$X \subseteq \mathbb{P}^n$ an algebraic set

**Question:** How many hypersurfaces of degree $d$ contain $X$?

$Z(f) \supseteq X \iff f \in I(X)$

**Notation**

$I(X)_d = \{f \in I(X) : f \text{ homog. of degree } d\}$

$I(X) \text{ homogeneous } \Rightarrow I(X) = \bigoplus_{d \geq 0} I(X)_d$

**Rephrasing:** What is the size of $I(X)_d$?
Examples: $X \subseteq \mathbb{P}^2$, $d = 2$

$$f = a_0 x^2 + a_1 xy + a_2 xz + a_3 y^2 + a_4 yz + a_5 z^2$$

$X = \{ (0, 0, 1) \}$

$$f(0, 0, 1) = 0 \iff a_5 = 0$$

One linear condition on the vector of coefficients.

$X = \{ (0, 0, 1), (1, 0, 1) \}$

$$f(0, 0, 1) = 0 \iff a_5 = 0$$

$$f(1, 0, 1) = 0 \iff a_0 + a_2 + a_5 = 0$$

Two linear conditions on the vector of coefficients.
Examples: $X \subseteq \mathbb{P}^2$, $d = 2$

$$f = a_0 x^2 + a_1 xy + a_2 xz + a_3 y^2 + a_4 yz + a_5 z^2$$

$X = \{(0,0,1), (1,0,1), (2,0,1)\}$

$$f(0,0,1) = 0 \iff a_5 = 0$$
$$f(1,0,1) = 0 \iff a_0 + a_2 + a_5 = 0$$
$$f(2,0,1) = 0 \iff 4a_0 + 2a_2 + a_5 = 0$$

Equivalent to

$$a_0 = a_2 = a_5 = 0.$$ 

Three linear conditions on the vector of coefficients.
Examples: \( X \subseteq \mathbb{P}^2, \quad d = 2 \)

\[
f = a_0 x^2 + a_1 xy + a_2 xz + a_3 y^2 + a_4 yz + a_5 z^2
\]

\[
X = \{(0, 0, 1), (1, 0, 1), (2, 0, 1), (3, 0, 1)\}
\]

\[
f(0, 0, 1) = 0 \iff a_5 = 0
\]
\[
f(1, 0, 1) = 0 \iff a_0 + a_2 + a_5 = 0
\]
\[
f(2, 0, 1) = 0 \iff 4a_0 + 2a_2 + a_5 = 0
\]
\[
f(3, 0, 1) = 0 \iff 9a_0 + 3a_2 + a_5 = 0
\]

Equivalent to

\[
a_0 = a_2 = a_5 = 0.
\]

Still, three linear conditions on the vector of coefficients.
A Question

The Hilbert function

Hilbert’s theorem

Examples: $X \subseteq \mathbb{P}^2$, $d = 2$

$$f = a_0 x^2 + a_1 xy + a_2 xz + a_3 y^2 + a_4 yz + a_5 z^2$$

$X = \{(0, 0, 1), (1, 0, 1), (2, 0, 1), (3, 0, 1), (4, 0, 1)\}$

$$a_0 = a_2 = a_5 = 0.$$ 

Again, three linear conditions on the vector of coefficients.
Equivalent questions

\( X \subseteq \mathbb{P}^n \) an algebraic set

- How many hypersurfaces of degree \( d \) contain \( X \)?
- What is the size of \( I(X)_d \)?
- How many conditions are placed on a hypersurface of degree \( d \) by requiring it to contain \( X \)?
The Hilbert function

$$S = k[x_0, \ldots, x_n]$$

$$S_d = \{ f \in S : f \text{ homog., degree } d \}$$

**Definition**

The **homogeneous coordinate ring** for $$X \subseteq \mathbb{P}^n$$ is

$$S(X) = S/I(X)$$

Its degree $$d$$ part is

$$S(X)_d = S_d/I(X)_d.$$
An exact sequence

\[0 \rightarrow I(X) \rightarrow S \rightarrow S(X) \rightarrow 0\]

restricted to degree \(d\)

\[0 \rightarrow I(X)_d \rightarrow S_d \rightarrow S(X)_d \rightarrow 0\]

rank-nullity theorem

\[\dim_k S(X)_d = \dim_k S_d - \dim_k I(X)_d\]
Definition
The Hilbert function of $X \subseteq \mathbb{P}^n$ is

$$H_X(d) = \dim_k S(X)_d = \dim_k S_d - \dim_k I(X)_d.$$  

Questions
- How many hypersurfaces of degree $d$ contain $X$?
- What is the size of $I(X)_d$?
- How many conditions are placed on a hypersurface of degree by requiring it to contain $X$?
- What is the Hilbert function of $X$?
Example 1

\[
\begin{align*}
X &= \mathbb{P}^n, \quad I(X) = (0) \\
S(X) &= S/(0) = S = k[x_0, \ldots, x_n] \\
H_{\mathbb{P}^n}(d) &= \dim S_d = \dim k[x_0, \ldots, x_n]_d.
\end{align*}
\]
\[ n = 2, \quad S = k[x_0, x_1, x_2] \]

<table>
<thead>
<tr>
<th>(d)</th>
<th>basis for (S_d)</th>
<th>(H_{\mathbb{P}^2}(d))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(x_0, x_1, x_2)</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>(x_0^3, x_0^2x_1, \ldots, x_2^3)</td>
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</tr>
<tr>
<td>4</td>
<td>\ldots</td>
<td>15</td>
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</table>
boxes-and-dots argument

One-to-one correspondence

\[
\begin{align*}
&x_0 \quad \bullet \quad x_1 \quad x_1 \quad \bullet \quad x_2 \quad \leftrightarrow \quad x_0 x_1^2 x_2 \\
&x_0 \quad x_0 \quad \bullet \quad x_1 \quad \bullet \quad x_2 \quad \leftrightarrow \quad x_0^2 x_1 x_2 \\
&x_0 \quad x_0 \quad \bullet \quad \bullet \quad x_0 \quad x_0 \quad \leftrightarrow \quad x_0^2 x_0^2 \\
&\bullet \quad \bullet \quad x_2 \quad x_2 \quad x_2 \quad x_2 \quad \leftrightarrow \quad x_2^4
\end{align*}
\]

Hilbert function for \( X = \mathbb{P}^n \)

In general, need \( n \) dots (for \( n + 1 \) indeterminates), and \( n + d \) slots:

\[
H_{\mathbb{P}^n}(d) = \binom{n + d}{n}
\]
Example 2

\[ X = \{(0, 0, 1), (1, 0, 1), (2, 0, 1), (3, 0, 1), (4, 0, 1)\} \]

CoCoA results

\[
\begin{align*}
  d : & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \ldots \\
  H_X(d) : & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 5 \quad 5 \quad \ldots 
\end{align*}
\]
Example 3

\[ X = \text{twisted cubic} \subset \mathbb{P}^3 \]

CoCoA results

\[ H_X(d) = 3d + 1 \quad \text{for } d \geq 0 \]

<table>
<thead>
<tr>
<th>( d )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tbody>
<tr>
<td>( \dim_k k[x, y, z, w] )</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>20</td>
<td>35</td>
<td>56</td>
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<tr>
<td>( H_X(d) )</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>16</td>
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<tr>
<td>( \dim \mathcal{I}(X)_d )</td>
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<td>0</td>
<td>3</td>
<td>10</td>
<td>22</td>
<td>40</td>
</tr>
</tbody>
</table>
Example 4

Hypersurface

\[ l(X) = (f), \quad f \in S = k[x_0, \ldots, x_n], \quad \deg f = e \]

\[ 0 \rightarrow S \xrightarrow{\cdot f} S \rightarrow S(X) \rightarrow 0 \]
\[ 0 \rightarrow S_{d-e} \xrightarrow{\cdot f} S_d \rightarrow S(X)_d \rightarrow 0 \]

\[ H_X(d) = \dim S(X)_d = \begin{cases} 
\dim_k S_d & \text{for } d < e \\
\dim_k S_d - \dim_k S_{d-e} & \text{for } d \geq e \\
\binom{n+d}{n} & \text{for } d < e \\
\binom{n+d}{n} - \binom{n+d-e}{n} & \text{for } d \geq e 
\end{cases} \]
Example 4 continued

\[ H_X(d) = \dim S(X)_d = \begin{cases} 
\binom{n+d}{n} & \text{for } d < e \\
\binom{n+d}{n} - \binom{n+d-e}{n} & \text{for } d \geq e
\end{cases} \]

**HW:** For \( d \geq e \),

\[ H_X(d') = \frac{e}{(n-1)!} d^{n-1} + \text{lots} \]
Theorem (Hilbert)

For $X \subseteq \mathbb{P}^n$, there exists a unique polynomial $P_X(d)$ such that

$$P_X(d) = H_X(d)$$

for $d \gg 0$.

- $P_X(d)$ is called the Hilbert polynomial of $X$;
- $\deg P_X = \dim X$;
- the leading coefficient of $P_X(d)$ is
  $$\frac{e}{(\dim X)!}$$

where $\deg X = e$. 