Let T denote the first of the two components above. The second component further decomposes (coloning with and adding b_{04}):

$$\Big(L_1 + N_1 + C_1 + (b_{01} - b_{12}^d b_{04}, b_{02}, b_{03}, b_{12}^d - b_{1i}^d, b_{11} - b_{14})$$
$$+ c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i} - b_{2j}), b_{11} - b_{2i}^{d^2} b_{14})\Big)$$
$$\cap \Big(L_1 + N_1 + C_1 + (b_{01}, b_{02}, b_{03}, b_{04}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2})$$
$$+ c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i} - b_{2j}), b_{11} - b_{2i}^{d^2} b_{14})\Big).$$

Let these two ideals be V and W, in that order. By Fact 1.6 and Proposition 7.3:

Proposition 7.4: *When $n \geq 3$,*

$$Ass\left(\frac{R}{K(n,d)}\right) \subseteq \{G_{ir\Lambda}, G_{jr}, G_{kr\Lambda'}, G_{12\Lambda\alpha} | i = 8, 10, 14; j = 6, 7, 9; k = 5, 11, 13\}$$
$$\cup\, Ass\left(\frac{R}{S}\right) \cup Ass\left(\frac{R}{T}\right) \cup Ass\left(\frac{R}{V}\right) \cup Ass\left(\frac{R}{W}\right),$$

where α varies over all dth roots of unity and Λ, Λ' over all subsets of $\{1,2,3,4\}$ with $|\Lambda'| > 0$. ∎

But $W = K(n-1, d^2) + C_1 + (b_{01}, b_{02}, b_{03}, b_{04})$, so the induction on n gives all of its associated primes. It remains to compute the associated primes of the other three ideals, namely of S, T and V. For each one of these ideals U (U varying over S, T and V), by Fact 1.6,

$$\text{Ass}\left(\frac{R}{U}\right) \subseteq \text{Ass}\left(\frac{R}{U + (b_{11}^{d^2})}\right) \cup \text{Ass}\left(\frac{R}{U : b_{11}^{d^2}}\right).$$

Modulo the other generators of U, $L_1 + N_1$ is contained in $(b_{11}^{d^2})$, so that the three $U + (b_{11}^{d^2})$ are as follows:

$$S + (b_{11}^{d^2}) = (b_{11}^{d^2}, c_{11}, c_{14} - c_{11}, c_{13} - c_{12}, b_{02} - b_{12}b_{03}, b_{12} - b_{1i}, b_{01} - b_{12}^d b_{04}, c_{2i}(1 - b_{2i})),$$
$$T + (b_{11}^{d^2}) = C_1 + (b_{11}^{d^2}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}, b_{12} - b_{1i}) + c_{2i}(b_{12}(1 - b_{2i}), c_{2j}(b_{2i} - b_{2j})),$$
$$V + (b_{11}^{d^2}) = C_1 + (b_{11}^{d^2}, b_{01} - b_{12}^d b_{04}, b_{02}, b_{03}, b_{12}^d - b_{1i}^d, b_{11} - b_{14})$$
$$+ c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i} - b_{2j}), b_{11}(1 - b_{2i}^{d^2})).$$

These contribute the following possible associated primes of $K(n,d)$:

$G_{15\Lambda} = (c_{11}, c_{13}-c_{12}, c_{14}, b_{01}, b_{02}, b_{11}, b_{12}, b_{13}, b_{14}) + (c_{2i}|i \notin \Lambda) + (1-b_{2i}|i \in \Lambda)$(from S),

$G_{16\Lambda} = C_1 + (b_{01}, b_{02}, b_{11}, b_{12}, b_{13}, b_{14}) + (c_{2i}|i \notin \Lambda) + (1-b_{2i}|i \in \Lambda)$ (from T),

$G_{17\Lambda} = C_1 + (b_{01}, b_{02}, b_{11}, b_{12}, b_{13}, b_{14}) + (c_{2i}|i \notin \Lambda) + (b_{2i}-b_{2j}|i,j \in \Lambda)$ (from T).

The associated primes obtained from V are not so easily read off, so we need to first decompose $V + (b_{11}^{d^2})$. By Fact 1.4 it suffices to decompose

$$\begin{aligned}
&(b_{11}^{d^2}, b_{12}^d - b_{11}^d, b_{12}^d - b_{13}^d) + c_{2i}(b_{12} - b_{2i}b_{13}, c_{2j}(b_{2i}-b_{2j}), b_{11}(1-b_{2i}^{d^2}))\\
&= \bigcap_{\Lambda}\big((b_{11}^{d^2}, b_{12}^d - b_{11}^d, b_{12}^d - b_{13}^d) + (c_{2i}|i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i}-b_{2j}, b_{11}(1-b_{2i}^{d^2})|i,j \in \Lambda)\big)\\
&= \big(C_2 + (b_{11}^{d^2}, b_{12}^d - b_{11}^d, b_{12}^d - b_{13}^d)\big)\\
&\quad \cap \bigcap_{\Lambda \neq \emptyset}\big((b_{11}^{d^2}, b_{13}^d - b_{11}^d) + (c_{2i}|i \notin \Lambda)\\
&\qquad + (b_{12} - b_{2i}b_{13}, b_{2i}-b_{2j}, b_{11}(1-b_{2i}^{d^2}), b_{13}^d(b_{2i}^d - 1)|i,j \in \Lambda)\big)\\
&= \big(C_2 + (b_{11}^{d^2}, b_{12}^d - b_{11}^d, b_{12}^d - b_{13}^d)\big)\\
&\quad \bigcap_{\Lambda \neq \emptyset}\big((b_{11}, b_{13}^d) + (c_{2i}|i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i}-b_{2j}|i,j \in \Lambda)\big)\\
&\quad \bigcap_{\Lambda \neq \emptyset}\big((b_{11}^d, b_{13}^d) + (c_{2i}|i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i}-b_{2j}, (1-b_{2i}^{d^2})/(b_{2i}^d - 1)|i,j \in \Lambda)\big)\\
&\quad \cap \bigcap_{\Lambda \neq \emptyset}\big((b_{11}^{d^2}, b_{13}^d - b_{11}^d) + (c_{2i}|i \notin \Lambda) + (b_{12} - b_{2i}b_{13}, b_{2i}-b_{2j}, 1-b_{2i}^d|i,j \in \Lambda)\big),
\end{aligned}$$

from which one can easily read off the associated primes. Then by adding $C_1 + (b_{01} - b_{12}^d b_{04}, b_{02}, b_{03}, b_{11} - b_{14})$ to these primes, the contribution of $V + (b_1^{d^2})$ to the associated primes of $K(n,d)$ is:

$$\begin{aligned}
G_{18\Lambda} &= C_1 + (b_{01}, b_{02}, b_{03}, b_{11}, b_{12}, b_{13}, b_{14}) + (c_{2i}|i \notin \Lambda) + (b_{2i} - b_{2j}|i,j \in \Lambda),\\
G_{19\Lambda'\alpha} &= C_1 + (b_{01}, b_{02}, b_{03}, b_{11}, b_{12}, b_{13}, b_{14})\\
&\quad + (c_{2i}|i \notin \Lambda') + (b_{2i} - \alpha|i \in \Lambda'), \alpha^{d^2} = 1, \alpha^d \neq 1,\\
G_{20\Lambda'\alpha} &= C_1 + (b_{01}, b_{02}, b_{03}, b_{11}, b_{12}, b_{13}, b_{14}) + (c_{2i}|i \notin \Lambda') + (b_{2i} - \alpha|i \in \Lambda'), \alpha^d = 1,
\end{aligned}$$

where Λ varies over all and Λ' over all non-empty subsets of $\{1,2,3,4\}$. This establishes

Proposition 7.5: *For $n \geq 3$,*

$$\begin{aligned}
Ass\left(\frac{R}{K(n,d)}\right) &\subseteq \{G_{i\Lambda}, G_j, G_{k\Lambda'}|i = 8,10,14,15,16,17,18; j = 6,7,9; k = 5,11,13\}\\
&\cup \{G_{12\Lambda\alpha}, G_{19\Lambda'\alpha'}, G_{20\Lambda'\alpha}\} \cup Ass\left(\frac{R}{S:b_{11}^{d^2}}\right) \cup Ass\left(\frac{R}{T:b_{11}^{d^2}}\right) \cup Ass\left(\frac{R}{V:b_{11}^{d^2}}\right)
\end{aligned}$$

$$\cup\, Ass\left(\frac{R}{K(n-1,d^2)+C_1+(b_{01},b_{02},b_{03},b_{04})}\right),$$

where α varies over all dth roots of unity, α' over all d^2 roots of unity which are not dth roots of unity, Λ varies over all subsets of $\{1,2,3,4\}$, and Λ' varies over all non-empty subsets of $\{1,2,3,4\}$. ■

Finally we analyze the three $U : b_{11}^{d^2}$ (where $U = S, T, V$, and $n \geq 3$). The computation of $U : b_{11}^{d^2}$ is is not difficult, as modulo $b_{11}^d - b_{14}^d \in U$, $L_1 + N_1$ equals $b_{11}^{d^2}\hat{L}$ for the following ideal $\hat{L}$:

$$\hat{L} = \big((g_{rj}|r \geq 4) \text{ with } c_{11} = 1\big) + D_2 + (c_{22}(b_{21} - b_{24}), c_{22}c_{3i}(b_{12} - b_{2i}b_{13})).$$

In particular, $\hat{L}$ contains D_2. Thus $S : b_{11}^{d^2}$ equals

$$\hat{L} + (c_{11} - b_{12}^{d^2}c_{12}, c_{14} - c_{11}, c_{13} - c_{12}, b_{02} - b_{12}b_{03}, b_{12} - b_{1i}, b_{01} - b_{12}^d b_{04}, c_{22}(1 - b_{2i}))$$

which decomposes as follows:

$$\bigcap_{t=2}^{n}\big(D_2 + \cdots + D_{t-1} + C_t + B_{2,t-1} + (c_{11} - b_{12}^{d^2}c_{12}) + (c_{14} - c_{11}, c_{13} - c_{12}, b_{02} - b_{12}b_{03}, b_{12} - b_{1i}, b_{01} - b_{12}^d b_{04})\big).$$

Thus $S : b_{11}^{d^2}$ contributes the following possible associated prime ideals of $K(n,d)$, for $t = 2\ldots,n$:

$$G_{21t} = D_2 + \cdots + D_{t-1} + C_t + B_{2t-1} + (c_{11} - b_{12}^{d^2}c_{12}, c_{14} - c_{11}, c_{13} - c_{12}, b_{02} - b_{12}b_{03}, b_{12} - b_{1i}, b_{01} - b_{12}^d b_{04}).$$

Similarly, $T : b_{11}^{d^2}$ equals

$$\hat{L} + C_1 + \big(b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}, b_{12} - b_{1i}, c_{22}(1 - b_{2i})\big),$$

which contributes the following possible associated prime ideals of $K(n,d)$ for $t = 2\ldots,n$:

$$G_{22t} = C_1 + D_2 + \cdots + D_{t-1} + C_t + B_{2t-1} + \big(b_{02} - b_{12}b_{03}, b_{12} - b_{1i}, b_{01} - b_{12}^d b_{04}\big).$$

And lastly,

$$\begin{aligned} V : b_{11}^{d^2} &= \hat{L} + C_1 + \big(b_{01} - b_{12}^d b_{04}, b_{02}, b_{03}, b_{11} - b_{14}\big) \\ &\quad + \big(b_{12}^d - b_{1i}^d\big) + c_{22}\left(b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j}, 1 - b_{2i}^d\right)\big) \\ &= \Big(C_1 + C_2 + \big(b_{01} - b_{12}^d b_{04}, b_{02}, b_{03}, b_{11} - b_{14}, b_{12}^d - b_{1i}^d\big)\Big) \\ &\quad \bigcap_{t=3}^{n}\Big(C_1 + D_2 + \cdots + D_{t-1} + C_t + B_{3t-1} \end{aligned}$$

$$+ \big(b_{01} - b_{12}^d b_{04}, b_{02}, b_{03}, b_{11} - b_{14}, b_{12}^d - b_{1i}^d, b_{12} - b_{2i}b_{13}, b_{2i} - b_{2j}, 1 - b_{2i}^d)\big),$$

which contributes the following possible associated prime ideals of $K(n,d)$ for $t = 3\ldots,n$, and with $\alpha^d = \beta^d = 1$:

$$\begin{aligned} G_{23,2\alpha\beta} &= C_1 + C_2 + \big(b_{01} - b_{12}^d b_{04}, b_{02}, b_{03}, b_{12} - \alpha b_{13}, b_{11} - \beta b_{13}, b_{11} - b_{14}\big)\,, \\ G_{23t\alpha\beta} &= C_1 + D_2 + \cdots + D_{t-1} + C_t + B_{3t-1} + \big(b_{01} - b_{12}^d b_{04}, b_{02}, b_{03}\big) \\ &\quad + (b_{12} - \alpha b_{13}, b_{11} - \beta b_{13}, b_{11} - b_{14}, b_{2i} - \alpha)\,. \end{aligned}$$

Thus we have proved:

Theorem 7.6: *For $n \geq 3$,*

$$\begin{aligned} Ass\left(\frac{R}{K(n,d)}\right) &\subseteq \{G_{i\Lambda}, G_j, G_{k\Lambda'} | i = 8, 10, 14, 15, 16, 17, 18; j = 6, 7, 9; k = 5, 11, 13\} \\ &\cup \{G_{12\Lambda\alpha}, G_{19\Lambda'\alpha'}, G_{20\Lambda'\alpha}\} \cup \{G_{jt}, G_{23t\alpha\beta} | j = 21, 22; t = 2\ldots,n\} \\ &\cup Ass\left(\frac{R}{K(n-1,d^2) + C_1 + (b_{01}, b_{02}, b_{03}, b_{04})}\right), \end{aligned}$$

where α and β vary over all dth roots of unity, α' varies over all d^2 roots of unity which are not dth roots of unity, Λ varies over all subsets of $\{1,2,3,4\}$, and Λ' varies over all non-empty subsets of $\{1,2,3,4\}$. ■

It remains to find the associated primes of $K(n,d)$ when $n = 2$. By Proposition 7.2, it remains to find the associated primes of $K(2,d) : b_{04}^d c_{12}$, which was computed on page 31:

$$\begin{aligned} &= b_{12}^{d^2}\big(b_{04}^d c_{11} - b_{01}^d c_{12}, b_{01}^d(c_{13} - c_{12}), (b_{12} - b_{n-1,i})b_{03}, (b_{12}^d - b_{1i}^d)b_{04}, c_{13} - c_{12}\big) \\ &\quad + (c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, b_{12} - b_{13}, b_{02} - b_{12}b_{03}, c_{12}b_{12}^{d^2}(b_{12} - b_{11})) \\ &\quad + (c_{12}b_{12}^{d^2}(b_{12} - b_{14}), b_{01} - b_{12}^d b_{04}, c_{12}b_{11} - c_{13}b_{14}, b_{11}b_{13}^{d^2} - b_{14}b_{12}^{d^2}) \\ &= b_{12}^{d^2}\big((b_{12} - b_{11})c_{12}, b_{11} - b_{14}, (b_{12} - b_{1i})b_{03}, (b_{12}^d - b_{1i}^d)b_{04}, c_{13} - c_{12}\big) \\ &\quad + (c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, b_{12} - b_{13}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}, c_{12}b_{11} - c_{13}b_{14}). \end{aligned}$$

By Fact 1.6, any associated prime of $K(2,d) : b_{04}^d c_{12}$ is associated either to $(K(2,d) : b_{04}^d c_{12}) + (b_{12}^{d^2})$ or to $K(2,d) : b_{04}^d c_{12} b_{12}^{d^2}$. The former ideal is

$$= (b_{12}^{d^2}, c_{11}, c_{14}, b_{12} - b_{13}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}, c_{12}b_{11} - c_{13}b_{14}),$$

which is a primary ideal, so this contributes only the following prime ideal to a list of possibly associated primes of $K(2,d)$:

$$G_9 = (c_{11}, c_{14}, b_{12}, b_{13}, b_{01}, b_{02}, c_{12}b_{11} - c_{13}b_{14}).$$

Next we analyze $K(2,d) : b_{04}^d c_{12} b_{12}^{d^2}$:

$$K(2,d) : b_{04}^d c_{12} b_{12}^{d^2} = \big((b_{12} - b_{11})c_{12}, b_{11} - b_{14}, (b_{12} - b_{1i})b_{03}, (b_{12}^d - b_{1i}^d)b_{04}, c_{13} - c_{12}\big)$$

$$+ (c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, b_{12} - b_{13}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}, c_{12}b_{11} - c_{13}b_{14})$$

$$= \big((b_{12} - b_{11})c_{12}, b_{11} - b_{14}(b_{12} - b_{1i})b_{03}, (b_{12}^d - b_{1i}^d)b_{04}, c_{13} - c_{12}\big)$$

$$+ (c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, b_{12} - b_{13}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}).$$

This decomposes further as (first coloning with and adding c_{12}):

$$= \Big(b_{12} - b_{1i}, c_{13} - c_{12}, c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}\Big)$$

$$\cap \Big(C_1 + (b_{11} - b_{14}, b_{12} - b_{13}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}, (b_{12} - b_{11})b_{03}, (b_{12}^d - b_{11}^d)b_{04})\Big)$$

$$= \Big(b_{12} - b_{1i}, c_{13} - c_{12}, c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}\Big)$$

$$\cap \Big(C_1 + (b_{11} - b_{1i}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04})\Big)$$

$$\cap \Big(C_1 + (b_{11} - b_{14}, b_{12} - b_{13}, b_{02}, b_{03}, (b_{01} - b_{12}^d b_{04}, (b_{12}^d - b_{11}^d)b_{04})\Big)$$

$$= \Big(b_{12} - b_{1i}, c_{13} - c_{12}, c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}\Big)$$

$$\cap \Big(C_1 + (b_{11} - b_{1i}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04})\Big)$$

$$\cap \Big(C_1 + (b_{11} - b_{14}, b_{12} - b_{13}, b_{02}, b_{03}, b_{01} - b_{12}^d b_{04}, b_{12}^d - b_{11}^d)\Big)$$

$$\cap \Big(C_1 + (b_{11} - b_{14}, b_{12} - b_{13}, b_{01}, b_{02}, b_{03}, b_{04})\Big).$$

From this it is easy to read off the associated primes:

$$G_{21,2} = \Big(b_{12} - b_{1i}, c_{13} - c_{12}, c_{11} - b_{12}^{d^2} c_{12}, c_{14} - c_{11}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}\Big)$$

$$G_{22,2} = C_1 + (b_{11} - b_{1i}, b_{02} - b_{12}b_{03}, b_{01} - b_{12}^d b_{04}),$$

$$G_{23,2,1,\alpha} = C_1 + (b_{11} - b_{14}, b_{12} - b_{13}, b_{02}, b_{03}, b_{01} - b_{12}^d b_{04}, b_{12} - \alpha b_{11}), \alpha^d = 1,$$

$$G_{24} = C_1 + (b_{11} - b_{14}, b_{12} - b_{13}, b_{01}, b_{02}, b_{03}, b_{04}.$$

This completes finding an upper bound on the set $Ass(R/K(n,d))$ for all $n \geq 2$:

Theorem 7.7: *For $n = 2$, $Ass\left(\frac{R}{K(n,d)}\right) \subseteq \{G_{5\Lambda'}, G_j, G_{i,2}, G_{23,2,1,\alpha}, G_{24} | i = 21, 22; j = 6, 7, 9\}$, where α varies all dth roots of unity.*

8. The associated primes of the Mayr-Meyer ideals

By Fact 1.4, any prime ideal associated to $K(n,d)$, after adding $C_0 + (s,f)$, is possibly associated to $J(n,d)$. In the previous section we obtained a set of prime ideals possibly associated to $K(n,d)$. This set consists of 20 variously subscripted families and the ideals associated to $K(n-1,d^2) + C_1 + (b_{01}, b_{02}, b_{03}, b_{04})$, where $K(n-1,d^2)$ involves the variables $c_{ri}, r \geq 2$, and $b_{ri}, r \geq 1$. From these families by Fact 1.4 then one easily constructs the corresponding families of prime ideals, which are possibly associated to $J(n,d)$. To list these families, as usual, Λ varies over all subsets of $\{1,2,3,4\}$ and Λ' over all non-empty subsets of $\{1,2,3,4\}$. Also, we will use the ideals

$$T_r = (s,f) + C_0 + \cdots + C_r + (b_{ti} | i = 1,\ldots,4; t = 0,\ldots,r-1).$$

With this then the list of prime ideals possibly associated to $K(n,d)$ lifts to the following prime ideals possibly associated to $J(n,d)$:

$$Q_{5r\Lambda'} = T_r + (b_{r1}, b_{r4}) + (c_{r+1,i} | i \notin \Lambda) + (b_{r2} - b_{r+1,i}b_{r3}, b_{r+1,i} - b_{r+1,j} | i,j \in \Lambda)\,,$$
$$\text{height } 8r+12, 0 \le r \le n-2,$$
$$Q_{6r} = T_r + C_{r+1} + (b_{r1}b_{r3}^{d^{2^r}} - b_{r4}b_{r2}^{d^{2^r}}), \text{height } 8r+11, 0 \le r \le n-2,$$
$$Q_{7r} = T_r + (c_{r+1,1}, c_{r+1,2}, c_{r+1,4}, b_{r1}, b_{r2}, b_{r+1,3}, b_{r+1,4}),$$
$$\text{height } 8r+13, 0 \le r \le n-2,$$
$$Q_{8r\Lambda} = T_r + (c_{r+1,1}, c_{r+1,4}, b_{r1}, b_{r2}, b_{r+1,2}, b_{r+1,3}, c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4})$$
$$+ (c_{r+2,i} | i \notin \Lambda) + (1 - b_{r+2,i} | i \in \Lambda), \text{height } 8r+17, 0 \le r \le n-3,$$
$$Q_{9r} = T_r + (c_{r+1,1}, c_{r+1,4}, b_{r1}, b_{r2}, b_{r+1,2}, b_{r+1,3}, c_{r+1,2}b_{r+1,1} - c_{r+1,3}b_{r+1,4}),$$
$$\text{height } 8r+13, 0 \le r \le n-2,$$
$$Q_{10r\Lambda} = T_r + (c_{r+1,1}, c_{r+1,3}, c_{r+1,4}, b_{r1}, b_{r2}, b_{r+1,1}, b_{r+1,2})$$
$$+ (c_{r+2,i} | i \notin \Lambda) + (b_{r+2,i} | i \in \Lambda), \text{height } 8r+17, 0 \le r \le n-3,$$
$$Q_{11r\Lambda'} = T_r + (c_{r+1,1}, c_{r+1,3}, c_{r+1,4}, b_{r1}, b_{r2}, b_{r+1,1}, b_{r+1,2}, b_{r+1,3})$$
$$+ (c_{r+2,i} | i \notin \Lambda') + (b_{r+2,i} - b_{r+2,j} | i,j \in \Lambda'), \text{height } 8r+17, 0 \le r \le n-3,$$
$$Q_{12r\Lambda\alpha} = T_r + C_1 + (b_{01}, b_{02}, b_{03}, b_{12}, b_{13}) + (c_{2i} | i \notin \Lambda) + (b_{2i} - \alpha | i \in \Lambda), \alpha^{d^{2^r}} = 1,$$
$$\text{height } 8r+19, 0 \le r \le n-3,$$
$$Q_{13r\Lambda'} = T_r + C_{r+1} + (b_{r1}, b_{r2}, b_{r3}, b_{r+1,2}, b_{r+1,3})$$
$$+ (c_{r+2,i} | i \notin \Lambda') + (b_{r+2,i} - b_{r+2,j} | i,j \in \Lambda'), \text{height } 8r+18, 0 \le r \le n-3,$$
$$Q_{14r\Lambda} = T_r + C_{r+1} + (b_{r1}, b_{r2}, b_{r3}, b_{r4}, b_{r+1,2}, b_{r+1,3}) + (c_{r+2,i} | i \notin \Lambda)$$
$$+ (b_{r+2,i} - b_{r+2,j} | i,j \in \Lambda), \text{height } 8r+19+\delta_{\lambda=\emptyset}, 0 \le r \le n-3,$$
$$Q_{15r\Lambda} = T_r + (c_{r+1,1}, c_{r+1,3} - c_{r+1,2}, c_{r+1,4}, b_{r1}, b_{r2}, b_{r+1,1}, b_{r+1,2}, b_{r+1,3}, b_{r+1,4})$$
$$+ (c_{r+2,i} | i \notin \Lambda) + (1 - b_{r+2,i} | i \in \Lambda), \text{height } 8r+19, 0 \le r \le n-3,$$
$$Q_{16r\Lambda} = T_r + C_{r+1} + (b_{r1}, b_{r2}, b_{r+1,1}, b_{r+1,2}, b_{r+1,3}, b_{r+1,4})$$
$$+ (c_{r+2,i} | i \notin \Lambda) + (1 - b_{r+2,i} | i \in \Lambda), \text{height } 8r+20, 0 \le r \le n-3,$$
$$Q_{17r\Lambda} = T_r + C_{r+1} + (b_{r1}, b_{r2}, b_{r+1,1}, b_{r+1,2}, b_{r+1,3}, b_{r+1,4}) + (c_{r+2,i} | i \notin \Lambda)$$

$$+ (b_{r+2,i} - b_{r+2,j} | i,j \in \Lambda), \text{height } 8r+19+\delta_{\lambda=\emptyset}, 0 \le r \le n-3,$$

$$Q_{18r\Lambda} = T_r + C_{r+1} + (b_{r1}, b_{r2}, b_{r3}, b_{r+1,1}, b_{r+1,2}, b_{r+1,3}, b_{r+1,4}) + (c_{r+2,i} | i \notin \Lambda)$$
$$+ (b_{r+2,i} - b_{r+2,j} | i,j \in \Lambda), \text{height } 8r+20+\delta_{\lambda=\emptyset}, 0 \le r \le n-3,$$

$$Q_{19r\Lambda'\alpha} = T_r + C_{r+1} + (b_{r1}, b_{r2}, b_{r3}, b_{r+1,1}, b_{r+1,2}, b_{r+1,3}, b_{r+1,4})$$
$$+ (c_{r+2,i} | i \notin \Lambda') + (b_{r+2,i} - \alpha | i \in \Lambda'), \alpha^{d^{2^{r+1}}} = 1, \alpha^{d^{2^r}} \ne 1,$$
$$\text{height } 8r+21, 0 \le r \le n-3,$$

$$Q_{20\Lambda'\alpha} = T_r + C_{r+1} + (b_{r1}, b_{r2}, b_{r3}, b_{r+1,1}, b_{r+1,2}, b_{r+1,3}, b_{r+1,4})$$
$$+ (c_{r+2,i} | i \notin \Lambda') + (b_{r+2,i} - \alpha | i \in \Lambda'), \alpha^{d^{2^r}} = 1,$$
$$\text{height } 8r+21, 0 \le r \le n-3,$$

$$Q_{21rt} = T_r + D_{r+2} + \cdots + D_{t-1} + C_t + B_{2t-1}$$
$$+ (c_{r+1,1} - b_{r+1,2}^{d^2} c_{r+1,2}, c_{r+1,4} - c_{r+1,1}, c_{r+1,3} - c_{r+1,2})$$
$$+ (b_{r2} - b_{r+1,2} b_{r3}, b_{r+1,2} - b_{r+1,i}, b_{r1} - b_{r+1,2}^d b_{r4}),$$
$$\text{height } 7t + r + 4\delta_{t<n}, 0 \le r \le n-2,$$

$$Q_{22rt} = T_r + C_{r+1} + D_{r+2} + \cdots + D_{t-1} + C_t + B_{2t-1}$$
$$+ \left(b_{r2} - b_{r+1,2} b_{r3}, b_{r+1,2} - b_{r+1,i}, b_{r1} - b_{r+1,2}^d b_{r4}\right),$$
$$\text{height } 7t + r + 1 + 4\delta_{t<n}, 0 \le r \le n-2,$$

$$Q_{23r,n-2\alpha\beta} = T_r + C_{r+1} + C_{r+2} + \left(b_{r1} - b_{r+1,2}^{d^{2^r}} b_{r4}, b_{r2}, b_{r3}, b_{r+1,1} - b_{r+1,4}\right)$$
$$+ \left(b_{r+1,2} - \alpha b_{r+1,3}, b_{r+1,1} - \beta b_{r+1,3}\right), \alpha^{d^{2^r}} = \beta^{d^{2^r}} = 1, \text{height } 8n,$$

$$Q_{23rt\alpha\beta} = T_r + C_{r+1} + D_{r+2} + \cdots + D_{t-1} + C_t + B_{3t-1} + \left(b_{r1} - b_{r+1,2}^d b_{r4}, b_{r2}, b_{r3}\right)$$
$$+ \left(b_{r+1,2} - \alpha b_{r+1,3}, b_{r+1,1} - \beta b_{r+1,3}, b_{r+1,1} - b_{r+1,4}, b_{r+2,i} - \alpha\right),$$
$$\alpha^{d^{2^r}} = \beta^{d^{2^r}} = 1, \text{height } 7t + r + 2 + 4\delta_{t<n}, 0 \le r \le n-3,$$

$$Q_{24} = T_{n-1} + (b_{n-1,1} - b_{n-1,4}, b_{n-1,2} - b_{n-1,3}), \text{height } 8n.$$

Thus finally:

Theorem 8.1: *With $n \ge 2$, the set of embedded primes of the Mayr-Meyer ideal $J = J(n,d)$ is contained in the set*

$$\{Q_{1\Lambda}, Q_{2\Lambda'\alpha}, Q_{3\Lambda'}, Q_{24}\} \bigcup \{Q_{4,2\alpha\beta}\delta_{n=2} | \alpha^d = \beta^d = 1, \alpha \ne \beta\}$$
$$\bigcup_{r=2}^{n} \{Q_{4r\alpha\beta\gamma}\delta_{n>2} | \alpha^d = \beta^d = \gamma^d = 1, |\{\alpha,\beta,\gamma\}| > 0\}$$
$$\bigcup_{r=0}^{n-2} \{Q_{5r\Lambda'}, Q_{jr}, Q_{krt} | j = 6,7,9; k = 21,22; t = r+2, \ldots, n\}$$

$$\bigcup_{r=0}^{n-3}\{Q_{ir\Lambda}, Q_{kr\Lambda'} | i = 8, 10, 14, 15, 16, 17, 18; k = 11, 13\}$$
$$\bigcup_{r=0}^{n-3}\{Q_{12r\Lambda\alpha}, Q_{19r\Lambda'\alpha'}, Q_{20r\Lambda'\alpha} | \alpha^{d^{2^r}} = 1, \alpha'^{d^{2^{r+1}}} = 1, \alpha'^{d^{2^r}} \neq 1\}$$
$$\bigcup_{r=0}^{n-3}\{Q_{23rt\alpha\beta} | t = r+2, \ldots, n; \alpha^{d^{2^r}} = \beta^{d^{2^r}} = 1\} \bigcup \{Q_{23,n-2,n,1\alpha} | \alpha^{d^{2^{n-2}}} = 1\},$$

where Λ *varies over all subsets of* $\{1, 2, 3, 4\}$*, and* Λ' *varies over all non-empty subsets of* $\{1, 2, 3, 4\}$*.* ■

Remark 8.2: *It was proved in Sections 2, 3 and 4 that the prime ideals* $Q_{1\Lambda}, Q_{2\Lambda'\alpha}, Q_{3\Lambda'}$*,* $Q_{42\alpha\beta}$ *are indeed associated to* J*.*

The last theorem proves that the Mayr-Meyer ideal $J(n,d)$ has at most $160n - 270 + 31d + n(n-1) + \delta_{n=2}(d^2 - d) + \delta_{n>2}(d^3 - d)(n-1) + 31(d^{2^1} + \cdots + d^{2^{n-3}}) + (n-1)d^{2^1} + (n-2)d^{2^2} + \cdots + 3d^{2^{n-3}} + 18d^{2^{n-2}}$ embedded prime ideals. Also, it proves that for all $n \geq 2$, none of the maximal ideals is associated to the Mayr-Meyer ideals.

Much is left to be done to answer the Bayer-Huneke-Stillman question. I end this paper with a list of questions:

1. Some of the prime ideals in Theorem 8.1 may not be associated to $J(n,d)$. Find the exact set of embedded primes of $J(n,d)$. In particular, determine if the set of associated primes of $J(n,d)$ is truly doubly exponential in n.
2. Determine if any of the associated prime ideals of $J(n,d)$ play a crucial role in the doubly exponential behavior. The prime ideals $Q_{23,n-2,n,1,\alpha}$ and Q_{24} are likely candidates.
3. The ideal $J(n,d)+(s,f)^2+\sum_{r=0}^{n-1}(c_{r1}, c_{r2}, c_{r3}, c_{r4})^2$ exhibits the same doubly exponential syzygetic behavior as $J(n,d)$. It has height $2+4n$, whereas $J(n,d)$ has height 2. What are the associated primes of this larger ideal?

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