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Lecture 7: Hilbert functions

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$X \subseteq \mathbb{P}^n$ an algebraic set

Question: How many hypersurfaces of degree d contain X?

$$Z(f) \supseteq X \quad \Longleftrightarrow \quad f \in I(X)$$

Notation

$$I(X)_d = \{f \in I(X) : f \text{ homog. of degree } d\}$$

$$I(X) \text{ homogeneous} \implies \quad I(X) = \bigoplus_{d \geq 0} I(X)_d$$

Rephrasing: What is the size of $I(X)_d$?

Examples: $X \subseteq \mathbb{P}^2$, $d = 2$

$$f = a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$$

$X = \{(0,0,1)\}$

$$f(0,0,1) = 0 \iff a_5 = 0$$

One linear condition on the vector of coefficients.

$X = \{(0,0,1),(1,0,1)\}$

$$f(0,0,1) = 0 \iff a_5 = 0$$
$$f(1,0,1) = 0 \iff a_0 + a_2 + a_5 = 0$$

Two linear conditions on the vector of coefficients.

Examples: $X \subseteq \mathbb{P}^2, \quad d = 2$

$$f = a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$$

$X = \{(0,0,1),(1,0,1),(2,0,1)\}$

$$\begin{aligned} f(0,0,1) = 0 &\iff a_5 = 0 \\ f(1,0,1) = 0 &\iff a_0 + a_2 + a_5 = 0 \\ f(2,0,1) = 0 &\iff 4a_0 + 2a_2 + a_5 = 0 \end{aligned}$$

Equivalent to

$$a_0 = a_2 = a_5 = 0.$$

Three linear conditions on the vector of coefficients.

Examples: $X \subseteq \mathbb{P}^2, \quad d = 2$

$$f = a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$$

$X = \{(0,0,1), (1,0,1), (2,0,1), (3,0,1)\}$

$$\begin{aligned}
f(0,0,1) = 0 \quad &\iff \quad a_5 = 0 \\
f(1,0,1) = 0 \quad &\iff \quad a_0 + a_2 + a_5 = 0 \\
f(2,0,1) = 0 \quad &\iff \quad 4a_0 + 2a_2 + a_5 = 0 \\
f(3,0,1) = 0 \quad &\iff \quad 9a_0 + 3a_2 + a_5 = 0
\end{aligned}$$

Equivalent to

$$a_0 = a_2 = a_5 = 0.$$

Still, three linear conditions on the vector of coefficients.

Examples: $X \subseteq \mathbb{P}^2, \quad d = 2$

$$f = a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$$

$X = \{(0,0,1),(1,0,1),(2,0,1),(3,0,1),(4,0,1)\}$

$$a_0 = a_2 = a_5 = 0.$$

Again, three linear conditions on the vector of coefficients.

Equivalent questions

$X \subseteq \mathbb{P}^n$ an algebraic set

- How many hypersurfaces of degree d contain X?
- What is the size of $I(X)_d$?
- How many conditions are placed on a hypersurface of degree d by requiring it to contain X?

$$\begin{aligned} S &= k[x_0, \ldots, x_n] \\ S_d &= \{f \in S : f \text{ homog., degree } d\} \end{aligned}$$

Definition

The homogeneous coordinate ring for $X \subseteq \mathbb{P}^n$ is

$$S(X) = S/I(X)$$

Its degree d part is

$$S(X)_d = S_d/I(X)_d.$$

An exact sequence

$$0 \to I(X) \to S \to S(X) \to 0$$

restricted to degree d

$$0 \to I(X)_d \to S_d \to S(X)_d \to 0$$

rank-nullity theorem

$$\dim_k S(X)_d = \dim_k S_d - \dim_k I(X)_d$$

Definition

The Hilbert function of $X \subseteq \mathbb{P}^n$ is

$$H_X(d) = \dim_k S(X)_d = \dim_k S_d - \dim_k I(X)_d.$$

Questions

- How many hypersurfaces of degree d contain X?
- What is the size of $I(X)_d$?
- How many conditions are placed on a hypersurface of degree by requiring it to contain X?
- What is the Hilbert function of X?

Example 1

$$
\begin{aligned}
X &= \mathbb{P}^n, \quad I(X) = (0) \\
S(X) &= S/(0) = S = k[x_0, \ldots, x_n] \\
H_{\mathbb{P}^n}(d) &= \dim S_d = \dim k[x_0, \ldots, x_n]_d.
\end{aligned}
$$

$$n = 2, \qquad S = k[x_0, x_1, x_2]$$

d	basis for S_d	$H_{\mathbb{P}^2}(d)$
0	1	1
1	x_0, x_1, x_2	3
2	$x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2$	6
3	$x_0^3, x_0^2x_1, \ldots, x_2^3$	10
4	$\ldots$	15

boxes-and-dots argument

One-to-one correspondence

$$\boxed{x_0}\ \boxed{\bullet}\ \boxed{x_1}\ \boxed{x_1}\ \boxed{\bullet}\ \boxed{x_2} \longleftrightarrow x_0x_1^2x_2$$

$$\boxed{x_0}\ \boxed{x_0}\ \boxed{\bullet}\ \boxed{x_1}\ \boxed{\bullet}\ \boxed{x_2} \longleftrightarrow x_0^2x_1x_2$$

$$\boxed{x_0}\ \boxed{x_0}\ \boxed{\bullet}\ \boxed{\bullet}\ \boxed{x_0}\ \boxed{x_0} \longleftrightarrow x_0^2x_2^2$$

$$\boxed{\bullet}\ \boxed{\bullet}\ \boxed{x_2}\ \boxed{x_2}\ \boxed{x_2}\ \boxed{x_2} \longleftrightarrow x_2^4$$

Hilbert function for $X = \mathbb{P}^n$

In general, need n dots (for $n+1$ indeterminates), and $n+d$ slots:

$$H_{\mathbb{P}^n}(d) = \binom{n+d}{n}$$

Example 2

$$X = \{(0,0,1),(1,0,1),(2,0,1),(3,0,1),(4,0,1)\}$$

CoCoA results

$$\begin{array}{rcccccccc} d: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ H_X(d): & 1 & 2 & 3 & 4 & 5 & 5 & 5 & \dots \end{array}$$

Example 3

$$X = \text{twisted cubic} \subset \mathbb{P}^3$$

CoCoA results

$$H_X(d) = 3d + 1 \quad \text{for } d \geq 0$$

d :	0	1	2	3	4	5
$\dim_k k[x, y, z, w]$:	1	4	10	20	35	56
$H_X(d)$:	1	4	7	10	13	16
$\dim I(X)_d$:	0	0	3	10	22	40

Example 4

Hypersurface

$$I(X) = (f), \qquad f \in S = k[x_0, \dots, x_n], \qquad \deg f = e$$

$$0 \longrightarrow S \xrightarrow{\cdot f} S \longrightarrow S(X) \longrightarrow 0$$
$$0 \longrightarrow S_{d-e} \xrightarrow{\cdot f} S_d \longrightarrow S(X)_d \longrightarrow 0$$

$$H_X(d) = \dim S(X)_d = \begin{cases} \dim_k S_d & \text{for } d < e \\ \dim_k S_d - \dim_k S_{d-e} & \text{for } d \geq e \end{cases}$$

$$= \begin{cases} \binom{n+d}{n} & \text{for } d < e \\ \binom{n+d}{n} - \binom{n+d-e}{n} & \text{for } d \geq e \end{cases}$$

Example 4 continued

$$H_X(d) = \dim S(X)_d = \begin{cases} \binom{n+d}{n} & \text{for } d < e \\ \binom{n+d}{n} - \binom{n+d-e}{n} & \text{for } d \geq e \end{cases}$$

HW: For $d \geq e$,

$$H_X(d) = \frac{e}{(n-1)!} d^{n-1} + \text{lots}$$

Theorem (Hilbert)

For $X \subseteq \mathbb{P}^n$, there exists a unique polynomial $P_X(d)$ such that

$$P_X(d) = H_X(d)$$

for $d \gg 0$.

- $P_X(d)$ is called the Hilbert polynomial of X;
- $\deg P_X = \dim X$;
- the leading coefficient of $P_X(d)$ is

$$\frac{e}{(\dim X)!}$$

  where $\deg X = e$.