Problem Set 2

⋆ = more important problem □ = challenge

Again, \( k \) denotes a field, and \( R = k[x_1, \ldots, x_n] \).

⋆ 1. Intersections and products of ideals. Let \( I \) and \( J \) be ideals of an arbitrary ring \( B \).

(a) Show that \( I \cap J \supseteq IJ \). Given an example to show that the inclusion can be proper.

(b) Show that if \( I + J = B \), then \( I \cap J = IJ \).

(c) If \( B = R = k[x_1, \ldots, x_n] \), show that \( Z(I \cap J) = Z(IJ) \).

Thus, from this exercise and results from Problem Set 1, you should see that

\[
Z(I_1) \cup \cdots \cup Z(I_m) = Z(I_1 \cap \cdots \cap I_m) = Z(I_1 \cdots I_m)
\]

and

\[
\cap _\alpha Z(I_\alpha) = Z(\cup _\alpha I_\alpha) = Z(\sum _\alpha I_\alpha),
\]

where \( \alpha \) runs over an arbitrary index set. The notation \( \sum _\alpha I_\alpha \) denotes the collection of finite sums of the form \( \sum _\alpha f_\alpha \) with \( f_\alpha \in I_\alpha \).

⋆ 2. The Nullstellensatz.

For the following problems, assume that \( k \) is algebraically closed.

(a) Let \( f_1, \ldots, f_m \in R \). Show that the system of equations \( f_1 = \cdots = f_m = 0 \) has no solutions iff 1 is an \( R \)-linear combination of the \( f_i \):

\[
1 = \sum _{i=1}^n g_i f_i
\]

for some polynomials \( g_i \in R \). The implication still runs in one direction, even if \( k \) is not algebraically closed. Which one?

(b) Show that an ideal \( I \subset R \) is maximal iff \( I = (x_1 - a_1, \ldots, x_n - a_n) \) for some \( (a_1, \ldots, a_n) \in \mathbb{A}^n_k \). Show by example that this result does not hold if \( k \) is not algebraically closed. We will go over this problem during the lecture.

(c) If \( I \) is an ideal of \( R \), not equal to \( R \), then \( Z(I) \neq \emptyset \). (This result is called the “weak Nullstellensatz”. ) Again, show this result does not hold if \( k \) is not algebraically closed.

□ 3. Is 1 in the ideal \( (x^2 + y - 3, xy^2 + 2x, y^3) \)? Does the answer depend on \( k \)?
4. Give the decomposition of $X = Z(z - xy, z^2 - xy)$ into irreducibles over $\mathbb{R}$. List the corresponding prime ideals.

5. (a) Show that it possible for $I(\mathbb{A}^n_k) \neq (0)$.
   * (b) Show that $I(\mathbb{A}^n_k) = (0)$ if $k$ is infinite. (Hint: induction on $n$.)

6. Cartesian products. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be algebraic sets.
   
   (a) Show that
   
   $$X \times Y := \{(p_1, \ldots, p_n, q_1, \ldots, q_m) \in \mathbb{A}^{n+m} : (p_1, \ldots, p_n) \in X \text{ and } (q_1, \ldots, q_m) \in Y\}$$
   
   is an algebraic set.

   □ (b) Show that if $X$ and $Y$ are varieties, so is $X \times Y$.

7. Let $B$ be a ring. If $B[x]$ is Noetherian, does it follow that $B$ is Noetherian? (Hint: Consider the mapping $\phi: B[x] \to B$ sending $f(x) \to f(0)$. To show an ideal $I$ of $B$ is finitely generated, look at $\phi^{-1}(I)$.)

8. Show that $\{(t, \cos(t)) : t \in \mathbb{R}\}$ is not an algebraic set. (What can you say about a polynomial $f(x, y)$ such that $f(t, \cos(t)) = 0$ for all $t \in \mathbb{R}$?)