Note: A couple of the exercises use the notion of a reduced Gröbner basis. A Gröbner basis G is *reduced* if (i) the leading coefficient of each element of G is 1, and (ii) for each $f \in G$, no monomial of f lies in the collection of initial terms of elements of $G \setminus \{f\}$. It may be formed by computing a Gröbner basis then using the division algorithm among its elements. The relevant CoCoA command is `ReducedGBasis`.

$\star$ 1. Let $J \subseteq I$ be ideals of $k[x_1, \ldots, x_n]$. Prove that $\mathrm{in}_>(J) = \mathrm{in}_>(I)$ if and only if $J = I$.

$\star$ 2. Compute by hand a deglex Gröbner basis for the ideal

$$I = (xy - z^3, y^2 - xz).$$

What is $\mathrm{in}(I)$?

$\star$ 3. Without the aid of a computer, show that $1 \in (x^2 - y, y^2 - x, x - y + 1)$ by writing 1 explicitly in terms of the generators of the ideal. (Hint: start to compute a Gröbner basis to find nice elements of the ideal.)

$\star$ 4. Prove that the Buchberger algorithm terminates. (Hint: at any point in the Buchberger algorithm, we have a set of polynomials, G_{old}, that is being grown into a Gröbner basis. The algorithm terminates unless there is an s-polynomial, s_{ij}, with nonzero remainder upon division by G_{old}. In that case, we add the remainder to G_{old} to get a larger set G_{new}. For any set, G, of polynomials, let $\mathrm{in}(G)$ be the set of initial terms of elements of G, and let $(\mathrm{in}(G))$ be the ideal generated by those initial terms. Prove that

$$(\mathrm{in}(G_{\mathrm{old}})) \subsetneq (\mathrm{in}(G_{\mathrm{new}})).$$

Then use the fact that polynomial rings over a field are Noetherian.)

$\star$ 5. Solve the following system of equations by using a computer to find a Gröbner basis with respect to lex ordering (an elimination ordering), then back substituting:

$$x^3 - yz^2 = y^2 - x = z^3 - x^2 = 1 - yz = 0.$$

What happens if you change the order of the indeterminates?

6. (Kreuzer and Robbiano) **Graph Colorings.**

Let $\Gamma = (V, E)$ be a graph with vertices V and edges E. Suppose $|V| = n$ and there is at most one edge between any two vertices. You are given 3 colors and asked if there is a way to color the vertices of the graph so that no edge connects two vertices of the same color. Call these colorings "acceptable".

Identify the colors with the elements of the field $k = \mathbb{Z}/3\mathbb{Z}$. Let $S = k[x_1, \ldots, x_n]$, and identify a coloring of the graph with an element of $\mathbb{A}^n_k$ in the natural way.

(a) Show that the zeros of the ideal $(x_1^3 - x_1, \ldots, x_n^3 - x_n)$ are exactly the set of all colorings of Γ.

(b) Prove that in a coloring, the vertices i and j have different colors if and only if the coloring satisfies the equation $x_i^2 + x_i x_j + x_j^2 - 1$.

(c) To simplify the equations a bit, assume that the first and second vertices are connected by an edge and that they have colors 0 and 1, respectively. What are the corresponding equations?

(d) Draw some simple graphs, make the corresponding "coloring ideal", and compute a reduced Gröbner basis with respect to lex ordering to find all the acceptable colorings.

(e) Consider the graph formed by connecting the center of a regular 7-gon to its vertices. It has 8 vertices and 14 edges. Use Gröbner bases to show that this graph has no acceptable colorings.

7. (Cox, Little, and O'Shea) For a surprise, use a computer to find a Gröbner basis for the ideal
$$I = (x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1)$$
first using lex ordering, then using revlex.

8. Mora showed that the reduced Gröbner basis with respect to revlex for the ideal
$$I = (x^{n+1} - yz^{n-1}w, xy^{n-1} - z^n, x^n z - y^n w)$$
contains the polynomial
$$z^{n^2+1} - y^{n^2} w.$$
Thus, the degrees of the elements of a Gröbner basis can be much larger than the degrees of the original generators of the ideal.

(a) Verify this for some small values of n.

(b) Can you prove this?

(c) Can you write down the whole reduced Gröbner basis?