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Game Theory and Experimental Games,

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Two-Person, Zero-Sum Games

4.1 Strictly Competitive Games

A ZERO-SUM game, as the term suggests, is one in which the payoffs to the players in any outcome add up to zero; what one player gains, the other(s) must necessarily lose. A zero-sum game, in other words, is a closed system within which nothing of value to the players is created or destroyed: utilities merely "change hands" when the game is played. If there are just two players, this means that their interests are diametrically opposed, because an outcome that is favourable for one is bound to be correspondingly unfavourable for the other and vice versa. Since one player can gain only at the expense of the other, there are no prospects of mutually profitable collaboration, and models of two-person zero-sum conflicts are therefore described as *strictly competitive games*. They have proved especially amenable to formal analysis, and the most significant contributions to mathematical game theory pertain to them.

A wide variety of economic, political, military, and interpersonal conflicts correspond to strictly competitive games. Most two-person sporting contests and indoor games are strictly competitive in the zero-sum sense, and examples from less frivolous spheres of life are not difficult to find. Two television networks competing for audiences, two politicians or political parties competing for votes, two armies competing for territory, or two parents competing for the custody of their child after a divorce may have diametrically opposed interests which can reasonably be modelled by a two-person, zero-sum game. It is a common mistake, however, to regard all competitive interactions as zero-sum, since in reality the protagonists' interests are seldom strictly opposed. In most wars, for example, an outcome involving mutual annihilation with no gains on either side represents a loss for both contenders. Wars are seldom zero-sum, although isolated battles may be, as I shall presently show. Strictly competitive conflicts certainly occur in everyday life, but mixed-motive interactions are undoubtedly more common.

There are two major classes of strictly competitive games. Those in which both players have a finite number of pure strategies are called *finite*, two-person, zero-sum games, and methods are available for solving them. A solution consists of a specification of a rational way in which each of the players should choose among their available strategies, and a payoff, known as the *value* of the game, which results if both are rational according to the dictates of the theory. *Infinite*, two-person, zero-sum games, in which at least one of the players has an infinity of pure strategies from which to choose, sometimes possess formal solutions in this sense and sometimes do not. These games are, however, rather rare, and they are not dealt with in detail in this chapter. A clear introduction to the mathematical theory of infinite games has been given by Owen (1968, chap. 4).

The essential ideas of two-person, zero-sum game theory are outlined in the following sections. Two ways of representing the abstract structures of such games are explained with concrete examples in Section 4.2. The following two sections centre on the fundamental ideas behind the minimax solution. Sections 4.5 and 4.6 are concerned with special techniques for solving games. In Section 4.7 the problem is confronted of games in which information about the players' preferences is incomplete or non-quantitative, and Section 4.8 contains a brief summary of the chapter.

4.2 Extensive and Normal Forms

An incident from the Second World War, known by military historians as the Battle of Bismarck Sea, can be modelled by the simplest type of two-person, zero-sum game. The following account is based on a classic paper by Haywood (1954).

In February 1943, during the critical phase of the struggle in the south-western Pacific, the Allies received intelligence reports indicating that the Japanese were planning a troop and supply convoy to reinforce their army in New Guinea. The convoy could sail either north of the island of New Britain where rain and poor visibility were almost certain, or south of the island, where the weather would probably be fair. By either route, the trip would take 3 days. General Kenney, who controlled the Allied forces in the area, was ordered by the supreme commander, General MacArthur, to attack the convoy with the objective of inflicting maximum destruction. General Kenney had to decide whether to concentrate the bulk of his reconnaissance aircraft on the northern or the southern route. Once the convoy was sighted, it would be bombed continuously until its arrival in New Guinea.

The players in this example were General Kenney and the Japanese commander. The options from which each had to choose were the northern and southern routes. The outcomes were the numbers of days of bombing resulting from each possible combination of choices. Kenney's staff estimated that if the reconnaissance aircraft were concentrated mainly on the northern route, then the convoy would probably be sighted after 1 day whether it

sailed north or south, and would therefore be subjected to 2 days of bombing in either case. If the aircraft were concentrated mainly on the southern route, on the other hand, then either 1 or 3 days of bombing would result depending on whether the Japanese sailed north or south respectively. The number of days of bombing may be interpreted as Kenney's gains and the Japanese commander's losses. Since the Japanese payoffs are just the negatives of Kenney's, the corresponding game is obviously zero-sum. All of the essential information is summarized in Figure 4.1.

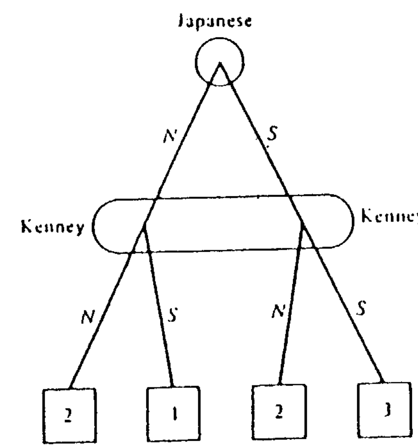

FIG. 4.1 Extensive form of the Battle of Bismarck Sea game.

In Figure 4.1 the *extensive form* of the game is represented in the most convenient way, by means of a *game tree*. The vertices of the game tree correspond to choice points and are labelled with the names of the players to whom they belong, while the branches represent the options between which the players choose. The initial (topmost) vertex is labelled "Japanese", indicating that the Japanese commander has the first move, and the succeeding vertices are labelled "Kenney". The terminal points of the game tree represent the outcomes which are reached after each player has moved in accordance with the rules of the game. If both choose south, for example, the outcome is 3 days of bombing as shown at the right-hand terminal point.

The vertices are enclosed in loops to indicate the *information sets* to which they belong. When making a move, a player cannot distinguish between choice points enclosed within a single information set. The initial vertex in Figure 4.1 is in an information set all on its own, but both of the vertices labelled "Kenney" are enclosed in a single information set. The purpose of this is to show that, at the time of choosing, Kenney does not know whether the left- or the right-hand vertex has been reached; in other words he does not know whether the Japanese commander has chosen to sail north or south. This means that the game is one of imperfect information—the players move simultaneously or in ignorance of any preceding moves—and it could have been portrayed equally well with the initial vertex labelled "Kenney" and the succeeding "Japanese" vertices enclosed in a single information set.

If the game were one of perfect information, on the other hand, then every vertex would be enclosed in a separate information set, like the initial vertex in Figure 4.1, and the sequence of moves represented in the game tree would assume significance. The extensive form of a perfect-information game such as chess, for example, would have to be represented in this way, although in practice it is impossible to do this for chess because the number of branches runs to billions after only a few moves. It is nonetheless possible to *imagine* a tree representing the extensive form of any finite game even if it cannot be drawn, and this is all that is required for some purposes. For a clear introduction to game trees, with several examples, see Singleton and Tyndall (1974, chaps 1 and 2).

A more compact method of representing a finite, strictly competitive game is by means of a rectangular array of numbers, called a *payoff matrix*, which displays the *normal form* of the game. Each row corresponds to one of Player I's pure strategies, and each column to one of Player II's pure strategies. Since a pure strategy is a complete plan of action specifying in advance how a player will move at any choice point which may arise in the course of play (see Chapter 1), the normal form allows a game involving a sequence of moves to be depicted statically, with the players simultaneously choosing a single row and column. The matrix element at the intersection of each row and column conventionally represents the payoff to Player I (the row chooser), and Player II's payoffs are simply negatives of those shown in the matrix. Any finite game in extensive form can be reduced to its equivalent normal form without loss of strategically relevant information. Matrix 4.1 shows the normal form of the Battle of Bismarck Sea game. The normal

Matrix 4.1
The Battle of Bismarck Sea

		Japanese N	Japanese S
Kenney	N	2	2
	S	1	3

form of this game is extremely simple since each player chooses between only two options under conditions of imperfect information; the players'

pure strategies are therefore simply their options. More complex examples are discussed later in this chapter.

4.3 Games With Saddle-Points

How might General Kenney, if he was rational, have analysed the game depicted in Matrix 4.1? At first glance, his optimal strategy seems to depend upon enemy intentions. If the Japanese decide to sail north of the island, then Kenney's best counter-strategy is to search north and obtain the outcome of 2 days of bombing. If they sail south, then Kenney does better by searching south and obtaining 3 days of bombing. But Kenney has no foreknowledge of the Japanese commander's intentions, so at this level of analysis his best counter-strategy remains indeterminate.

Rational generalship, however, is supposed to be based on enemy *capabilities* rather than enemy *intentions*. Kenney might therefore examine each of his available options from the viewpoint of the worst possible outcomes that could follow, given the options open to the Japanese. This pessimistic approach leads to the following conclusions: by searching north, Kenney is assured 2 days of bombing, whereas if he searches south the worst possible outcome is only 1 day of bombing. In other words, Kenney's minimum possible payoff if he chooses the northern strategy is 2 days, and the minimum possible if he chooses the southern strategy is 1 day of bombing. It follows that by choosing the northern strategy Kenney maximizes his minimum possible payoff. This choice is therefore called his *maximin* strategy; it has the property of ensuring the best of the worst possible outcomes. By choosing it, Kenney can guarantee that the payoff will not be less than two; this figure is his *maximin* value.

The Japanese commander can analyse the game in an analogous fashion. His objective is to minimize rather than to maximize the number of days of bombing. If he sails north, the worst that can happen—the maximum amount of bombing—is 2 days, and if he sails south, the worst possible outcome is 3 days of bombing. In order to ensure the best of the worst possible outcomes, he must therefore sail north. This choice corresponds to the Japanese *minimax* strategy since it minimizes the maximum possible payoff to the enemy. In this game, the minimum that the column player can guarantee (the minimax) is equal to the maximum that the row player can guarantee (the maximin): both are equal to 2 days of bombing.

According to the minimax principle of game theory, the optimal strategies available to the players in this type of zero-sum game are their maximin and minimax strategies. (For convenience, both are commonly referred to as minimax strategies.) In the Battle of Bismarck Sea game, players who are rational according to the minimax principle will choose their northern strategies, and the value of the game is 2 days of bombing since this is the payoff that results from rational play on both sides. These minimax strategies were in fact chosen by General Kenney and the Japanese commander in February 1943, and the Japanese suffered a disastrous defeat. The outcome cannot be attributed to any strategic error on the part of the Japanese commander; it was inherent in the unfair payoff structure of the game, whose value was positive and hence favourable to Kenney. In the event, the outcome would have been no better for the Japanese had they chosen differently, and Kenney, for his part, would have obtained a worse outcome (1 day of bombing instead of two) if he had chosen his non-minimax strategy.

The players' northern strategies are optimal because they intersect in an *equilibrium point* of the game. An equilibrium point is an outcome that gives neither player any cause to regret his choice of strategy when his opponent's choice is revealed. It represents a rational solution to any strictly competitive game for the following reason: it determines a minimax strategy for each player which yields the best possible payoff—the value of the game—against an opponent who also chooses optimally, and a payoff at least as good against one who chooses non-optimally. It should be borne in mind, however, that a minimax strategy does not necessarily exploit irrational play on the part of an opponent to maximum advantage. If there is good reason to suspect that the opponent will choose a non-minimax strategy, then a non-minimax choice may be the best counter-strategy. If, for example, Kenney suspected that the Japanese planned to sail south, then his best counter might be to search south although this choice would be irrational according to formal game theory. The minimax principle offers a persuasive definition of rational choice provided that there are no reasons for suspecting that one's adversary is irrational. Against an irrational adversary, however, it loses some of its force.

The equilibrium point of the Battle of Bismarck Sea game is easy to find because the payoff matrix contains a *saddle-point*. This technical term derives from the fact that a saddle is normally situated on a horse's back at the point which has minimum height on the animal's nose-to-tail axis and maximum height on its flank-to-flank axis. A saddle-point of a payoff matrix is an element that is a minimum in its row and a maximum in its column. The top left-hand element in Matrix 4.1 has this property, and it therefore corresponds to the equilibrium point of the game. Both of the numbers in row N are row minima because there is no number in this row less than 2. Only one of them, however, is also a column maximum, namely the left-hand one; therefore the NN cell is a unique saddle-point. If the payoff matrix of any strictly competitive game has a saddle-point, then its value is the value of the game and the players' minimax strategies are simply the corresponding row and column. If no saddle-point exists, then the minimum that Player I can guarantee by choosing a pure strategy—the maximin—is bound to be less

than the maximum that II can guarantee, namely the minimax. If maximin and minimax are equal, and only then, a saddle-point exists in the payoff matrix. The fact that a saddle-point necessarily corresponds to the intersection of the players' minimax strategies is particularly useful for solving games that are more complicated than the Battle of Bismarck Sea game.