PRIMER FOR THE ALGEBRAIC GEOMETRY OF SANDPILES

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ABSTRACT. This is a draft of a primer on the algebraic geometry of the Abelian Sandpile Model. VERSION:July, 11, 2009.

1. Introduction

This is a draft of a primer on the algebraic geometry of sandpiles based on lectures given in an undergraduate Topics in Algebra course at Reed College in the fall of 2008. It is assumed that the reader has no background in algebraic geometry. The second section of these notes gives an introduction to the Abelian Sandpile Model. What might be novel here is the treatment of burning configurations for directed multigraphs, but the main idea is due to [9]. Section 3 is a summary of the theory of lattice ideals as needed for the sequel. The first paper on the algebraic geometry of sandpiles of which we are aware is *Polynomial ideals for sandpiles and their Gröbner bases*, by Cori, Rossin, and Salvy [3]. That paper defines the toppling ideal of a undirected graph and computes a Gröbner basis for the ideal with respect to a certain natural monomial ordering. Section 4 extends their work, putting it in the context of lattice ideals and generalizing the Gröbner basis result to the case of directed multigraphs. It turns out that any lattice ideal whose zero set is finite is the lattice ideal corresponding to some directed multigraph. Section 6 gives an explicit description of the zero set of the toppling ideal. It is a generic orbit of a faithful representation of the sandpile group of the graph. The affine Hilbert function of the toppling ideal is defined in terms of the sandpile group. Matthew Baker et al. proved a Riemann-Roch theorem for undirected graphs. Using their language, in section 7, we see that the minimal free resolution of the homogeneous toppling ideal is graded by divisors on the graph modulo linear equivalence. The Betti numbers are determined by the simplicial homology of complexes forming the supports of complete linear systems on the graph. Finally, in Section 8, we completely characterize directed multigraphs whose homogeneous toppling ideals are complete intersection ideals. Further, we give a method of constructing directed multigraphs whose homogeneous toppling ideals are arithmetically Gorenstein.

The reader may be interested in `www.reed.edu/~davidp/sand`. Among other things, this site has links to programs that are useful in doing sandpile calculations. One of these is a package for the mathematical software system Sage [10]. The online manual for the package contains a brief overview of this paper and more, along with examples done within Sage. Following the Google Summer of Code link will take the reader to programs being developed to visualize and analyze the Abelian Sandpile Model.

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Thanks to all the students in Math 412.

In this draft version of the primer, I have not yet included examples. Sorry about that! For now, the reader is directed to the Sage Sandpile package referenced above.

2. Sandpiles

Here we outline the theory of sandpile groups. For a solid introduction with proofs of the various claims made in this section, the reader is referred to [6].

2.1. **Graph theory.** Let Γ be a finite, directed multigraph with vertices V and the multiset of edges E. Loops are allowed. For vertices $v, w \in V$, define the weight function,

$$\mathrm{wt}(v,w) = \text{ the number of edges between } v \text{ and } w.$$

If $e \in E$ is an edges from vertex v to vertex w, we write $e = (v,w)$ with $\mathrm{tail}(e) = v$ and $\mathrm{head}(e) = w$. For $v \in V$,

$$\begin{aligned} d_v = \mathrm{outdeg}(v) &= \sum_{w \in V} \mathrm{wt}(v,w) \\ \mathrm{indeg}(v) &= \sum_{w \in V} \mathrm{wt}(w,v). \end{aligned}$$

A vertex $s \in V$ is a *sink* if $d_v = 0$. If, in addition, there is a directed path from each vertex to s, then s is a *global sink*. Note that a global sink is unique if it exists. The graph Γ is *undirected* if $\mathrm{outdeg}(v) = \mathrm{indeg}(v)$ for all $v \in V$. If Γ is undirected and $s \in \Gamma$, by Γ *with sink* s, denoted Γ_s we will mean the directed graph obtained from Γ by removing outgoing edges from s.

For any finite set X, let

$$\mathbb{Z}X = \{\sum_{x \in X} a_x\, x : x \in \mathbb{Z}\}$$

be the free abelian group on X. Restricting to nonnegative coefficients gives $\mathbb{N}X$. For $a, b \in \mathbb{Z}X$, we define $|a| = \sum_{x \in X} a_x$ and $a \geq b$ if $a_x \geq b_x$ for all $x \in X$. The *support* of a is

$$\mathrm{supp}(a) = \{x \in X : a_x \neq 0\}.$$

We use similar notation for any integer vectors.

Definition 2.1. The *Laplacian* of Γ is the operator $\Delta\colon \mathbb{Z}^V \to \mathbb{Z}^V$ defined by

$$\Delta\phi(v) = \sum_{(w,v) \in E} (\phi(v) - \phi(w))$$

for $\phi \in \mathbb{Z}^V$ and $v \in V$.

The *standard basis* for $\mathbb{Z}^V$ is $\{v^*\}_{v \in V}$ where

$$v^*(w) = \delta(v,w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{if } v \neq w. \end{cases}$$

We have

$$\Delta v^* = d_v v^* - \sum_{w \in V} \mathrm{wt}(w,v) w^*.$$

Fixing an ordering $v_1, \dots, v_n$ on the vertices gives an isomorphism

$$\begin{aligned} \mathbb{Z}^V &\approx \mathbb{Z}^n \\ v_i^* &\mapsto e_i \end{aligned}$$

where e_i is the i-th standard basis vector. With respect to this isomorphism, Δ becomes a matrix with

$$\Delta_{ij} = \begin{cases} d_{v_i} - \mathrm{wt}(v_i, v_i) & \text{if } i = j \\ -\mathrm{wt}(v_i, v_j) & \text{if } i \neq j. \end{cases}$$

Consider the matrix $D = \mathrm{diag}(d_{v_1}, \dots, d_{v_n})$ and the *adjacency matrix*, A, where $A_{ij} = \mathrm{wt}(v_i, v_j)$. Then

$$\Delta = D - A.$$

Let $\widetilde{V}$ denote the nonsink vertices of a digraph Γ. Since $\widetilde{V} \subset V$, there are two natural maps between $\mathbb{Z}^V$ and $\mathbb{Z}^{\widetilde{V}}$. There is the restriction map

$$\begin{array}{rcl} \rho\colon \mathbb{Z}^V & \to & \mathbb{Z}^{\widetilde{V}} \\ \phi & \mapsto & \phi|_{\widetilde{V}} \end{array}$$

and the extension map

$$\iota\colon \mathbb{Z}^{\widetilde{V}} \to \mathbb{Z}^V$$

where

$$\iota(\phi)(v) = \begin{cases} \phi(v) & \text{if } v \in \widetilde{V} \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.2. The *reduced Laplacian* of Γ is the operator $\tilde{\Delta}\colon \mathbb{Z}^{\tilde{V}} \to \mathbb{Z}^{\tilde{V}}$ such that $\tilde{\Delta} = \rho \circ \Delta \circ \iota$.

Ordering V, hence also $\widetilde{V}$, the matrix representing $\widetilde{\Delta}$ is formed from the matrix representing Δ by removing rows and columns indexed by sinks. The transpose of the reduced Laplacian is the mapping

$$\widetilde{\Delta}^t\colon \mathbb{Z}\widetilde{V} \to \mathbb{Z}\widetilde{V}$$

obtained by dualizing $\widetilde{\Delta}$. Thus, for $v \in \mathbb{Z}\widetilde{V}$,

$$\widetilde{\Delta}^t v = d_v v - \sum_{w \in \widetilde{V}} \mathrm{wt}(v, w) w.$$

Having ordered V, the matrix for $\widetilde{\Delta}^t$ is the transpose of the matrix for $\widetilde{\Delta}$.

2.2. **The Sandpile Group.**

Definition 2.3. A *configuration* on Γ is an element of $\mathbb{N}\widetilde{V}$. A configuration $c = \sum_{v \in \widetilde{V}} c_v v$ is *stable* at a vertex $v \in \widetilde{V}$ if $c_v < d_v$. Otherwise, it is *unstable* or *active*. A configuration is *stable* if it is stable at each $v \in \widetilde{V}$.

We think of a configuration c as a pile of sand on Γ having c_v grains of sand at vertex v.

Remark 2.4. In the literature, a configuration is sometimes defined to be an element of $\mathbb{N}^V$. We prefer the dual semigroup, $\mathbb{N}V = \hom_{\mathbb{N}}(\mathbb{N}^V, \mathbb{N})$, for categorical reasons. (Consider the case of a subgraph, giving a subset of vertices $W \subseteq V$. There is a natural induced mapping $\mathbb{N}^V \to \mathbb{N}^W$ and hence, dualizing, $\mathbb{N}V \to \mathbb{N}W$.)

If c is unstable at v, we can *fire* or *topple* c at v to get a new configuration $\tilde{c}$ defined by

$$\tilde{c}_w = \begin{cases} \sigma_v - d_v + \mathrm{wt}(v, v) & \text{if } w = v \\ \sigma_w + \mathrm{wt}(v, w) & \text{if } w \neq v \end{cases}$$

for each $w \in \tilde{V}$. In other words,

$$\tilde{c} = \sigma - (d_v v - \textstyle\sum_{w \in \widetilde{V}} \mathrm{wt}(v,w)w).$$

Hence, $\tilde{c} = c - \widetilde{\Delta}^t v$: firing v amounts to subtracting the v-th column of $\widetilde{\Delta}$ from the configuration.

For configurations a and b, write $a \to b$ if b can be obtained from a by a sequence of firings.

It turns out that stabilizations are independent of the order of firings. To be precise, suppose $a \to b$ and $a \to b'$ where b and b' are stable configurations. Then $b = b'$. Further, the number of times each vertex fires in the stabilization process is independent of the order of firings. So the following definition makes sense.

Definition 2.5. Let a be a configuration on Γ and suppose $a \to b$ where b is stable. The *firing script* (also *firing vector* or just *script*) for $a \to b$ is $\sigma \in \mathbb{N}\widetilde{V}$ where σ_v is the number of times the vertex v fires as a stabilizes to b.

If the configuration a has a stabilization, it is denoted a°.

Lemma 2.6. *If Γ has a global sink s, then every configuration on Γ has a stabilization.*

> **Assumption:** For the rest of this section assume that Γ has a global sink, s.

Let $\mathcal{M}$ denote the set of stable configurations on Γ. Then $\mathcal{M}$ is a commutative monoid under *stable addition*

$$a \circledast b = (a+b)^\circ.$$

Thus, the operation is addition in $\mathbb{N}\widetilde{V}$ followed by stabilization. The identity is the zero configuration.

Definition 2.7. A configuration r is *accessible* if for each configuration s, there exists a configuration t such that $t + s \to r$. If, in addition, r is stable, then r is *recurrent*.

We now give a nice characterization of the recurrent elements.

Definition 2.8. The *maximal stable configuration* on Γ is the configuration

$$c_{\max} = \sum_{v \in \widetilde{V}} (d_v - 1)v$$

Proposition 2.9. *A configuration r is recurrent if and only if there exists a configuration s such that*

$$r = (s + c_{\max})^\circ.$$

It is not hard to see that the recurrent elements form a submonoid of $\mathcal{M}$. In fact, they form a group.

Theorem 2.10. *The collection of recurrent elements of Γ form a group under stable addition, denoted $\mathcal{S}(\Gamma)$ and called the* sandpile group *for Γ.*

Given what was said above, the sandpile group can be found by adding configurations to $c_{\max}$ and stabilizing. Considering a graph consisting of unconnected vertices connected to a common sink with edges of various weights, one sees that every finite abelian group is the sandpile group for some graph. Although the zero configuration is the identity for $\mathcal{M}$, it is seldom the identity for Γ. The following is an easy exercise:

Proposition 2.11. *The following are equivalent:*

(1) *the configuration* $\vec{0}$ *is recurrent;*
(2) *every stable configuration is recurrent;*
(3) Γ *is acyclic.*

We now give another description of the sandpile group.

Definition 2.12. The *Laplacian lattice*, $\mathcal{L} \subset \mathbb{Z}V$, is the image of Δ^t. The reduced Laplacian lattice, $\tilde{\mathcal{L}} \subset \mathbb{Z}\widetilde{V}$, is the image of $\widetilde{\Delta}^t$. The *critical group* for Γ is

$$\mathcal{C}(\Gamma) = \mathbb{Z}\widetilde{V}/\tilde{\mathcal{L}}.$$

Theorem 2.13. *There is an isomorphism of abelian groups*

$$\begin{array}{rcl} \mathcal{S}(\Gamma) & \to & \mathcal{C}(\Gamma) \\ c & \mapsto & c + \tilde{\mathcal{L}} \end{array}$$

Thus, each element of $\mathbb{Z}\widetilde{V}$ is equivalent to a unique recurrent element modulo the reduced Laplacian lattice. The identity of the sandpile group is the recurrent configuration in $\tilde{\mathcal{L}}$. In light of this theorem, we routinely identify these two groups, although it is sometimes useful to maintain the distinction.

Remark 2.14. A word about category theory, again. Suppose that $\Gamma' = (V', E')$ is another finite directed multigraph with global sink and with reduced Laplacian lattice $\tilde{\mathcal{L}}'$. Let $\Psi : \Gamma' \to \Gamma$ be a mapping of graphs that maps the sink of Γ' to that of Γ. Applying $\hom_{\mathbb{Z}}(\,\cdot\,, \mathbb{Z})$ to the natural induced map $\mathbb{Z}^{\Gamma} \to \mathbb{Z}^{\Gamma'}$ yields $\Psi_*\colon \mathbb{Z}V' \to \mathbb{Z}V$. If $\Psi(\tilde{\mathcal{L}}') \subseteq \mathcal{L}$, there is an induced mapping of sandpile groups. This would seem to be the right set of morphisms, then, for a category of sandpile groups.

Remark 2.15. Babai [1] has noted another characterization of the sandpile group: it is the principal semi-ideal in $\mathcal{M}$ generated by $c_{\max}$, which turns out to be the intersection of all the semi-ideals of $\mathcal{M}$.

2.3. **Superstables.** Firing only unstable vertices of a configuration ensures that no vertex ends up with a negative amount of sand. Consider a graph containing vertices v_1 and v_2 connected by edges (v_1, v_2) and (v_2, v_1). If c is a configuration with $c_{v_1} = d_{v_1} - 1$ and $c_{v_2} = d_{v_2} - 1$, then firing either vertex would leave a deficit of sand on that vertex. However, if we were allowed to simultaneously fire both vertices, the result would be a configuration, i.e., and element of $\mathbb{N}\widetilde{V}$.

Definition 2.16. A *firing script* is an element $\mathbb{N}\widetilde{V}$. If $c \in \mathbb{N}\widetilde{V}$ is a configuration and σ is a script, *firing the script* σ yields $c' = c - \widetilde{\Delta}^t\sigma \in \mathbb{Z}\widetilde{V}$. The firing is *legal* for c if $c' \in \mathbb{N}\widetilde{V}$, i.e., if c' is a configuration.

Thus, scripts and configurations are technically the same. However, saying σ is a script connotes that it is a potential firing script for a configuration.

Definition 2.17. A configuration is *superstable* if it has no legal firing scripts.

The following can be found in [6] for the case of Eulerian graphs, but it is true for the more general directed graphs considered in these notes:

Theorem 2.18. *A configuration c is superstable if and only if $c_{\max} - c$ is recurrent.*

We say that the superstables are *dual* to the recurrents.

2.4. **Burning configurations.** Speer's script algorithm [9] generalizes the burning algorithm of Dhar, testing whether a configuration is recurrent. We present a variation on Speer's algorithm using burning configurations. A proof for this result appears in the class notes for the Topics in Algebra class, available online [8].

Definition 2.19. A configuration b is a *burning configuration* if it has the following three properties:

(1) $b \in \tilde{\mathcal{L}}$
(2) $b \geq 0$, i.e., b is a configuration;
(3) for all $v \in \widetilde{V}$, there exists a path to v from some element of $\operatorname{supp}(b)$.

If b is a burning configuration, we call $\sigma_b = (\widetilde{\Delta}^t)^{-1}b$ the *script* or the *firing vector* for b.

Theorem 2.20. *Let b be the burning configuration with script σ. Then*

(1) *$(kb)^\circ$ is the identity configuration for $k \in \mathbb{N}$ large;*
(2) *A configuration c is recurrent if and only if $(c+b)^\circ = c$;*
(3) *A configuration c is recurrent if and only if the firing vector for $b + c \to (b+c)^\circ$ is σ;*
(4) *$\sigma \geq \vec{1}$.*

Theorem 2.21. *There exists a unique burning configuration b with script $\sigma_b = \widetilde{\Delta}^{-1}b$ having the following property: if $\sigma_{b'}$ is the script for a burning configuration b', then $\sigma_{b'} \geq \sigma_b$. For this b, we have:*

(1) *For all $v \in \tilde{V}$, $b_v < d_v$ unless v is a source, i.e., $\operatorname{indeg}(v) = 0$, in which case $b_v = d_v$. Thus, b is stable unless Γ has a source, and in any case, $b_v \leq d_v$ for all v.*
(2) *$\sigma_b \geq \vec{1}$ with equality iff Γ has no "selfish" vertices, i.e., no vertex $v \in \tilde{V}$ with $\operatorname{indeg}(v) > d_v := \operatorname{outdeg}(v)$.*

We call this b the minimal burning configuration *and its script, σ_b, the* minimal burning script.

3. Lattice ideals

Our reference for this section is [7]. Let A be a finitely generated abelian group, and let $a_1, \dots, a_n$ be a collection of elements generating A. Define $\phi\colon \mathbb{Z}^n \to A$ by $\phi(e_i) = a_i$, and denote its kernel by L. Let $\{t_a : a \in A\}$ be indeterminates, and let

$$\mathbb{C}[A] = \operatorname{Span}_{\mathbb{C}}\{t_a : a \in A\}$$

be the group algebra of A; hence, $t_a t_b = t_{a+b}$ for elements $a, b \in A$. Let $R := \mathbb{C}[x_1, \dots, x_n]$ and define a surjection of rings

$$\begin{aligned} \psi\colon R &\to \mathbb{C}[A] \\ x_i &\mapsto t_{a_i} \end{aligned}$$

For $c \in \mathbb{N}^n$, we define $x^c = \prod_i x^{c_i}$. Then, $\psi(x^c)$ is the group algebra element t_b where $b = \sum_{i=1}^n a_{c_i}$.

For $u \in \mathbb{Z}^n$, we write $u = u^+ - u^-$ with $u^+, u^- \in \mathbb{N}^n$ having disjoint support.

Theorem 3.1.

(1) *The kernel of* ψ *is the* lattice ideal

$$I(L) := \mathrm{Span}_{\mathbb{C}}\{x^u - x^v : u, v \in \mathbb{N}^n, u - v \in L\}.$$

(The vector space span, above, forms an ideal.) Hence, ψ *induces an isomorphism* $R/I(L) \approx \mathbb{C}[A]$.

(2) *If* $\ell_1, \ldots, \ell_k$ *are generators for the* $\mathbb{Z}$*-module,* L*, then* $I(L)$ *is the saturation of*

$$J = \langle x^{\ell_i^+} - x^{\ell_i^-} : i = 1, \ldots, k\rangle$$

with respect to the ideal generated by the product of the indeterminates, $\prod_{i=1}^n x_i$*. Thus,*

$$I(L) = \{f \in R : (\textstyle\prod_{i=1}^n x_i)^m f \in J \text{ for some } m \in \mathbb{N}\}.$$

Let $U \subset \mathbb{N}^n$ such that $X := \{x^u : u \in U\}$ is a $\mathbb{C}$-vector space basis for $R/I(L)$. Letting $g := (a_1, \ldots, a_n) \in A^n$,

$$\psi(X) = \{t_{u\cdot g} : u \in U\} = \{t_a : a \in A\},$$

the last equality holding since $R/I(L)$ and $\mathbb{C}[A]$ are isomorphic as vector spaces via ψ. Thus, ψ induces a bijection of X with A with which we endow X with the structure of a group isomorphic to A. For $u, v \in U$, we define $x^u x^v = x^w$ where w is the unique element of U for which $w \cdot g = (u + v) \cdot g$.

A choice of a monomial ordering on R gives a natural choice for U, namely, those $u \in \mathbb{N}^n$ such that x^u is not divisible by the initial term of any element of $I(L)$, e.g., not divisible by the initial term of any element of a Gröbner basis for $I(L)$. This will be discussed in §5.

4. Toppling ideals

Let Γ be a directed graph with global sink s. Identify the vertices with $\{1, \ldots, n+1\}$ with $n+1$ representing the sink. To avoid ambiguity, we will sometimes denote vertex i by v_i. By ordering the vertices, we thus have the exact sequence for the sandpile group of Γ,

$$0 \to \mathbb{Z}^n \xrightarrow{\widetilde{\Delta}^t} \mathbb{Z}^n \to \mathcal{S}(\Gamma) \to 0.$$

Definition 4.1. The *toppling ideal* is the lattice ideal for $\mathrm{im}(\widetilde{\Delta}^t) = \ker(\mathbb{Z}^n \to \mathcal{S}(\Gamma))$,

$$I(\Gamma) := \mathrm{Span}_{\mathbb{C}}\{x^u - x^v : u = v \bmod \tilde{\mathcal{L}}\} \subset R = \mathbb{C}[x_1, \ldots, x_n].$$

Thus,

$$R/I(\Gamma) \approx \mathbb{C}[\mathcal{S}(\Gamma)].$$

(What happens if we use finite fields instead of $\mathbb{C}$ for the polynomial ring, R?)

For each nonsink vertex i, define the *toppling polynomial*

$$t_i = x_i^{d_i - \mathrm{wt}(i,i)} - \textstyle\prod_{j \neq i} x_j^{\mathrm{wt}(i,j)}.$$

Proposition 4.2. *The ideal* $I(\Gamma)$ *is generated by the toppling polynomials,* $\{t_i\}_{i=1}^n$*, and the polynomial* $x^b - 1$ *where* b *is any burning configuration.*

Proof. Let $J = (t_i : i = 1, \ldots, n) + (x^\beta - 1)$. It is clear that $J \subseteq I(\Gamma)$, and by Theorem 3.1, Part (2), $I(\Gamma)$ is the saturation of J with respect to the ideal $(x_1 \cdots x_n)$. So it suffices to show that J is already saturated with respect to that ideal. Suppose that $(x_1 \cdots x_n)^k f \in J$ for some $f \in R$ and for some k. For each positive integer m

consider the monomial $x^{\beta m}$. We think of this monomial as a configuration of sand with $\beta_i m$ grains of sand on vertex i. If vertex i of this configuration is unstable, we think of firing the vertex as replacing $x_i^{\beta_i m}$ by $x_i^{\beta_i m - d_i} \prod_{j \neq i} x_j^{\mathrm{wt}(i,j)}$. Performing this replacement in $x^{\beta m}$ gives an equivalent monomial modulo J. Recall that every vertex of Γ is connected by a directed path from a vertex in the support of β. Thus, by taking m large enough and firing appropriate vertices, we arrive at a monomial x^γ, equivalent to $x^{\beta m}$ modulo J and corresponding to a configuration with at least k grains of sand at each vertex. Write $x^\gamma = x^\delta (x_1 \cdots x_n)^k$ for some monomial x^δ. Modulo J, we have

$$\begin{aligned} 0 &= (x_1 \cdots x_n)^k f \\ &= x^\delta (x_1 \cdots x_n)^k f \\ &= x^\gamma f \\ &= x^{\beta m} f \\ &= f. \end{aligned}$$

Thus, $f \in J$, as required. □

Remark 4.3. As in the proof of the above theorem, we can identify a monomial x^σ with a sandpile configuration on Γ. If $\sigma \to \tau$ as configurations, then $x^\sigma = x^\tau$ in $R/I(\Gamma)$.

Remark 4.4. The toppling ideal was introduced by Cori, Rossin, and Salvy [3]. They considered only undirected graphs and defined the ideal via generators. For an undirected graph, the all-1s configuration is a burning configuration; so Proposition 4.2 shows that our definition coincides with theirs in the case of an undirected graph.

Definition 4.5. Let $f \in R = \mathbb{C}[x_1, \dots, n]$, and let x_{n+1} be another indeterminate. The *homogenization* of f with respect to x_{n+1} is the homogeneous polynomial

$$f^h := x_{n+1}^{\deg f} f\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right).$$

If $I \subseteq R$ is an ideal, the *homogenization* of I with respect to x_{n+1} is the ideal

$$I^h := (f^h : f \in I).$$

Now consider the exact sequence corresponding to the full Laplacian,

$$0 \to \mathbb{Z}^{n+1} \xrightarrow{\Delta^t} \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}/\mathcal{L} \to 0$$

recalling the notation for the Laplacian lattice, $\mathcal{L} := \mathrm{im}(\Delta^t)$. Let $S = \mathbb{C}[x_1, \dots, x_{n+1}]$ and consider the lattice ideal for $\mathcal{L}$,

$$I(\Gamma)^h := \mathrm{Span}_{\mathbb{C}}\{x^u - x^v : u = v \bmod \mathcal{L}\} \subset S = \mathbb{C}[x_1, \dots, x_{n+1}].$$

The following proposition justifies our choice of notation.

Proposition 4.6. *The ideal $I(\Gamma)^h$ is the homogenization of $I(\Gamma)$ with respect to x_{n+1}.*

Remark 4.7. In general, $I(\Gamma)^h$ is not the ideal generated by the homogenizations of the toppling polynomials, t_i, and the polynomial $x^\beta - 1$.

Theorem 4.8. *Let $\tilde{\mathcal{L}}$ be any submodule of $\mathbb{Z}^n$ having rank n. Then there exists a directed multigraph with global sink whose reduced Laplacian lattice is $\tilde{\mathcal{L}}$. Every lattice ideal defining a finite set of points is the lattice ideal associated with the reduced Laplacian of some directed multigraphs with global sink.*

Proof. The proof is an algorithm due to John Wilmes. It will be included in these notes by the end of August 2009. ☐

5. Gröbner bases of toppling ideals

The book Ideals, varieties, and algorithms [4] is recommended as a general reference for the theory of Gröbner bases need for this section. Let $R = \mathbb{C}[x_1, \ldots, x_n]$.

Definition 5.1. A *monomial order*, $>$, on R is a total ordering on the monomials of R satisfying

(1) If $x^a > x^b$, then $x^{c+a} > x^{c+b}$ for all $c \geq 0$;
(2) $1 = x^0$ is the smallest monomial.

Example 5.2. The following are the most common examples of monomial orders:

(1) Lexicographic ordering, `lex`, is defined by $x^a > x^b$ if the left-most nonzero entry of $a - b$ is positive (i.e., more of the earlier indeterminates).
(2) Degree lexicographic ordering, `deglex`, is defined by $x^a > x > b$ if $|a| > |b|$ or if $|a| = |b|$ and the left-most entry of $a - b$ is positive (i.e., order by degree and break ties with `lex`).
(3) Degree reverse lexicographic ordering, `grevlex`, is defined by $x^a > x^b$ if $|a| = |b|$ or if $|a| = |b|$ and the right-most entry of $a - b$ is negative (i.e., order by degree then break ties by checking which monomial has fewer of the later indeterminates).

A monomial multiplied by a constant is called a *term*. Once a monomial ordering is fixed, then we write $\alpha\, x^a > \beta\, x^b$ for two terms if $x^a > x^b$. Let $f \in R$, then f is a sum of terms corresponding to distinct monomials. We denote the leading term—the largest term with respect to the chosen monomial ordering—by $\mathrm{LT}(f)$.

Definition 5.3. Fix a monomial ordering on R and let $f, g \in R$. The *S-polynomial* for the pair (f, g) is

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(f)}\, f - \frac{\mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(g)}\, g.$$

Definition 5.4. Fix a monomial ordering on R, and let I be an ideal of R. A finite subset G of I is a *Gröbner basis* for I with respect to the given monomial ordering if any of the following equivalent conditions hold:

(1) $(\mathrm{LT}(g) : g \in G) = (\mathrm{LT}(f) : f \in I)$.
(2) For all $f \in I$, there is a $g \in G$ such that $\mathrm{LT}(g)$ divides $\mathrm{LT}(f)$.
(3) Each $f \in I$ may be reduced to 0 by G, i.e., by repeatedly reducing by elements of G.
(4) For all $g, g' \in G$, the S-polynomial $S(g, g')$ reduces to 0 by G and G is a generating set for I.

The last criterion is essentially Buchberger's algorithm for calculating a Gröbner basis.

Let $G = \{g_1, \ldots, g_m\}$ be the Gröbner basis for an ideal $I \subseteq R$ with respect to some monomial ordering, and let $f \in \mathbb{R}$. If f has a term m divisible by $\mathrm{LT}(g_i)$ for some i, then replace f by $f - \frac{m}{\mathrm{LT}(g_i)}\, g_i$. Repeating this process one arrives at a remainder r that is unique with respect to the property that (i) $r = f + g$ for some $g \in I$ and (ii) r has no terms divisible by any leading term of an element of G. We call this remainder the *reduction* or *normal form* of f with respect to the Gröbner basis G.

Notation 5.5. The reduction of f with respect to G is denoted by $f \,\%\, G$. If $g \in R$, we write $f \,\%\, g$ for the special case in which $I = (g)$ and $G = \{g\}$.

Definition 5.6. Fix a monomial ordering on R and let I be an ideal of R. The set of monomials of R that are not divisible by the leading term of a Gröbner basis element for I with respect to the given ordering is called the *normal basis* for R/I.

By Macaulay's theorem, a normal basis is a vector space basis for R/I.

We now introduce the appropriate monomial ordering for sandpiles, due to Cori, Rossin, and Salvy, [3].

Definition 5.7. A *sandpile monomial ordering* on R is any graded reverse lexicographical ordering in which $x_i > x_j$ if the length of the shortest path from vertex j to the sink is no greater than that for i, i.e., vertex j is no further from the sink than is vertex i.

> **Assumption:** For the rest of this section, we will number the nonsink vertices of Γ and fix a sandpile monomial ordering on R so that $x_i > x_j$ if $i < j$.

The utility of the above convention becomes apparent when one considers topplings of configurations.

Proposition 5.8. *Let $a, b \in \mathbb{Z}\widetilde{V}$ be distinct configurations on Γ such that $a \to b$, i.e., b is obtained from a by a sequence of vertex firings. Then, $x^a > x^b$ with respect to the sandpile monomial ordering we have fixed on R.*

Proof. Each vertex firing deceases the size of the corresponding monomial. The reason is that either the vertex firing shoots sand into the sink, decreasing the total degree of the corresponding monomial, or shoots sand to a vertex closer to the sink, in which case the corresponding monomial has more of the later indeterminates. □

We now proceed to compute a Gröbner basis for the toppling ideal. Let

$$\begin{array}{rcl} t\colon \mathbb{Z}\widetilde{V} & \to & R \\ \ell & \mapsto & x^{\ell^+} - x^{\ell^-}. \end{array}$$

Then define $\mathcal{T} = t \circ \widetilde{\Delta}^t : \mathbb{Z}\widetilde{V} \to R$. It follows that $\mathcal{T}(v_i)$ is the i-th toppling polynomial, defined earlier, and for any configuration c, we have $x^c \,\%\, \mathcal{T}(v_i) = x^{c'}$ where c' is the configuration obtained from c by firing v_i until v_i is stable. Morever, if σ is a script, then $x^c \,\%\, \mathcal{T}(\sigma)$ yields the monomial corresponding to the configuration formed by firing σ as many times as legal from c.

Theorem 5.9. *Let b be a burning configuration, and let σ_b be its script. Then*

$$G_b = \{\mathcal{T}(\sigma) : 0 \le \sigma \le \sigma_b\}$$

is a Gröbner basis for $I(\Gamma)$.

Proof. We have $\mathrm{im}(\mathcal{T}) \subset I(\Gamma)$ by definition of $I(\Gamma)$. On the other hand, $\mathcal{T}(v_i)$ is the i-th toppling polynomial and $\mathcal{T}(\sigma_b) = x^b - 1$. So $I(\Gamma) = \mathrm{Span}_{\mathbb{C}}\{\mathrm{im}(\mathcal{T})\}$ by Proposition 4.2.

We need to show that all S-polynomials of G_b reduce to 0 by G_b. Let σ_1 and σ_2 be scripts with $\sigma_1, \sigma_2 \leq \sigma_b$. Write

$$\mathcal{T}(\sigma_i) = x^{c_i^+} - x^{c_i^-}$$

for $i = 1, 2$ where c_i^- is the configuration obtained from c_i^+ by firing script σ_i. Hence, $x^{c_i^+}$ is the leading term of $\mathcal{T}(\sigma_i)$ for each i. Define

$$x^{a_i} = \frac{\mathrm{lcm}(x^{c_1^+}, x^{c_2^+})}{x^{c_i^+}}$$

for $i = 1, 2$ so that $a_1 + c_1^+ = a_2 + c_2^+ = c$ for some configuration c. We must show that the S-polynomial,

$$\begin{aligned} S(\sigma_1, \sigma_2) &= x^{a_1}\mathcal{T}(\sigma_1) - x^{a_2}\mathcal{T}(\sigma_2) \\ &= x^{a_2 + c_2^-} - x^{a_1 + c_1^-}, \end{aligned}$$

reduces to 0. Since both scripts σ_1 and σ_2 are legal from c, so is the script $\sigma = \max(\sigma_1, \sigma_2)$ defined by $\sigma_v = \max(\sigma_{1,v}, \sigma_{2,v})$. Note that $\sigma \leq \sigma_b$. Letting c' be the configuration obtained by firing $\max(\sigma, \tau)$, we have the sequence of script firings

$$a_i + c_i^+ \xrightarrow{\sigma_i} a_i + c_i^- \xrightarrow{\sigma - \sigma_i} c$$

for $i = 1, 2$, which shows that the S-polynomial reduces to 0 using the elements $\mathcal{T}(\sigma - \sigma_i)$ for $i = 1, 2$. □

Remark 5.10. In the case of an undirected graph, one may take the burning script to be the vector whose components are all ones. Thus, the script firings that are relevant in constructing the Gröbner basis, described in the statement of the previous theorem, can be identified with firing subsets of vertices (none more than once). The paper [3] goes further, in this case, to describe a *minimal* Gröbner basis, i.e., one in which each member has the property that none if its terms is divisible by the leading term any other member. It consists of the subset of the Gröbner basis elements described in the previous theorem corresponding to $X \subseteq \widetilde{V}$ such that the subgraphs of Γ induced by X and by $\widetilde{V} \setminus X$ are each connected. It would be interesting to see if this result could be generalized to the case of directed graphs.

Theorem 5.11. *Each configuration is equivalent to a unique superstable configuration modulo $\tilde{\mathcal{L}}$, and*

$$\{x^c : c \textit{ is a superstable configuration}\}$$

is the normal basis for $R/I(\Gamma)$ with respect to the sandpile monomial ordering.

Proof. Two configurations are equivalent modulo $\tilde{\mathcal{L}}$ if and only if their corresponding monomials are equivalent modulo $I(\Gamma)$. In detail, first let $c_1, c_2 \in \mathbb{N}^n$ and suppose

$$c_1 - c_2 = \ell = \ell^+ - \ell^- \in \tilde{\mathcal{L}}.$$

Then $c_1 \geq \ell^+$ and $c_2 \geq \ell^-$. Define $e = c_1 - \ell^+ = c_2 - \ell^- \geq 0$. Then

$$x^{c_1} - x^{c_2} = x^e(x^{\ell^+} - x^{\ell^-}) \in I(\Gamma).$$

Conversely, if $x^{c_1} - x^{c_2} \in I(\Gamma)$, we may write

$$x^{c_1} - x^{c_2} = \sum_i x^{e_i}(x^{\ell_i^+} - x^{\ell_i^-})$$

for some $e_i \in \mathbb{N}^n$ and $\ell_i \in \tilde{\mathcal{L}}$. It follows that $c_1 - c_2 = \sum_i e_i\,\ell_i \in \tilde{\mathcal{L}}$.

Now let c be any configuration. Since $x^c \,\%\, \mathcal{T}(\sigma) = x^{c'}$ where c' is obtained by firing the script σ as many times as is legal, the normal form for x^c with respect to the sandpile monomial ordering is superstable. Since the normal form is unique, so is this superstable element. □

As noted in §4, we have $R/I(\Gamma) \approx \mathbb{C}[\mathcal{S}(\Gamma)]$. Hence, by the previous theorem, we see that the sandpile group can be thought of as the set of superstables where the sum of superstables c_1 and c_2 is taken to be $\log(x^{c_1}x^{c_2} \,\%\, I(\Gamma))$.

6. Zeros of the toppling ideal

Given any ideal $I \in R = \mathbb{C}[x_1, \ldots, x_n]$, the set of zeros of I is

$$Z(I) = \{p \in \mathbb{C}^n : f(p) = 0 \text{ for all } f \in I\}.$$

Let $R_{\leq d}$ denote the vector space of polynomials in R of degree at most d, and let $I_{\leq d}$ be the subspace $I \cap R_{\leq d}$.

Definition 6.1. The *affine Hilbert function* of I is $H\colon \mathbb{N} \to \mathbb{N}$, given by

$$H(d) = \operatorname{codim}_{\mathbb{C}} I_{\leq d} = \dim_{\mathbb{C}} R_{\leq d} - \dim_{\mathbb{C}} I_{\leq d}.$$

In this section, our goal is to describe the set of zeros of the toppling ideal.

Proposition 6.2. *The set of zeros of the toppling ideal, $I(\Gamma)$, is finite.*

Proof. We have seen that

$$R/I(\Gamma) \approx \mathbb{C}[\mathcal{S}(\Gamma)],$$

and thus, $R/I(\Gamma)$ is a finite-dimensional vector space over C. For each indeterminate $x_i \in R$, consider the powers $1, x_i, x_i^2, \ldots$. By finite-dimensionality, the image of these powers in the quotient ring are linearly dependent. This means there is a polynomial f_i such that $f_i(x_i) \in I(\Gamma)$. Each f_i will have a finite number of zeros, and thus, for each i, we see that the there are a finite number of possible i-th coordinates for any zero of the toppling ideal. □

Remark 6.3. In fact, the i-th coordinates of the zeros of the toppling ideal are the eigenvalues of the multiplication mapping

$$\begin{array}{ccc} R/I(\Gamma) & \to & R/I(\Gamma) \\ g & \mapsto & x_i g \end{array}$$

It would be worth following up on this idea, reading David Cox's article, *Solving equations via algebras*, in the book *Algorithms and Computation in Mathematics*, [5], applying the ideas there to toppling ideals.

We are going to take a different tack.

6.1. **Orbits of representations of abelian groups.**

6.1.1. *Affine case.* Let $\{a_1, \ldots, a_n\}$ be generators (not necessarily distinct) for a finite abelian group, A. Consider the exact sequence

$$\begin{array}{rcl} 0 \to L \to \mathbb{Z}^n & \to & A \to 0 \\ e_i & \mapsto & a_i \end{array} \tag{6.1}$$

where L is defined as the kernel of the given mapping $\mathbb{Z}^n \to A$. Take duals, i.e., applying $\mathrm{Hom}_{\mathbb{Z}}(\ \cdot\ , \mathbb{C}^\times)$, gives the sequence

$$1 \leftarrow L^* \leftarrow (\mathbb{C}^\times)^n \leftarrow A^* \leftarrow 1, \tag{6.2}$$

where A^* is the character group of A.

Remark 6.4.

(1) Exactness of (6.2) is not immediate. The exactness at $L^* \leftarrow (\mathbb{C}^\times)^n$ follows because $\mathbb{C}^\times$ is a *divisible* abelian group. An abelian group B is divisible if for all $a \in B$ and positive integers n there exists a $b \in b$ such that $nb = a$. (So for the multiplicative group $\mathbb{C}^\times$, each element has an n-th root.) Applying $\mathrm{Hom}_{\mathbb{Z}}(\ \cdot\ , B)$ to an exact sequence of abelian groups ($\mathbb{Z}$-modules) always gives an exact sequence precisely when B is divisible. The proof of this, in general, is not immediate. However, in the case in which we are most concerned, the exactness is easy to establish. Suppose $A = \mathcal{S}(\Gamma)$ is the sandpile group of a directed graph with global sink, and suppose L is the reduced Laplacian lattice, $\tilde{\mathcal{L}} = \mathrm{im}(\widetilde{\Delta}^t) \hookrightarrow \mathbb{Z}^n$. We would like to show that the natural map, given by composition,

$$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^\times) \to \mathrm{Hom}(\tilde{\mathcal{L}}, \mathbb{C}^\times)$$

is surjective. Let $\phi\colon \tilde{\mathcal{L}} \to \mathbb{C}^\times$ be given. Since the reduced Laplacian has full rank, given $v \in \mathbb{Z}^n$, there exist unique rational numbers α_ℓ such that $v = \sum_\ell \alpha_\ell \ell$, with the sum going over a basis for $\tilde{\mathcal{L}}$ (say, over the columns of the reduced Laplacian). Then define $\tilde{\phi}\colon \mathbb{Z}^n \to \mathbb{C}^\times$ by $\tilde{\phi}(v) = \sum_\ell \phi(\ell)^{\alpha_\ell}$.

(2) To be explicit, denote the mapping $\mathbb{Z}^n \to A$ by ϕ. Then part of sequence (6.2) is

$$\begin{array}{ccccc} A^* & \to & \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^\times) & \approx & (\mathbb{C}^\times)^n \\ \chi & \mapsto & \chi \circ \phi & \mapsto & (\chi(a_1), \ldots, \chi(a_n)). \end{array}$$

We get an n-dimensional representation of A^*:

$$\rho\colon A^* \to (\mathbb{C}^\times)^N \to \mathrm{GL}(\mathbb{C}^n)$$

given by

$$\rho(\chi) = \mathrm{diag}(\chi(a_1), \ldots, \chi(a_n)).$$

In other words, the choice of generators for A induces a homomorphism of A^* into group of invertible $n \times n$ matrices over $\mathbb{C}$. (Since every n-dimensional representation of A^* over $\mathbb{C}$ is a direct sum of characters of A^*, i.e., of elements of $A^{**} \approx A$. So this section can be regarded as saying something about representations of A^*, in general.)

For each $z \in \mathbb{C}^n$, define the *orbit of z under ρ* to be

$$\mathcal{O}_\rho(z) = \{\rho(\chi)z : \chi \in A^*\} = \{(\chi(a_1)z_1, \ldots, \chi(a_n)z_n) : \chi \in A^*\}.$$

We will assume that no coordinate of z is zero, in which case by by scaling coordinates of $\mathbb{C}^n$, we may assume for our purposes that $z = (1, \dots, 1)$. Thus, we are interested in the orbit of the 1-vector,

$$\mathcal{O} = \{\rho(\chi) : \chi \in A^*\} = \{(\chi(a_1), \dots, \chi(a_n)) : \chi \in A^*\}.$$

Theorem 6.5. *Let $R = \mathbb{C}[x_1, \dots, x_n]$ and consider*

$$I = \{f \in R : f(\mathcal{O}) = 0\},$$

the ideal of polynomials vanishing on the orbit. Let $R_{\leq d}$ denote the finite-dimensional vector space of polynomials of degree at most d, and let $I_{\leq d} = I \cap R_{\leq d}$. Then

(1)
$$I = I(L) = \mathrm{Span}_{\mathbb{C}}\{x^u - x^v : u = v \bmod L\};$$

(2) *The affine Hilbert function of I is given by*
$$H(d) = \left|\{\textstyle\sum_{i=1}^n n_i a_i : n_i \geq 0 \text{ for all } i \text{ and } \sum_i n_i \leq d\}\right|.$$

Proof. This proof is due to the author and Donna Glassbrenner. It appears in [2]. Consider the matrix $M^{(d)}$ with rows indexed by A^* and columns indexed by the monomials of $R_{\leq d}$ (arranged in lexicographical order so that $M^{(d)}$ is naturally nested in $M^{(d+1)}$), with

$$M^{(d)}_{\chi, x^u} = \prod_{i=1}^n \chi^{u_i}(a_i)$$

Recall the isomorphism

$$\begin{array}{ccc} A & \to & A^{**} \\ a & \mapsto & \bar{a} \end{array}$$

where $\bar{a}(\chi) := \chi(a)$. Thus, we can write

$$M^{(d)}_{\chi, x^u} = \prod_{i=1}^n \bar{a}_i^{u_i}(\chi) = \bar{a}^u(\chi)$$

where $\bar{a}^u := \prod_{i=1}^n \bar{a}_i^{u_i} \in A^{**}$. The x^u-th column of $M^{(d)}$ has entries $\bar{a}^u(\chi)$ as χ varies over A^*. In other words, it is the list of all values of the function $\bar{a}^u$. Thus, at least as far as linear algebra is concerned, the x^u-th column *is* $\bar{a}^u$. Since distinct characters are linearly independent, it follows that any linear dependence relations are the result of columns that are equal.

Now, the x^u-th and x^v-th columns of $M^{(d)}$ are equal exactly when $\bar{a}^u = \bar{a}^v$ are equal. This occurs exactly when $\sum_i u_i a_i = \sum_i v_i a_i$, which we write as $(u - v) \cdot a = 0$ where $a := (a_1, \dots, a_n)$. In light of exact sequence (6.1), this condition is equivalent to $u - v \in L$.

A vector $(\alpha_u) \in \ker M^{(d)}$ if and only if

$$\sum_u \alpha_u \prod_{i=1}^n \chi^{u_i}(a_i) = 0$$

for all $\chi \in A^*$. Thus, $(\alpha_u) \in \ker M^{(d)}$ if and only if the polynomial $p = \sum_u \alpha_u x^u$ vanishes on $\mathcal{O}$, i.e., $p \in I$. Thus, elements of $I_{\leq d}$ correspond exactly with linear combinations among the columns of $M^{(d)}$. As these relations are due to equality among columns, as already noted, part 1 follows. For part 2, note that we have just shown that

$$\dim I_{\leq d} = \dim R_{\leq d} - \mathrm{rank}\ M^{(d)}.$$

Since distinct characters are linearly independent,

$$\operatorname{rank} M^{(d)} = \left|\{\textstyle\sum_{i=1}^{n} n_i a_i : n_i \geq 0 \text{ for all } i \text{ and } \sum_i n_i \leq d\}\right|.$$

□

Back to the case of the toppling ideal, the exact sequence

$$0 \to \mathbb{Z}^n \xrightarrow{\tilde{\Delta}^t} \mathbb{Z}^n \to \mathcal{S}(\Gamma) \to 0$$

has the form of exact sequence (6.1). The generators a_i are the configurations having exactly one grain of sand.

Corollary 6.6.

(1) *The toppling ideal is the set of polynomials vanishing on an orbit, $\mathcal{O}$, of a faithful representation of $\mathcal{S}(\Gamma)^*$.*
(2) *The set of zeros of the toppling ideal is the finite set, $\mathcal{O}$. It thus has the symmetry of $\mathcal{S}(\Gamma)^*$, which is isomorphic to the sandpile group.*
(3) *If H_Γ is the affine Hilbert function of the toppling ideal, then $H_\Gamma(d)$ is*
   (a) *the number of elements of $\mathbb{Z}^n/\tilde{\mathcal{L}}$ represented by configurations containing at most d grains of sand;*
   (b) *the number of recurrent configurations c such that $|c| \geq |c_{\max}| - d$.*

Proof. Part (1) follows directly from the first part of Theorem 6.5. For part (2), since $\mathcal{O}$ is a finite collection of points in $\mathbb{C}^n$, and $I(\Gamma) = I(\mathcal{O})$, it is a basic result of algebraic geometry that the set of zeros of $I(\Gamma)$ is $\mathcal{O}$. Part (3) is immediate from the second part of the theorem and the fact that r is recurrent if and only if $c_{\max} - r$ is superstable. □

6.1.2. *Projective case.* An ideal J in $S = \mathbb{C}[x_1, \ldots, x_{n+1}]$ is *homogeneous* if it has a set of homogeneous generators. The set of zeros of J is a subset of projective space:

$$Z(J) = \{p \in \mathbb{P}^n : f(p) = 0 \text{ for all homogeneous } f \in J\}.$$

The ring S/J is graded by the integers: $(S/J)_d := S_d/J_d$.

Definition 6.7. The *Hilbert function* of S/J is $H\colon \mathbb{N} \to \mathbb{N}$, given by

$$H(d) = \dim_{\mathbb{C}}(S/J)_d.$$

With notation as above, define the *homogenization* of L as

$$L^h = \left\{ \begin{pmatrix} \ell \\ -|\ell| \end{pmatrix} \in \mathbb{Z}^{n+1} : \ell \in L \right\}$$

where $|\ell| = \sum_{i=1}^n \ell_i$. Consider the exact sequence

$$0 \to L^h \to \mathbb{Z}^{n+1} \xrightarrow{M} A \oplus \mathbb{Z} \to 0.$$

where

$$M = \begin{pmatrix} a_1 & \ldots & a_n & 0 \\ 1 & \ldots & 1 & 1 \end{pmatrix}.$$

Apply $\operatorname{Hom}(\,\cdot\,, \mathbb{C}^\times)$ to get

$$\begin{array}{rcl} 1 \to A^* \times \mathbb{C}^\times & \to & (\mathbb{C}^\times)^{n+1} \to (L^h)^* \to 0 \\ (\chi, z) & \mapsto & (\chi(a_1), \ldots, \chi(a_n), 1)z \end{array}$$

We get a corresponding representation

$$\begin{array}{rcl} A^* \times \mathbb{C}^\times & \to & \mathrm{GL}(\mathbb{C}^{n+1}) \\ (\chi, z) & \mapsto & \mathrm{diag}(\chi(a_1)z, \ldots, \chi(a_n)z, z) \end{array}$$

The orbit of $(1, \ldots, 1)$ under this representation is

$$\mathcal{O}^h = \{(\chi(a_1), \ldots, \chi(a_n), 1) \in \mathbb{P}^n : \chi \in A^*\} \subset \mathbb{P}^n.$$

Thus, $\mathcal{O}^h$ is the *projective closure* of the orbit $\mathcal{O}$ from the previous section.

Theorem 6.8. *Let $a^h = (a_1, \ldots, a_n, 0)$.*

(1) *The homogeneous ideal defining $\mathcal{O}^h$ is the lattice ideal for L^h, the saturation of the lattice ideal for L:*

$$I^h = \{x^u - x^v : u = v \bmod L^h\}.$$

(2) *The Hilbert function for $\mathcal{O}^h$ is*
   (a)
$$H(d) = \left|\{s \cdot a^h \in A : s \in \mathbb{N}^{n+1} \text{ such that } |s| = d\}\right|.$$
   (b) *the same as the Hilbert function for I.*

Remark 6.9.

(1) I^h is not necessarily given by saturating the standard generators for the toppling ideal I.
(2) A Gröbner basis for I^h with respect to a toppling order is the homogenization of the corresponding Gröbner basis for I.

If $L = \tilde{\mathcal{L}}$, the reduced Laplacian lattice of Γ, then L^h is the full Laplacian lattice, $\mathcal{L}$. As in the previous section, we have

Corollary 6.10.

(1) *The homogenization of the toppling ideal is the ideal generated by all homogeneous polynomials vanishing on an orbit, $\mathcal{O}^h$ of a faithful representation of $(\mathbb{Z}^{n+1}/\mathcal{L})^*$.*
(2) *The set of zeros of the homogenization of the toppling ideal is the finite set $\mathcal{O}^h$, having the symmetry of $\mathcal{S}(\Gamma)^*$.*

7. Resolutions

This section will be added by the end of August 2009. The minimal free resolution of the homogeneous toppling ideal is graded by *divisors* in the sense of Baker, et al. The Betti numbers are given by the simplicial homology of complexes associated with complete linear systems of divisors.

8. Gorenstein toppling ideals

At least part of this section will be added by the end of August 2009. It will contain a characterization of graphs whose homogeneous toppling ideals are complete intersection ideals. It will also give a method of constructing graphs whose toppling ideals are Gorenstein.

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