Putnam Practice Problems

October 4, 2009

0.1 Prove that the sequence (in base-10 notation) $11, 111, 1111, 11111, \ldots$ contains no squares.

0.2 At first, a room is empty. Each minute, either four people enter or one person leaves. After exactly 1 hour, could the room contain 101 people?

1. Let $a_1, a_2, a_3, \ldots$ be a strictly increasing sequence of positive integers. Show that for each a_p, there exist infinitely many a_m such that

$$a_m = xa_p + ya_q$$

where x, y are positive integers and $q > p$.

2. Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

3. Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2, 1 = 1^2 + 0^2, 2 = 1^2 + 1^2$.]

4. The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

5. Let n be a positive integer. Characterize those n for which the integral

$$\int_0^{2\pi} \cos(x)\cos(2x)\ldots\cos(nx)dx$$

is nonzero.

6. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

7. Let $(x, y) \in (0, 1) \times (0, 1)$ be chosen at random with uniform distribution. For any $\alpha \in \mathbb{R}$, let $[\alpha]$ denote the closest integer to α: for example $[1/3] = 0$ and $[2/3] = 1$. We adopt the convention that $[n/2] = (n-1)/2$ in the case where n is an odd integer. What is the probability that $[x/y]$ is even? Express your answer in the form $a + b\pi$, where a, b are rational numbers.