PCMI 2008 Undergraduate Summer School

Lecture 8: The Syzygy Theorem

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Review: Hilbert function of a hypersurface

$f \in S = k[x_0, \ldots, x_n]$ homog., $\deg f = e$

$I = (f)$

$$0 \longrightarrow S \xrightarrow{\cdot f} S \longrightarrow S/I \longrightarrow 0$$

$$0 \longrightarrow S_{d-e} \xrightarrow{\cdot f} S_d \longrightarrow (S/I)_d \longrightarrow 0$$

$$H_X(d) = \binom{n+d}{n} - \binom{n+d-e}{n} \quad \text{where } \binom{m}{n} = 0 \text{ if m<n}$$

$$P_X(t) = \binom{n+t}{n} - \binom{n+t-e}{n} \quad \text{where } \binom{t}{n} = \frac{t(t-1)\cdots(t-n+1)}{n!}$$

Some notation

Twists

$S(-e) = S$ but with the degrees shifted by e:

$$S(-e)_d := S_{d-e}$$

$$0 \longrightarrow S(-e) \xrightarrow{\cdot f} S \longrightarrow S/I \longrightarrow 0$$

with all mappings preserving degrees.

$$H_X(d) = \binom{n+d}{n} - \binom{n+d-e}{n}$$

Remember

$$0 \longrightarrow S(-e) \xrightarrow{\cdot f} S \longrightarrow S/I \longrightarrow 0$$

$$0 \longrightarrow S_{d-e} \xrightarrow{\cdot f} S_d \longrightarrow (S/I)_d \longrightarrow 0$$

$$H_X(d) = \binom{n+d}{n} - \binom{n+d-e}{n}$$

Goal

Generalize this method of computation of the Hilbert function.

Graded modules

Definition

An *S*-module is a “vector space” M having scalars in S rather than in a field. The module M is graded if

$$M = \bigoplus_d M_d, \qquad S_e M_d \subseteq M_{d+e}$$

for all d, e.

Examples

- S
- $I \subseteq S$, homogeneous ideal
- S/I, where I is homogeneous
- $S \oplus S$

Free modules

$$S^2 = S \oplus S = \{(f, g) : f, g \in S\}$$

$$\begin{aligned} (f, g) + (f', g') &= (f + f', g + g') \\ h(f, g) &= (hf, hg) \end{aligned}$$

$$S^m = S^{\oplus m} = \underbrace{S \oplus \cdots \oplus S}_{m \text{ times}}.$$

A finitely generated graded free S-module is a graded module M with a homogeneous basis: a graded module isomorphic to

$$\bigoplus_{i=1}^{m} S(-d_i)$$

for some m and twists, $-d_i$.

Non-example

$$I = (x, y) \subset S = k[x, y]$$

is not free. For instance,

$$\begin{array}{rcl} S(-1) \oplus S(-1) & \to & I \\ e_1 = (1, 0) & \mapsto & x \\ e_2 = (0, 1) & \mapsto & y \end{array}$$

is not an isomorphism. The element $ye_1 - xe_2$ is in the kernel:

$$ye_1 - xe_2 \quad \mapsto \quad y(x) - x(y) = 0.$$

Free resolutions

$$I \subseteq S, \quad \text{homogeneous}$$
$$M = S/I$$

Strategy for calculating H_M

Approximate M with free modules.

Free resolution of length ℓ

$$0 \longrightarrow F_\ell \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where each F_i is free.

Example

$$I = (x, y) \cap (z, w) = (xz, yz, xw, yw) \subset S = k[x, y, z, w]$$

$$\begin{array}{ccccccc}
0 \to S(-4) & \xrightarrow{\begin{pmatrix} -x \\ y \\ z \\ -w \end{pmatrix}} & S(-3)^4 & \xrightarrow{\begin{pmatrix} 0 & w & 0 & y \\ w & 0 & 0 & -x \\ 0 & -z & y & 0 \\ -z & 0 & -x & 0 \end{pmatrix}} & S(-2)^4 \\
& & & & \Big\downarrow {\scriptstyle (\, xz \;\; yz \;\; xw \;\; yw \,)} \\
0 & \longleftarrow & S/I & \longleftarrow & S
\end{array}$$

$$0 \to S(-4) \to S(-3)^4 \to S(-2)^4 \to S \to S/I \to 0$$

Example continued

$$I = (x, y) \cap (z, w) = (xz, yz, xw, yw) \subset S = k[x, y, z, w]$$

$$0 \to S(-4) \to S(-3)^4 \to S(-2)^4 \to S \to S/I \to 0$$

$$0 \to S(-4)_d \to S(-3)^4_d \to S(-2)^4_d \to S_d \to (S/I)_d \to 0$$

$$0 \to S_{d-4} \to S^4_{d-3} \to S^4_{d-2} \to S_d \to (S/I)_d \to 0$$

Hilbert function?

Proposition

Given an exact sequence of k-vector spaces

$$0 \to V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{\ell-1}} V_\ell \to 0$$

we have

$$\sum_{i=1}^{\ell} (-1)^i \dim_k V_i = 0.$$

$$0 \to V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{\ell-1}} V_\ell \to 0$$

Proof.

$$0 \to \ker \phi_i \to V_i \to \mathrm{im}\, \phi_i \to 0$$

$$0 \to \ker \phi_{i+1} \to V_{i+1} \to \mathrm{im}\, \phi_{i+1} \to 0$$

Rank-nullity:

$$\dim \ker \phi_i - \dim V_i + \dim \mathrm{im}\, \phi_i = 0$$

$$-\dim \ker \phi_{i+1} + \dim V_{i+1} - \dim \mathrm{im}\, \phi_{i+1} = 0$$

Exactness:

$$\mathrm{im}\, \phi_i = \ker \phi_{i+1}$$

□

Example continued

$$I = (x, y) \cap (z, w) = (xz, yz, xw, yw) \subset S = k[x, y, z, w]$$

$$0 \to S_{d-4} \to S^4_{d-3} \to S^4_{d-2} \to S_d \to (S/I)_d \to 0$$

$$\dim S_d = \binom{3+d}{3}$$

$$H(d) = \binom{3+d}{3} - 4\binom{3+d-2}{3} + 4\binom{3+d-3}{3} - \binom{3+d-4}{3}$$

$$H(d) = \binom{d+3}{3} - 4\binom{d+1}{3} + 4\binom{d}{3} - \binom{d-1}{3}$$

Example continued

$$I = (x, y) \cap (z, w) = (xz, yz, xw, yw) \subset S = k[x, y, z, w]$$

$$0 \to S_{d-4} \to S^4_{d-3} \to S^4_{d-2} \to S_d \to (S/I)_d \to 0$$

$$H(d) = \binom{d+3}{3} - 4\binom{d+1}{3} + 4\binom{d}{3} - \binom{d-1}{3}$$

$$\begin{aligned} P(t) &= \binom{t+3}{3} - 4\binom{t+1}{3} + 4\binom{t}{3} - \binom{t-1}{3} \\ &= 2t+2 \end{aligned}$$

Another example

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Res(S/Ideal(x^2y,wy,z^5xy));

0 --> S(-9) --> S(-4)(+)S^2(-8) --> S(-2)(+)S(-3)(+)S(-7) --> S
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Describe(It);
Mat([
  [yw, x^2y, xyz^5]
])
Mat([
  [x^2, xz^5, 0],
  [-w, 0, z^5],
  [0, -w, -x]
])
Mat([
  [z^5],
  [-x],
  [w]
])

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Theorem (Hilbert Syzygy Theorem)

If $S = k[x_0, \ldots, x_n]$, then every f.g. graded S-module has a free resolution of length $\leq n + 1$.

Minimal free resolution

$$0 \to F_\ell \to \cdots \to F_1 \to F_0 \to S/I \to 0$$

$$F_j = \bigoplus_{i=1}^{\beta_j} S(-d_{ij})$$

- projective dimension: $\mathrm{pd}(S/I) = \ell$
- Betti numbers: $\beta_\ell, \ldots, \beta_0$
- twists: $\{-d_{ij}\}$
- arithmetically Cohen-Macaulay: $\mathrm{pd}(S/I) = \mathrm{codim}(S/I)$.
- arithmetically Gorenstein: aCM and $(\beta_\ell, \ldots, \beta_0)$ is symmetric about its center

A big event in 2008

- Boij and Söderberg conjectures
- multiplicity conjecture of Huneke and Srinivasan

Proved by Eisenbud and Schreyer