1. Give a description of $Z(z - xy, z^2 - xy)$ over the real numbers: solve the equations and draw the picture.

2. Verify the following basic properties:
   (a) If $S \subseteq R$ and $I$ is the ideal generated by $S$, then $Z(S) = Z(I)$.
   (b) If $S \subseteq T \subseteq R$, then $Z(S) \supseteq Z(T)$.
   (c) If $X \subseteq Y \subseteq \mathbb{A}^n$, then $I(X) \supseteq I(Y)$.
   (d) For $S \subseteq R$ and $X \subseteq \mathbb{A}^n$, we have
      i. $I(Z(S)) \supseteq S$;
      ii. $Z(I(X)) \supseteq X$;
      iii. $Z(I(Z(S))) = Z(S)$ (so if $X$ is an algebraic set, then $Z(I(X)) = X$);
      iv. $I(Z(I(X))) = I(X)$ (so if $J$ is the ideal of an algebraic set, then $I(Z(J)) = J$);

3. (a) Show that if $X$ and $Y$ are algebraic subsets of $\mathbb{A}^n$, then $X = Y$ iff $I(X) = I(Y)$.
   (b) Let $X \subset \mathbb{A}^n$ be an algebraic set and let $p \in \mathbb{A}^n \setminus X$. Show there exists $f \in R$ such that $f(q) = 0$ for all $q \in X$ and $f(p) = 1$. (Hint: let $Y = X \cup \{p\}$ and apply the first part of this problem.)

4. Show that the following are algebraic sets:
   (a) $\{(t, t^2, t^3) : t \in k\}$.
   (b) $\{(\cos t, \sin t) : t \in \mathbb{R}\}$.

5. What are the algebraic subsets of $\mathbb{A}^{1}_k$?

6. Radical ideals.
   (a) Show that the radical of an ideal is an ideal. (Hint: if $f^s$ and $g^t$ are elements of an ideal $I \subseteq R$, then $(f + g)^{s+t} \in I$.)
   (b) Show that $I(X)$ is a radical ideal for all $X \subseteq \mathbb{A}^n$.
   (c) An ideal $I \subset R$ is a prime ideal if $fg \in I$ implies $f$ or $g$ is an element of $I$. Show that a prime ideal is radical.
   (d) If $f \in R$ is irreducible, does it follow that the ideal $(f)$ is radical?
(e) For any ideal $J \subseteq R$, we have $Z(J) = Z(\text{rad}(J))$ and $\text{rad}(J) \subseteq I(Z(J))$.

* 7. Zariski topology.

Let $M$ be any set. A topology on $M$ is a collection $\tau$ of subsets of $M$ such that

(a) $\emptyset \in \tau$;
(b) $M \in \tau$;
(c) $\tau$ is closed under arbitrary unions: if $U_\alpha \in \tau$ for $\alpha$ in some index set $A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$;
(d) $\tau$ is closed under finite intersections: if $U_1, \ldots, U_s \in \tau$ for some integer $s$, then $\bigcap_{i=1}^s U_i \in \tau$.

If $\tau$ is a topology on $M$, then the elements of $\tau$ are called the open sets of the topology. A subset of $M$ is closed if its complement is open.

For example, the usual topology on $\mathbb{R}^n$ is formed by calling a set open if it contains an open ball about each of its points.

(a) Show that $\emptyset$ and $\mathbb{A}^n$ are algebraic sets.
(b) Show that an arbitrary intersection of algebraic sets is an algebraic set.
(c) Show that a finite union of algebraic sets is an algebraic set. (Hint: show $Z(I) \cup Z(J) = Z(\{fg : f \in I, g \in J\})$.)
(d) Explain why it follows that the collection of complements of algebraic sets forms a topology on $\mathbb{A}^n$.
(e) The topology $\mathbb{A}^n$ described above is called the Zariski topology. It is the usual topology of interest to algebraic geometers. Draw some examples of (Zariski) open sets in $\mathbb{A}^2_{\mathbb{R}}$.
(f) Show that the Zariski topology is not Hausdorff in general. That is, given an example of points $p, q \in \mathbb{A}^n$ such that there are no open sets $U$ containing $p$ and $V$ containing $q$ with $U \cap V = \emptyset$. In other words, we cannot necessarily surround distinct points by disjoint open sets. This is quite a difference with the usual topology on $\mathbb{R}^n$.

8. Show that if $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$, then $\{p\}$ is an algebraic set. Show that finite subsets of $\mathbb{A}^n$ are algebraic sets.

9. Draw pictures of $Z(y - x^2)$ over $\mathbb{C}$ and over $\mathbb{Z}/7\mathbb{Z}$.

10. Calculate $Z(x^2 + y^2 - 1, (x - 3)^2 + y^2 - 1) \subset \mathbb{A}_k^2$ for $k = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}/7\mathbb{Z}$ (an intersection of two circles).