PCMI 2008 Undergraduate Summer School

Lecture 6: Projective space II

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Homogeneous Coordinates

$\mathbb{P}^n = \{\text{lines in } \mathbb{A}^{n+1}\}$

We name $\ell \in \mathbb{P}^n$, with any $p \in \ell$.

Equivalence relation on $\mathbb{A}^{n+1} \setminus \{0\}$

$$p \sim \lambda p \text{ for } \lambda \in k \setminus \{0\}$$

$$\mathbb{P}^n = \left(\mathbb{A}^{n+1} \setminus \{0\}\right) / \sim$$

Standard open cover

Definition

$$\begin{aligned} U_i = U_{x_i} &= \{(x_0, \ldots, x_n) \in \mathbb{P}^n : x_i \neq 0\} \\ &= Z(x_i)^c \end{aligned}$$

Charts

$$\begin{aligned} \phi_i \colon U_i &\to \mathbb{A}^n \\ (x_0, \ldots, x_n) &\mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}\right) \end{aligned}$$

inverse

$$\begin{aligned} \psi_i \colon \mathbb{A}^n &\to U_i \\ (x_1, \ldots, x_n) &\mapsto (x_1, \ldots, 1, \ldots, x_n) \end{aligned}$$

$\mathbb{P}^n$ is a manifold with atlas $\{(U_i, \phi_i)\}_{i=0}^n$.

Projective closure

$$\begin{array}{ccc} X \subseteq \mathbb{A}^n & \subset & \mathbb{P}^n \\ (x_1, \ldots, x_n) & \mapsto & (1, x_1, \ldots, x_n) \end{array}$$

$\bar{X}$ = smallest proj. alg. set of $\mathbb{P}^n$ containing X

$$I(\bar{X}) = ?$$

The homogenization of $f \in k[x_1, \ldots, x_n]$ with respect to x_0 is

$$f^h = x_0^{\deg f} f(\tfrac{x_1}{x_0}, \ldots, \tfrac{x_n}{x_0}).$$

Example

If $f(x, y) = x^2 - 2x^3y + y + 2$, then

$$\begin{aligned} z^4 f(\tfrac{x}{z}, \tfrac{y}{z}) &= z^4 \left(\left(\tfrac{x}{z}\right)^2 - 2\left(\tfrac{x}{z}\right)^3 \left(\tfrac{y}{z}\right) + 2 \right) \\ &= x^2z^2 - 2x^3y + yz^3 + 2z^4. \end{aligned}$$

Definition

The homogenization of an ideal $I \subseteq k[x_1, \ldots, x_n]$ with respect to x_0 is

$$I^h = (f^h : f \in I) \subseteq k[x_0, \ldots, x_n].$$

HW

$$I(\bar{X}) = I(X)^h$$

Example

$$X = Z(y - x^2), \qquad \bar{X} = Z(yz - x^2)$$

Example

twisted cubic

$$C = Z(y - x^2, z - x^3), \qquad \bar{C} = Z(wy - x^2, zw - xy, y^2 - xz).$$

Caution

$$X = Z(f_1, \ldots, f_m) \quad \not\Rightarrow \quad \bar{X} = Z(f_1^h, \ldots, f_m^h).$$

The finite part of a projective algebraic set

$$Y = Z(J) \subseteq \mathbb{P}^n, \qquad J \subseteq k[x_0, \ldots, x_n], \quad \text{homogeneous}$$

Choose some std. open set, say $U_0 \subset \mathbb{P}^n$.

The finite part of Y is

$$\begin{array}{rcl} Y_* = U_0 \cap Y \subset U_0 & \approx & \mathbb{A}^n \\ (y_0, \ldots, y_n) & \mapsto & (\frac{y_1}{y_0}, \ldots, \frac{y_n}{y_0}) \end{array}$$

$$\begin{array}{rcl} J_* & = & (f(1, x_1, \ldots, x_n) : f \in J) \\ U_0 \cap Y & = & Z(J_*) \end{array}$$

One-to-one correspondence

affine algebraic sets in $\mathbb{A}^n$ not equal to $\mathbb{A}^n$

$$\updownarrow$$

projective algebraic sets in $\mathbb{P}^n$
with no component contained in or containing $\{x_0 = 0\}$

$$\begin{array}{ccc} X & & Y_* \\ \downarrow & & \uparrow \\ \bar{X} & & Y \end{array}$$

Points at ∞

$$\begin{array}{ccc} X \subseteq \mathbb{A}^n & \subset & \mathbb{P}^n \\ (x_1, \ldots, x_n) & \mapsto & (1, x_1, \ldots, x_n) \end{array}$$

Definition

The points at ∞ on X are the points

$$\bar{X} \setminus X = \bar{X} \cap Z(x_0).$$

Examples

X	$\bar{X}$	$\bar{X} \setminus X$	∞
$y - x^2$	$zy - x^2$	$z = x^2 = 0$	$(0, 1, 0)$
$xy - 1$	$xy - z^2$	$z = xy = 0$	$(1, 0, 0)$ $(0, 1, 0)$
$x^2 + y^2 - 1$	$x^2 + y^2 - z^2$	$z = x^2 + y^2 = 0$	$(i, 1, 0)$ $(1, i, 0)$

HW

A plane conic is a circle iff it passes through the circular points at ∞: $(i, 1, 0)$, $(1, i, 0)$.

Changing coordinates

To see a point at infinity p on an alg. set X:

- Choose i such that $p_i \neq 0$, i.e., such that $p \in U_i$.
- Set $x_i = 1$ in the equations defining $\bar{X}$.

Example

$X = Z(y - x^2), \quad \bar{X} = Z(zy - x^2)$

$p = (0, 1, 0) \in U_y$

$\bar{X} \cap U_y \approx Z(z - x^2)$.

Example

$$X = Z(x^2 + y^2 - 1), \quad \bar{X} = Z(x^2 + y^2 - z^2)$$

$$p = (\pm i, 1, 0) \in U_y$$

$$\bar{X} \cap U_y \approx Z(x^2 + 1 - z^2)$$

Duality

Definition

hyperplane in $\mathbb{P}^n$:

$$H = Z(a_0 x_0 + \cdots + a_n x_n), \quad \text{not all } a_i = 0.$$

The set of all hyperplanes in $\mathbb{P}^n$ is called the dual projective space, denoted $(\mathbb{P}^n)^*$.

One-to-one correspondence

$$\begin{array}{ccc} \mathbb{P}^n & \approx & (\mathbb{P}^n)^* \\ p = (a_0, \ldots, a_n) & \leftrightarrow & H_p = Z(a_0 x_0 + \cdots + a_n x_n) \end{array}$$

Example of duality

Two distinct points determine a unique line in $\mathbb{P}^2$.

Two distinct lines determine a unique point in $\mathbb{P}^2$.

Proof.

Points $(a_0, a_1, a_2), (b_0, b_1, b_2)$ lie on line $c_0x + c_1y + c_2z = 0$ iff

$$(c_0, c_1, c_2) \in \ker \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

Dually, lines $a_0x + a_1y + a_2z = 0, b_0x + b_1y + b_2z = 0$ contain point (c_0, c_1, c_2) iff

$$(c_0, c_1, c_2) \in \ker \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

In either case, the kernel has dimension 1. $\square$

Conics

conic in $\mathbb{P}^2$

$$C = Z(a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2)$$

not all $a_i = 0$.

One-to-one correspondence

$$\begin{array}{ccc} \mathbb{P}^5 & \approx & \{\text{conics in } \mathbb{P}^2\} \\ (a_0, \ldots, a_5) & \leftrightarrow & Z(a_0x_0^2 + \cdots + a_5z^2) \end{array}$$

Fix $p = (p_0, p_1, p_2) \in \mathbb{P}^2$.

General conic: $a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$.

conics passing through p

$$\Big\{(a_0, \ldots, a_5) \in \mathbb{P}^5 :$$
$$a_0p_0^2 + a_1p_0p_1 + a_2p_op_2 + a_3p_1^2 + a_4p_1p_2 + a_5p_2^2 = 0\Big\}$$

hyperplane H_q with coefficients

$$q = (p_0^2, p_0p_1, p_0p_2, p_1^2, p_1p_2, p_2^2).$$

Veronese embedding

$$\begin{array}{rcl} \nu_2 : \mathbb{P}^2 & \to & \mathbb{P}^5 \\ (x, y, z) & \mapsto & (x^2, xy, xz, y^2, yz, z^2) \end{array}$$

$$\{\text{conics through } p \in \mathbb{P}^2\} = H_{\nu_2(p)} \in (\mathbb{P}^5)^*$$

Conics tangent to $x = 0$

$$\begin{aligned} f(x, y, z) &= a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 = 0 \\ x &= 0 \end{aligned}$$

$$f(0, y, z) = a_3y^2 + a_4yz + a_5z^2 = 0$$

The conics tangent to the line $x = 0$ form the quadric hypersurface

$$a_4^2 - 4a_3a_5 = 0$$

in $\mathbb{P}^5$.