1. For each of the polynomial mappings $X \rightarrow Y$, describe corresponding ring homomorphisms, $A(Y) \rightarrow A(X)$, using the notation of problem 2.

(a) 
$$
\phi : \mathbb{A}^2 \rightarrow \mathbb{A}^3 \\
(x, y) \mapsto (y - x^2, xy, x^3 + 2y^2)
$$

(b) $X = \mathbb{A}^1$ and $Y = Z((y - x^3, z - xy) \subset \mathbb{A}^3$
$$
\phi : X \rightarrow Y \\
t \mapsto (t, t^3, t^4)
$$

2. For each of the ring homomorphisms $A(Y) \rightarrow A(X)$, describe the corresponding morphism of algebraic sets, $X \rightarrow Y$, using the notation of problem 1.

(a) 
$$
\sigma : k[x, y] \rightarrow k[t] \\
x \mapsto t^2 - 1 \\
y \mapsto t(t^2 - 1)
$$

(b) 
$$
\sigma : k[s, t, u, w]/(s^2 - w, sw - tu) \rightarrow K[x, y, z]/(xy - z) \\
s \mapsto xy \\
t \mapsto yz \\
u \mapsto xz \\
w \mapsto z^2
$$

The morphism constructed here is a mapping of the saddle surface to a surface in $\mathbb{A}^4$.

3. Show that the mapping in 1b, above, is an isomorphism by showing that the induced mappings of coordinate rings is an isomorphism of rings.

4. In 2a, above, let $C$ denote the image of the corresponding mapping, $\phi : \mathbb{A}^1 \rightarrow \mathbb{A}^2$. Compute $I(C)$. Draw a picture of $C$ letting $k = \mathbb{R}$.

5. We have seen that the parabola, $Z(y - x^2)$, is isomorphic to $\mathbb{A}^1$. Show that the same is not true of the hyperbola, $Z(xy - 1)$. For a challenge consider the circle $Z(x^2 + y^2 - 1)$. Is it ever isomorphic to the parabola? What if the characteristic of $k$ is 2?

In a previous problem set, we discussed the Zariski topology on \( \mathbb{A}^n \). The closed sets of the topology are taken to be algebraic sets, i.e., sets of the form \( \mathbb{Z}(I) \) where \( I \) is an ideal of \( R = k[x_1, \ldots, x_n] \). Consider \( \mathbb{A}^n \) with the Zariski topology.

⋆ (a) Let \( X \subseteq \mathbb{A}^n \). Show that \( \mathbb{Z}(I(X)) \) is the closure of set \( X \). This means that \( \mathbb{Z}(I(X)) \) is the smallest closed set containing \( X \). (Show that if \( Y \) is a closed set containing \( X \), then \( Y \supseteq \mathbb{Z}(I(X)) \).)

(b) What is the closure (in the Zariski topology!) of the open unit disc centered at the origin in \( \mathbb{R}^2 \)?

(c) What is the closure of the set \( \{(n, n) : n \in \mathbb{Z}\} \) in \( \mathbb{A}^2_{\mathbb{Q}} \)?

(d) Show that over an algebraically closed field (or just an infinite field), the closure of any nonempty open set of \( \mathbb{A}^n \) is \( \mathbb{A}^n \), i.e., every nonempty open set is dense. (Again: this is quite a difference from the usual topology in the case of \( k = \mathbb{R} \) or \( \mathbb{C} \).)

7. Is the composition of two polynomial mappings necessarily a polynomial mapping?

8. Consider the curve \( C = \mathbb{Z}(y^3 - x^4) \subset \mathbb{A}^2 \). Find a mapping \( \mathbb{A}^1 \to C \) that is one-to-one and onto but not an isomorphism.

⋆ 9. Suppose that \( k \) is algebraically closed, and let \( X \subseteq \mathbb{A}^n_k \) be an algebraic set. Show that algebraic sets (respectively, varieties, points) contained in \( X \) are in one-to-one correspondence with radical ideals (respectively, prime ideals, maximal ideals) containing \( I(X) \).

10. Degeneracy loci.

(a) Show that the set of singular \( n \times n \) matrices forms an algebraic set in \( \mathbb{A}^{n^2} \). (An \( n \times n \) matrix is singular if it has rank less than \( n \).)

(b) Fix a nonnegative integer \( r \). Show that \( m \times n \) matrices of rank less than \( r \) form an algebraic set in \( \mathbb{A}^{mn} \).

□ (c) Show that the above algebraic sets are varieties.

11. Let \( \sigma : B \to A \) be a homomorphism of rings. Show that if \( p \subseteq A \) is a prime ideal, then \( \sigma^{-1}(p) \) is a prime ideal of \( B \). What if \( p \) is a maximal ideal?

12. Surjectivity.

Let \( \phi : X \to Y \) be a morphism of algebraic sets, and let \( \tilde{\phi} : A(Y) \to A(X) \) be the corresponding homomorphism of coordinate rings.

(a) Show that if \( \tilde{\phi} \) is onto, then \( X \) is isomorphic to the algebraic set \( \mathbb{Z}(\ker \tilde{\phi}) \subseteq Y \).

(b) Show that if \( \phi \) is onto, then \( \ker \tilde{\phi} = (0) \), i.e., \( \tilde{\phi} \) is one-to-one.