SYMBOLIC POWERS OF RADICAL IDEALS

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Abstract. Hochster proved several criteria for when for a prime ideal $P$ in a commutative Noetherian ring with identity, $P^n = P^{(n)}$ for all $n$. We generalize the criteria to radical ideals.

1. Introduction.

In [1], M. Hochster established several criteria for when for a prime ideal $P$ in a Noetherian ring $R$, the $n$th power $P^n$ of $P$ equals the $n$th symbolic power $P^{(n)}$ of $P$ for every positive integer $n$. He used a so-called test sequence of ideals in a polynomial ring over $R$ to determine whether $P^n = P^{(n)}$ for all $n$. We extend Hochster’s criteria to radical ideals.

Here is the set-up: let $R$ be a Noetherian domain and $P$ an ideal of $R$. Suppose that $\{a_1, a_2, \ldots, a_m\}$ is a generating set for $P$. Write the $m$-tuple as $\underline{p} = (a_1, a_2, \ldots, a_m)$. Let $S = R[x_1, x_2, \ldots, x_m]$, where $x_1, x_2, \ldots, x_m$ are indeterminates over $R$.

Definition 1.1. For an ideal $P = (a_1, \ldots, a_m)R$ of $R$, define recursively ideals of $S = R[x_1, \ldots, x_m]$:

$J_0(\underline{p}) = 0$ and $J_{n+1}(\underline{p}) = (\{\Sigma_{i=1}^m s_i x_i \mid s_i \in S \text{ and } \Sigma_{i=1}^m s_i a_i \in J_n(\underline{p})\})S$ for $n \geq 0$. We write $J_n$ for $J_n(\underline{p})$ and denote $J = \cup_{n=1}^\infty J_n$. We call the sequence of ideals

$PS + J_0, \; PS + J_1, \; \ldots, \; PS + J_n, \; \ldots$

the test sequence of the $m$-tuple $\underline{p}$.

Note that for each $n$, $J_n \subseteq J_{n+1}$. Since $R$ is Noetherian, $J = J_n$ for all large $n$. Hochster proved:

Theorem 1.2. [1, Theorem 1] With the above notation, the following are equivalent for a prime ideal $P$ in a Noetherian domain $R$:

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A. The associated graded ring of $R_P$ is a domain, and for every positive integer $n$, the $n^{th}$ symbolic and ordinary powers of $P$ agree.

B. The ideal $PS + J$ is prime.

C. For some integer $n$, $PS + J_n$ is a prime ideal of height $m$.

D. There is a height-$m$ prime ideal $Q$ of $S$ such that $Q \subseteq PS + J$.

E. Let $z$ be an indeterminate over $R$. Then $z$ is a prime element in the subring $R[z, a_1/z, \ldots, a_m/z]$ of $R[z, 1/z]$.

As a generalization, we analyze the situation in which $P$ is a radical ideal of a reduced Noetherian ring. We first define generalized symbolic powers of ideals. We then give some criteria regarding the equality of $P^n$ and $P^{(n)}$.

2. SOME BASIC RESULTS ABOUT TEST SEQUENCES

We start with some useful examples of test sequences:

**Lemma 2.1.** Let $R$ be a Noetherian ring and $P$ an ideal generated by a regular sequence $a_1, a_2, \ldots, a_m$. For the $m$-tuple $\mathbf{p} = (a_1, a_2, \ldots, a_m)$, denote $J_k = J_k(\mathbf{p})$. Then

$$J_1 = (x_j a_k - x_k a_j \mid 1 \leq j < k \leq m)S = J_2 = J_3 = \cdots = J.$$

**Proof.** The generators of $J_1$ are of the form $\sum_i s_i x_i$ such that $\sum_i s_i a_i = 0$. As $a_1, a_2, \ldots, a_m$ is a regular sequence, this means that the element $(s_1, \ldots, s_m) \in S^m$ is in the $S$-module generated by the Koszul relations $(0, \ldots, a_j, \ldots, -a_k, \ldots, 0)$, with $k < j$ and at most the $k$th and $j$th entries non-zero. Thus $J_1$ is generated by elements of the form $x_j a_k - x_k a_j$. It remains to prove that $J_1 = J_2$.

Let $\sum_i s_i x_i \in J_2$ with $\sum_i s_i a_i \in J_1$. Write $\sum_i s_i a_i = \sum_{j<k} l_{jk} (x_j a_k - x_k a_j)$ for some $l_{jk} \in S$. Then

$$\sum_{i=1}^m \left( s_i - \sum_{j=1}^{i-1} l_{ji} x_j + \sum_{k=i+1}^m l_{ik} x_k \right) a_i = 0,$$

so that

$$\sum_{i=1}^m s_i x_i = \sum_{i=1}^m \left( s_i - \sum_{j=1}^{i-1} l_{ji} x_j + \sum_{k=i+1}^m l_{ik} x_k \right) x_i \in J_1. \quad \square$$

In general, when the generating sequence does not form an $R$-sequence, the ideal $J_2$ may be bigger than $J_1$. One such example is given below:

**Example 2.2.** Let $R = k[y_1, y_2]$ be a polynomial in two variables over a field $k$. Let $P = (a_1, a_2, a_3)R$, where $a_1 = y_1^2$, $a_2 = y_1 y_2$, and $a_3 = y_2^2$. The generating sequence $(a_1, a_2, a_3)$ is not a regular sequence of $R$. In addition, $J_2 \neq J_1$.

**Proof.** The module of relations on $a_1, a_2, a_3$ in $S = R[x_1, x_2, x_3]$ is generated by $(y_2, -y_1, 0)$ and $(0, y_2, -y_1)$, so that $J_1 = (y_2 x_1 - y_1 x_2, y_2 x_2 - y_1 x_3)S \subseteq (y_1, y_2)S$. 

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The element $x_1x_3-x_2^2$ is therefore not in $J_1$. But $x_1x_3-x_2^2 \in J_2$ as $x_1y_2^2-x_2y_1y_2 = y_2(x_1y_2-y_1x_2) \in J_1$. \qed

Now let $S_r = R[x_1, \ldots, x_r]$ and consider an $r$-tuple $\mathbf{p}_r = (a_1, \ldots, a_r)$, where $a_1, \ldots, a_r \in R$. Similar as in Definition 1.1, we denote

$$J_{k+1}(\mathbf{p}_r) = (\{\Sigma_{i=1}^r s_i x_i \mid s_i \in S_r \text{ and } \Sigma_{i=1}^r s_i a_i \in J_k(\mathbf{p}_r)\}) S_r.$$

Lemma 2.3. Let $R$ be a Noetherian ring and $S = R[x_1, \ldots, x_m]$. Let $P = (a_1, a_2, \ldots, a_m)R$ be an ideal of $R$ and $\mathbf{p}_m = (a_1, a_2, \ldots, a_m)$. If $\sum_{i=r+1}^k g_i x_i = 0$, where $g_{r+1}, \ldots, g_k \in S$ and $r + 1 \leq k \leq m$, then $\sum_{i=r+1}^k g_i a_i \in J_1(\mathbf{p}_m)$.

Proof. It is trivial when $k = r + 1$. For $k > r + 1$, $\sum_{i=r+1}^k g_i x_i = 0$ implies $g_k = \sum_{i=r+1}^{k-1} h_i x_i$ for some $h_i \in S$ since $x_k$ is a regular element of $S$. Thus $\sum_{i=r+1}^{k-1} (g_i + h_i x_k) x_i = 0$. By induction hypothesis, $\sum_{i=r+1}^{k-1} (g_i + h_i x_k) a_i \in J_1(\mathbf{p}_m)$. On the other hand,

$$\sum_{i=r+1}^{k-1} (g_i + h_i x_k) a_i = \sum_{i=r+1}^{k-1} g_i a_i + \sum_{i=r+1}^{k-1} h_i (x_k a_i - x_i a_k) + \sum_{i=r+1}^{k-1} h_i x_i a_k$$

$$= \sum_{i=r+1}^k g_i a_i + \sum_{i=r+1}^{k-1} h_i (x_k a_i - x_i a_k) \in J_1(\mathbf{p}_m).$$

Since each $x_k a_i - x_i a_k$ is an element of $J_1(\mathbf{p}_m)$, $\sum_{i=r+1}^k g_i a_i \in J_1(\mathbf{p}_m)$. \qed

Lemma 2.4. Let $R$ be a Noetherian ring and $P = (a_1, a_2, \ldots, a_m)R$, an ideal of $R$. Assume $a_m = \sum_{i=1}^{m-1} b_i a_i$, where each $b_i \in R$. For the $m$-tuple $\mathbf{p}_m = (a_1, a_2, \ldots, a_m)$ and the $(m-1)$-tuple $\mathbf{p}_{m-1} = (a_1, a_2, \ldots, a_{m-1})$,

$$J_k(\mathbf{p}_m) = \left(J_k(\mathbf{p}_{m-1}) + (x_m - \sum_{i=1}^{m-1} b_i x_i)\right) R[x_1, \ldots, x_m]$$

and

$$J_k(\mathbf{p}_m) \cap R[x_1, \ldots, x_{m-1}] = J_k(\mathbf{p}_{m-1})$$

for all $k \geq 1$.

Proof. Let $\sum_{i=1}^m s_i x_i \in J_k(\mathbf{p}_m)$ such that $\sum_{i=1}^m s_i a_i \in J_{k-1}(\mathbf{p}_m)$. We want to show that $\sum_{i=1}^m s_i x_i$ is contained in the ideal generated by $J_k(\mathbf{p}_{m-1})$ and $x_m - \sum_{i=1}^{m-1} b_i x_i$ in $R[x_1, \ldots, x_m]$. We can write $\sum_{i=1}^m s_i x_i = \sum_{i=1}^{m-1} t_i x_i + (x_m - \sum_{i=1}^{m-1} b_i x_i)s$ for some $s \in S$ and $t_i \in R[x_1, \ldots, x_{m-1}]$. It suffices to prove that $\sum_{i=1}^{m-1} t_i x_i$ is in $J_k(\mathbf{p}_{m-1})$, or more generally that $J_k(\mathbf{p}_m) \cap R[x_1, \ldots, x_{m-1}] \subseteq J_k(\mathbf{p}_{m-1})$. \qed
Let \( f \in J_k(\mathfrak{p}_m) \cap R[x_1, \ldots, x_{m-1}] \). We may write \( f = \sum_{i=1}^{m} s_i x_i \) such that \( \sum_{i=1}^{m} s_i a_i \in J_{k-1}(\mathfrak{p}_m) \). For each \( i = 1, \ldots, m-1 \), we write \( s_i = t_i + f_i x_m \), where \( t_i \in R[x_1, \ldots, x_{m-1}] \) and \( f_i \in S \). Then \( \sum_{i=1}^{m} s_i x_i = \sum_{i=1}^{m-1} t_i x_i + x_m(s_m + \sum_{i=1}^{m-1} f_i x_i) \in R[x_1, \ldots, x_{m-1}] \) implies that \( s_m + \sum_{i=1}^{m-1} f_i x_i = 0 \) and \( \sum_{i=1}^{m} s_i a_i = \sum_{i=1}^{m-1} t_i a_i \in J_{k-1}(\mathfrak{p}_m) \cap R[x_1, \ldots, x_{m-1}] \). If \( k = 1 \), this says that \( \sum_{i=1}^{m-1} t_i a_i = 0 \in J_{k-1}(\mathfrak{p}_{m-1}) \), and if \( k > 1 \), then by induction \( \sum_{i=1}^{m-1} t_i a_i \in J_{k-1}(\mathfrak{p}_{m-1}) \). Thus for all \( k \geq 1 \), \( \sum_{i=1}^{m} s_i x_i = \sum_{i=1}^{m-1} t_i x_i \in J_k(\mathfrak{p}_{m-1}) \).

As a generalization of Lemma 2.1, we have

**Theorem 2.5.** Let \( R \) be a Noetherian ring and \( P = (a_1, \ldots, a_r)R \) an ideal of \( R \) which is also generated by \( a_1, \ldots, a_r \), where \( 0 < r < m \). Let \( \mathfrak{p}_m \) and \( \mathfrak{p}_r \) be as before. If \( a_1, a_2, \ldots, a_r \) forms a regular \( R \)-sequence, then

\[
J_k(\mathfrak{p}_m) = J_1(\mathfrak{p}_m)
\]

for all \( k \geq 1 \).

**Proof.** Since \( \{a_1, a_2, \ldots, a_r\} \) is a generating set of \( P \), for each \( i = r + 1, \ldots, m \), we can write \( a_i = \sum_{j=1}^{r} b_j a_j \) for some \( b_j \in R \). Let \( S = R[x_1, \ldots, x_m] \). Set \( c_i = x_i - \sum_{j=1}^{r} b_j x_j \in J_1(\mathfrak{p}_m) \) for each \( i = r + 1, \ldots, m \). By repeated application of Lemma 2.4, for all \( k \geq 1 \),

\[
J_k(\mathfrak{p}_m) = (J_k(\mathfrak{p}_r) + (c_{r+1}, \ldots, c_m)) S.
\]

By Lemma 2.1, \( J_k(\mathfrak{p}_r) = J_1(\mathfrak{p}_r) \) for all \( k \geq 1 \), which finishes the proof. \( \Box \)

This gives some information on the test sequence of prime ideals in a regular ring:

**Theorem 2.6.** Let \( R \) be a regular ring and \( P \) a prime ideal in \( R \). Then there exists a generating set \( \{a_1, \ldots, a_m\} \) of \( P \) such that with \( \mathfrak{p} = (a_1, \ldots, a_m) \), for all integers \( k \geq 1 \), \( J_k(\mathfrak{p}) R_P = J_1(\mathfrak{p}) R_P \).

More generally, whenever \( P \) is an ideal and \( U \) a multiplicatively closed subset such that \( U^{-1}P \) is generated by a regular sequence, there exists a generating set \( \{a_1, \ldots, a_m\} \) of \( P \) such that with \( \mathfrak{p} = (a_1, \ldots, a_m) \), for all integers \( k \geq 1 \), \( U^{-1} J_k(\mathfrak{p}) = U^{-1} J_1(\mathfrak{p}) \).

**Proof:** As \( U^{-1}P \) is generated by a regular sequence, there exists a generating set such that the first \( r \) generators form a maximal regular sequence after localization at \( U \). Let \( J_k(\mathfrak{p}) \) be the corresponding \( k \)-th test ideal of \( U^{-1}R \) for \( \mathfrak{p} \). Clearly \( U^{-1} J_k(\mathfrak{p}) = J_k(\mathfrak{p}) \). By Theorem 2.5, \( J_k(\mathfrak{p}) = J_1(\mathfrak{p}) \). Thus \( U^{-1} J_k(\mathfrak{p}) = U^{-1} J_1(\mathfrak{p}) \).

The first part follows as in a regular ring, \( PR_P \) is generated by a regular sequence. \( \Box \)
3. Criteria for Radical Ideals

In this section we generalize Hochster’s criterion to radical ideals, see Theorem 3.6.

Recall that $S = R[x_1, \ldots, x_m]$ and that $J_k = J_k(p)$ refers to the $k$th test ideal with respect to the $m$-tuple $p = (a_1, \ldots, a_m)$. Clearly if $U$ is a multiplicatively closed subset of $R$, then $U^{-1}J_k(p) = J_k(U^{-1}(p))$.

**Definition 3.1.** Let $R$ be a reduced Noetherian ring, $P$ an ideal of $R$ and $U$ a multiplicatively closed subset of $R$. We define the $n$th generalized symbolic power of $P$ with respect to $U$ to be $P^{(n)} = P^nU^{-1}R \cap R$.

If $P$ is a radical ideal with the minimal primes $p_1, p_2, \ldots, p_t$, then the $n$th generalized symbolic power of $P$ with respect to $U = R \setminus (p_1 \cup \cdots \cup p_t)$ is called the $n$th symbolic power of $P$.

In the proofs we will use the extended Rees algebra of $P$: $R' = R\left[z, \frac{P}{z}\right] = R\left[z, \frac{a_1}{z}, \frac{a_2}{z}, \ldots, \frac{a_m}{z}\right]$, where $z$ is an indeterminate over $R$. Note that $R'/zR' \cong \frac{R}{P} \oplus \frac{P}{P^2} \oplus \frac{P^2}{P^3} \oplus \cdots$, the associated graded ring of $P$.

For a ring $A$, we denote by $\mathcal{Z}(A)$ the set of all zero divisors of $A$. The following is well-known:

**Remark 3.2.** Let $R$ be a reduced Noetherian ring, $P$ an ideal of $R$, and $R'$ as above. Let $U$ be a multiplicatively closed set of $R$. Then

1. $\mathcal{Z}(A)$ is the union of all associated prime ideals of $A$.
2. For each $n \geq 0$, $P^n = z^n R' \cap R$, and $P^nU^{-1}R \cap R = z^nU^{-1}R' \cap R$.
3. For a fixed $n > 0$, $P^n = P^nU^{-1}R \cap R$ if and only if $(P^n:R_u) = P^n$ for all $u \in U$. In particular, $P = PU^{-1}R \cap R$ if $U \cap \mathcal{Z}(R/P) = \emptyset$.
4. If $U \cap \mathcal{Z}(R'/zR') = \emptyset$, then $zU^{-1}R' \cap R' = zR'$ and $\text{Rad}(zU^{-1}R') \cap R' = \text{Rad}(zR')$.

Our goal is to give similar criteria as those in [1] for radical ideals. First we establish some lemmas.

**Lemma 3.3.** Let $R$ be a Noetherian ring and $P = (a_1, a_2, \ldots, a_m)R$ an ideal. Let $R'$, $S$ and $J$ be as above. Then $R'/zR'$ is isomorphic to $S/(J + PS)$.

In particular, $PS + J$ is a radical ideal if and only if $zR'$ is a radical ideal.
Proof. Consider the surjective $R$-homomorphism $\phi$ from $S$ to $R'/zR'$, shown as composition below:

$$
\phi : \quad S \xrightarrow{\phi'} R' \xrightarrow{zR'} \frac{R'}{zR'} \cong \frac{R}{P} \oplus \frac{P}{P^2} \oplus \frac{P^2}{P^3} \oplus \cdots
$$

$$
\quad x_i \mapsto \frac{a_i}{z} \mapsto \frac{a_i + P^2}{P^2}.
$$

It suffices to prove that $\ker(\phi) = PS + J$. Note that each $a_i$ maps to 0 in $R/P$, so that $PS \subseteq \ker(\phi)$. Clearly $\phi'(J_1) = 0$. Suppose that $\phi'(J_n) = 0$. Let $\sum s_i x_i \in J_{n+1}$ be such that $\sum s_i a_i \in J_n$. As $z\phi'(\sum s_i x_i) = \phi'(\sum s_i a_i) = 0$, it follows that $\phi'(\sum s_i x_i) = 0$. Thus by induction, $J \subseteq \ker(\phi') \subseteq \ker(\phi)$. This proves that $PS + J \subseteq \ker(\phi)$. To prove the opposite inclusion, let $f \in \ker(\phi)$. As $\phi$ is a graded homomorphism and $PS + J$ is a homogeneous ideal, it suffices to assume that $f$ is a homogeneous element of $S$ of degree $d$. Write $f = \sum_{|\nu| = d} f_\nu x^\nu$ for some $f_\nu \in R$. As $f \in \ker(\phi)$, this means that $\sum_{|\nu| = d} f_\nu a^\nu \in P^{d+1}$. Write $\sum_{|\nu| = d} f_\nu a^\nu = \sum_{i=1}^m \sum_{|\mu| = d} r_{i\mu} a^\mu a_i$ for some $r_{i\mu} \in R$. By definition of test sequences then $\sum_{|\nu| = d} f_\nu x^\nu - \sum_{i=1}^m \sum_{|\mu| = d} r_{i\mu} x^\mu a_i \in J_d$, whence

$$
f = \sum_{|\nu| = d} f_\nu x^\nu - \sum_{i=1}^m \sum_{|\mu| = d} r_{i\mu} x^\mu a_i + \sum_{i=1}^m \sum_{|\mu| = d} r_{i\mu} x^\mu a_i \in J_d + PS \subseteq PS + J. \quad \square
$$

Lemma 3.4. Let $R$ be a Noetherian ring and $P$ an ideal. Let $U$ be an arbitrary multiplicatively closed subset of $R$. Then the following are equivalent:

1. $P^n U^{-1} R \cap R = P^n$ for every positive integer $n$, and the associated graded ring $\text{gr}_{U^{-1} P}(U^{-1} R)$ is reduced.
2. $zR'$ is a radical ideal and $U \cap \mathcal{Z}(R'/zR') = \emptyset$.

Proof. Assume the first statement. We first show that $U \cap \mathcal{Z}(R'/zR') = \emptyset$. Let $u \in U$ and $b \in R'$ such that $ub \in zR'$. Without loss of generality $b$ is a homogeneous element of $R'$ under the grading determined by the variable $z$. Thus we may write $b = b_0 z^{-n}$ for some integer $n$ and some $b_0 \in P^n$. If $n$ is negative, this means that $b_0 \in R$, $ub_0 \in P$, so that by assumption, $b_0 \in P$, whence $b = zR'$. Now let $n \geq 0$. Then $ub_0 \in z^{n+1} R' \cap R = P^{n+1}$ by Remark 3.2 (2). This implies that $b_0 \in P^{n+1} U^{-1} R \cap R = P^{n+1} = z^{n+1} R' \cap R'$, so that $b_0 \in z^{n+1} R'$ and thus $b \in zR'$. Hence $U \cap \mathcal{Z}(R'/zR') = \emptyset$.

By the assumption that the associated graded ring of $U^{-1} P$ is reduced and as $\text{gr}_P(R) = R'/zR'$, it follows that $zU^{-1} R'$ is a radical ideal. Thus by Remark 3.2 (4), $zR' = zU^{-1} R' \cap R' = \text{Rad}(zU^{-1} R') \cap R' = \text{Rad}(zR')$, so $zR'$ is a radical ideal of $R'$.

Next assume that the second statement holds. As $zR'$ is a radical ideal, $\text{gr}_P(R)$ is reduced, and so trivially $\text{gr}_{U^{-1} P}(U^{-1} R)$ is reduced.
Let \( b \in P^nU^{-1}R \cap R = z^nU^{-1}R' \cap R \). There exists \( u \in U \) such that \( ub \in z^nR' \). We have to prove that \( b \in P^n \). If not, then there exists an integer \( k < n \) such that \( b \in P^k \) and \( b \not\in P^{k+1} \). Then \( \frac{b}{z^n} \in R' \) and \( u \cdot \frac{b}{z^n} = \frac{ub}{z^n} \cdot z^{n-k} \in zR' \). Since \( u \) is not a zero divisor of \( R'/zR' \), then \( \frac{b}{z^n} \in zR' \), so that \( b \in z^{k+1}R' \cap R = P^{k+1} \), a contradiction. Thus necessarily \( k \geq n \) and \( b \in P^k \subseteq P^n \). \( \square \)

Lemma 3.5. Let \( P, S, J \) be as in the set-up, with \( P \) presented with \( m \) generators. Then all of the minimal primes of \( PS + J \) are of height \( m \). In particular, \( \text{ht}(PS + J) = m \).

Proof. Let \( \psi \) be the \( R[z] \)-homomorphism of \( S[z] \) onto \( R' = R[z, P/z] \) which takes \( x_i \) to \( \frac{x_i}{z} \) for each \( i \). Let \( I = \ker(\psi) \) and \( I_0 = (a_1 - x_1z, a_2 - x_2z, \ldots, a_m - x_mz)S[z] \), both ideals of \( S[z] \). Obviously, \( I_0 \subseteq I \). After inverting \( z \), both \( I \) and \( I_0 \) are generated by the regular sequence \( a_1 - x_1z, \ldots, a_m - x_mz \), so that \( I = \cup_{n \geq 0}(I_0 : z^n) \). This implies that \( z \) is not a zero divisor on \( S[z]/I \). It is easy to check that \( PS + J = (I + zS[z]) \cap S \).

We claim that every minimal prime of \( I \) is of height \( m \). When going up to the localization \( S[z, 1/z] \) of \( S[z] \) localized at \( z \), the minimal primes of \( I \) in \( S[z] \) correspond to the minimal primes of \( IS[z, 1/z] \) in \( S[z, 1/z] \) and the heights do not change since \( z \) is not a zero divisor of \( S[z]/I \). But \( IS[z, 1/z] = I_0S[z, 1/z] = (x_1 - a_1/z, x_2 - a_2/z, \ldots, x_m - a_m/z)S[z, 1/z] \), and obviously all of the minimal primes of \( I_0S[z, 1/z] \) are of height \( m \). Thus all the minimal primes of \( I \) in \( S[z] \) are of height \( m \). In addition, all minimal primes of \( (I + zS[z])S[z] \) are of height \( m + 1 \), again because \( z \) is not a zero divisor of \( S[z]/I \).

Let \( q \) be a minimal prime of \( PS + J \) in \( S \). In the polynomial ring \( S[z] \) over \( S \), \( qS[z] + zS[z] \) is a minimal prime of \( (PS + J + zS[z])S[z] = (I + zS[z])S[z] \), and so \( m + 1 = \text{ht}(qS[z] + zS[z]) = \text{ht}(qS) + 1 \). Hence \( \text{ht}(qS) = m \). \( \square \)

Now we give similar criteria as those in [1] for radical ideals:

Theorem 3.6. Let \( R \) be a reduced Noetherian ring and \( P = (a_1, \ldots, a_m) \), a radical ideal of \( R \). Let \( U = R \setminus (p_1 \cup \cdots \cup p_t) \) and \( S, z \) be as above. Recall that \( R' = R[z, Pz^{-1}] \). The following statements are equivalent:

A'. For every integer \( n > 0 \), \( P^n = P^{(n)} \), and the associated graded ring \( \text{gr}_{U^{-1}P}(U^{-1}R) \) is reduced.

B'. The ideal \( PS + J \) is a radical ideal of \( S \) and \( U \cap \mathcal{Z}(S/(PS + J)) = \emptyset \).

C'. For some positive integer \( n \), \( PS + J_n \) is a radical ideal of height \( m \) which has the same number of minimal primes as \( PS + J \) has, and \( U \cap \mathcal{Z}(S/(PS + J_n)) = \emptyset \). In this case, \( PS + J_n = PS + J \).

D'. The ideal \( PS + J \) contains a height-\( m \) radical ideal \( Q \) which has the same number of minimal primes as \( PS + J \) has, and \( U \cap \mathcal{Z}(S/Q) = \emptyset \). In this case, \( Q = PS + J \).

E'. The ideal \( zR' \) is a radical ideal of \( R' \) and \( U \cap \mathcal{Z}(R'/zR') = \emptyset \).
Proof. Lemma 3.4 gives the equivalence of A' and E' by setting $U = R \setminus (p_1 \cup \cdots \cup p_t)$. By the isomorphism in Lemma 3.3, B' and E' are equivalent.

By Lemma 3.5, all the minimal primes of $PS + J$ are of height $m$. If an ideal $Q$ of height $m$ is contained in $PS + J$ and has the same number of minimal primes as $PS + J$ does, then the minimal primes of $PS + J$ are exactly the minimal primes of $Q$. Thus $\text{Rad}(Q) = \text{Rad}(PS + J)$. Furthermore, if $Q$ is radical, then $Q = \text{Rad}(PS + J) \supseteq PS + J$, so that $Q = PS + J$. Whence the equivalences of B', C', and D' follow trivially.

Now it is clear that the statements A', B', C', D', and E' are all equivalent. □

**Remark 3.7.** Let $R$ be an integral domain, $P$ a prime ideal, and $U = R \setminus P$. The statements A' - E' are equivalent to the statements A - E in Theorem 1.2, respectively.

Proof. It is enough to show that the condition $U \cap \mathcal{Z}(R'/zR') = \emptyset$ in E' can be dropped with this special setting. From the isomorphism $\frac{R'}{zR'} \cong \frac{R}{P} \oplus \frac{R}{P^2} \oplus \cdots = \text{gr}_P R$, it is sufficient to show that $U \cap \mathcal{Z}(\text{gr}_P R) = \emptyset$. Let $b \in \text{gr}_P (R)$ be a non-zero homogeneous element of degree $n$, and let $ub = 0$ in $\text{gr}_P (R)$ for some $u \in U$. By assumption $zR'$ is an integral domain, i.e., $\text{gr}_P (R)$ is an integral domain. Since $b$ is non-zero, necessarily $u$ must be zero, i.e., $u \in P$, which contradicts its choice. □

We give two applications of Theorem 3.6.

**Corollary 3.8.** Let $R$ be a reduced Noetherian ring and $P$ a radical ideal generated by an $R$-sequence. Then $P^n = P^{(n)}$ for every positive integer $n$.

Proof. Assume that $P = (a_1, a_2, \ldots, a_m)R$, where $a_1, a_2, \ldots, a_m$ is an $R$-sequence. As in Theorem 3.6, we set $S = R[x_1, x_2, \ldots, x_m]$ and $U = R \setminus (p_1 \cup \cdots \cup p_t)$, where $p_1, p_2, \ldots, p_t$ are the minimal primes of $P$ in $R$.

Then $PS = (a_1, a_2, \ldots, a_m)S$ is a radical ideal of $S$ with the minimal primes $p_1 S, p_2 S, \ldots, p_t S$ in $S$. Furthermore, $(a_1, a_2, \ldots, a_m)$ is an $S$-sequence. For each $i$, $p_i S$ is of height $m$ because it is minimal over an ideal generated by an $S$-sequence of $m$ elements.

By Lemma 2.1, $J \subseteq PS$. So $PS + J = PS$. Furthermore, the isomorphism $S/PS \cong (R/P)[x_1, x_2, \ldots, x_m]$ implies that $U \cap \mathcal{Z}(S/PS) = \emptyset$. So the condition B' in Theorem 3.6 is satisfied. Therefore $P^n = P^{(n)}$ for every positive integer $n$. □

**Proposition 3.9.** Let $Y = (y_{ij})$ be a $(2 \times r)$ matrix of indeterminates ($r > 1$) and $R = k[[y_{ij}]]$ be the polynomial ring over a field $k$. Let $P$ be the ideal generated by the $2 \times 2$ permanents of $Y$, i.e.,

$P$ is generated by elements of form $y_1 y_2 + y_2 y_1$ ($i \neq j$). Then

(1) If $r = 2$ or $3$, $P^n = P^{(n)}$ for all $n \in \mathbb{N}$;

(2) If $r > 3$, there exists a positive integer $n$ such that $P^n \neq P^{(n)}$. 

Proof. It is shown in [3, Theorem 4.1] that \( P \) is a radical ideal with \( \text{ht}(P) = \min\{r, 2r - 3\} \) for \( r \geq 3 \), so that clearly \( \text{ht}(P) = \min\{r, 2r - 3\} \) for \( r \geq 2 \). For case \( r = 2 \) and \( r = 3 \), the number of generators of \( P \) is equal to the height of \( P \), so that the generating set of permanents forms a regular sequence. It follows from Corollary 3.8 that \( P^n = P^{(n)} \) for all \( n \).

For (2), suppose that \( P = (a_1, a_2, \ldots, a_{n(n-1)/2}) \), where \( a_1, a_2, \ldots, a_{n(n-1)/2} \) are the generating permanents and \( a_1 = y_{11}y_{22} + y_{21}y_{12} \). In [3] it is shown that \( P \) contains all products of three indeterminates chosen from three different columns but not from the same row. For example, both \( y_{11}y_{22}y_{23} \) and \( y_{21}y_{13}y_{24} \) are elements of \( P \). Let

\[
\alpha = y_{13}y_{23}y_{24}a_1 = y_{13}y_{23}y_{24}(y_{11}y_{22} + y_{21}y_{12}).
\]

Then \( \alpha \in P \). In addition, \( \alpha \notin P^2 \). This can be easily checked by Macaulay2. However, \( \alpha^2 \in P^3 \). This is because

\[
\alpha^2 = y_{23}(y_{11}y_{22}y_{13})(y_{11}y_{24}y_{23})(y_{13}y_{24}y_{22}) + 2y_{13}(y_{13}y_{22}y_{21})(y_{23}y_{24}y_{12})(y_{11}y_{24}y_{23}) + y_{13}(y_{13}y_{21}y_{24})(y_{23}y_{12}y_{21})(y_{24}y_{12}y_{23})
\]

and by above each of the nine elements in parentheses is in \( P \). So we can represent \( \alpha^2 \) as \( \alpha^2 = \sum_{i_1i_2i_3} l_{i_1i_2i_3}a_{i_1}a_{i_2}a_{i_3} \) with \( l_{i_1i_2i_3} \in R \). Let \( \beta = [(y_{13}y_{23}y_{24})^2x_1]\ x_1 - \sum_{i_1i_2i_3} (l_{i_1i_2i_3}a_{i_1}a_{i_2}a_{i_3})x_{i_3} \in S \). Note that \( [(y_{13}y_{23}y_{24})^2a_1]a_1 - \sum_{i_1i_2i_3} (l_{i_1i_2i_3}a_{i_1}a_{i_2}a_{i_3})a_{i_3} = \alpha^2 - \alpha^2 = 0 \), so \( [(y_{13}y_{23}y_{24})^2a_1]x_1 - \sum_{i_1i_2i_3} (l_{i_1i_2i_3}a_{i_1}a_{i_2})x_{i_3} \in J_1 \), which implies that \( \beta = [(y_{13}y_{23}y_{24})^2x_1]\ x_1 - \sum_{i_1i_2i_3} (l_{i_1i_2i_3}a_{i_1}a_{i_2})x_{i_3} \in J_2 \subseteq J \). This implies that \( (y_{13}y_{23}y_{24}x_1)^2 = \beta + \sum_{i_1i_2i_3} (l_{i_1i_2i_3}a_{i_1}a_{i_2})x_{i_3} \in J + PS \), i.e., \( y_{13}y_{23}y_{24}x_1 \in \sqrt{J + PS} \).

However, under the homomorphism from Lemma 3.3, \( y_{13}y_{23}y_{24}x_1 \) is sent to the element \( (y_{13}y_{23}y_{24}a_1 + P^2)/P^2 \) in the graded ring \( grPR \), which is nonzero. So \( y_{13}y_{23}y_{24}x_1 \) is not in the kernel \( J + PS \). Therefore, \( J + PS \) is not a radical ideal of \( S \). By Theorem 3.6, \( P^n \neq P^{(n)} \) for some positive integer \( n \). \( \square 

Example 3.10. Let \( k \) be a field and \( R = k[x, y, z] \), where \( x, y, z \) are indeterminates over \( k \). Let \( P = (x, y) \cap (x - 1, z) \cap (y, 1 - zx) \), a radical ideal. Then \( P^n = P^{(n)} \) for all positive integers \( n \).

Proof. Obviously, the three prime ideals \( p_1 = (x, y) \), \( p_2 = (x - 1, z) \), and \( p_3 = (y, 1 - zx) \) are comaximal and each of them is generated by an \( R \)-sequence. By Corollary 3.8, \( p_i^n = p_i^{(n)} \) for all positive integers \( n \) and for \( i = 1, 2, 3 \). Thus \( P^n = P^{(n)} \) for all \( n \).

An application of Corollary 3.8 shows also the following:

Example 3.11. Let \( k \) be a field and \( R = k[x, y, z, u, v]/(xy - uy) \), where \( x, y, z, u, v \) are indeterminates over \( k \), and let \( P = (xy - u, yz) \). Then \( P^n = P^{(n)} \) for all positive integers \( n \).
REFERENCES


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