In the following, let $S = k[x_0, \ldots, x_n]$.

$\star$ 1. Show that an ideal $I \subseteq S$ is homogeneous iff each I contains the homogeneous components of each of its elements.

$\star$ 2. If k is infinite and $f \in S$, show that f vanishes along a line through the origin iff each of its homogeneous components vanishes along the line.

3. Let $I \subsetneq S$ be a homogeneous ideal. Show that I is a prime ideal iff for all homogeneous polynomials f and g with $fg \in I$, either $f \in I$ or $g \in I$.

4. Show that the radical of a homogeneous ideal is homogeneous.

$\star$ 5. Duality.

A *hyperplane* in $\mathbb{P}^n$ is any projective algebraic set of the form $H = Z(a_0x_0 + \cdots + a_nx_n)$ with each $a_i \in k$ and not all $a_i = 0$. One may think of the coefficients as giving a point $p = (a_0, \ldots, a_n)$ in $\mathbb{P}^n$. We write $H^* = p$ and $p^* = H$.

(a) Show that this gives a well-defined, one-to-one correspondence between hyperplanes and points in $\mathbb{P}^n$.

(b) Show that $H^{**} = H$ for each hyperplane H and $p^{**} = p$ for each point p.

(c) Show that $p \in H$ iff $H^* \in p^*$.

$\star$ 6. Conics

A *conic* in the projective plane, $\mathbb{P}^2$, is any algebraic set of the form

$$Z(a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2)$$

with each $a_i \in k$ and not all $a_i = 0$. This sets up a one-to-one correspondence between points $(a_0, \ldots, a_5) \in \mathbb{P}^5$ and plane conics. Thus, projective 5-space can be thought of as the set of all plane conics. (Note: to think about these conics as conics in the "ordinary" affine plane, set $z = 1$ in the defining equation. For instance, $x^2+y^2-z^2 = 0$ gives the circle $x^2 + y^2 - 1 = 0$.)

(a) Fix a point $p \in \mathbb{P}^2$. Describe the set of points in $\mathbb{P}^5$ corresponding to conics in $\mathbb{P}^2$ passing through p. (To be more concrete, take $p = (0, 0, 1)$ or $p = (1, 1, 1)$ to start.)

(b) Describe the set of points in $\mathbb{P}^5$ corresponding to conics tangent to $x = 0$. (By "tangent" we mean that if $f = 0$ defines the conic, then the system given by $f = 0$ and $x = 0$ has only one solution, even assuming k is algebraically closed.)