PCMI 2008 Undergraduate Summer School

Lecture 1: The Dictionary

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Summer 2008

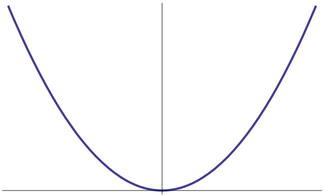
Introduction

Algebraic geometry is the study of solutions to systems of polynomial equations.

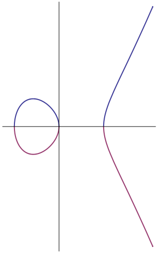
$$\begin{cases} f_1(x_1, \ldots, x_n) & = & 0 \\ f_2(x_1, \ldots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ f_m(x_1, \ldots, x_n) & = & 0 \end{cases}$$

Examples

$y - x^2 = 0$

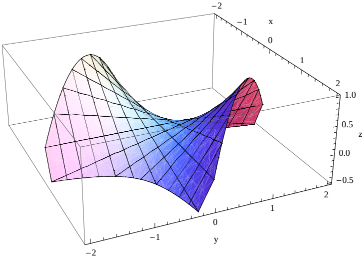


$y^2 - x^3 + x = 0$

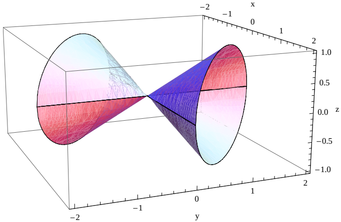


Examples

$z - xy = 0$

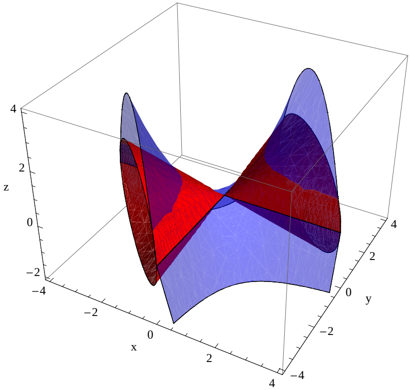


$z^2 - xy = 0$



Examples

$$
\begin{aligned}
z - xy &= 0 \\
z^2 - xy &= 0
\end{aligned}
$$



Basic notation and definitions

- k field, e.g. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/p\mathbb{Z}$.
- $R = k[x_1, \ldots, x_n]$, polynomial ring over k
- affine n-space

$$\mathbb{A}^n = \mathbb{A}^n_k = k^n = \{(a_1, \ldots, a_n) : a_i \in k \text{ for all } i\}.$$

zero sets

- $f_1, \ldots, f_m \in R$,

$$Z(f_1, \ldots, f_m) = \{p \in \mathbb{A}^n : f_1(p) = \cdots = f_m(p) = 0\}$$

- $E \subseteq R$,

$$Z(E) = \{p \in \mathbb{A}^n : f(p) = 0 \text{ for all } p \in E\} = \cap_{f \in E} Z(f)$$

Definition

$X \subseteq \mathbb{A}^n$ is an algebraic set if $X = Z(E)$ for some $E \subseteq R$.

Definition

The ideal of $X \subseteq \mathbb{A}^n$ is

$$I(X) = \{f \in R : f(p) = 0 \text{ for all } p \in X\}$$

Correspondence

Algebra		Geometry
subsets of R	$\longleftrightarrow$	subsets of $\mathbb{A}^n$
E	$\rightarrow$	$Z(E)$
$I(X)$	$\leftarrow$	X

Proposition

Given $E \subset R$, let J be the ideal generated by E:

$$\begin{aligned} J &= (E) \\ &= \{\sum f_i g_i : f_i \in R,\ g_i \in E\} \end{aligned}$$

Then

$$Z(S) = Z(J).$$

Better Correspondence

Algebra		Geometry
ideals of R	$\longleftrightarrow$	algebraic sets in $\mathbb{A}^n$
J	$\rightarrow$	$Z(J)$
$I(X)$	$\leftarrow$	X

Radical ideals

Definition

The radical of an ideal $I \subseteq R$ is

$$\operatorname{rad}(I) = \{f \in R : f^m \in I \text{ for some } m \in \mathbb{Z}_{>0}\}.$$

An ideal $I \subseteq R$ is radical if

$$I = \operatorname{rad}(I).$$

Note: For $X \subseteq \mathbb{A}^n$, the ideal $I(X)$ is radical.

Better Correspondence

Algebra		Geometry
radical ideals	$\longleftrightarrow$	algebraic sets
J	$\rightarrow$	$Z(J)$
$I(X)$	$\leftarrow$	X

Theorem

If k is algebraically closed, then this is a one-to-one correspondence.