1. Let $I \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal generated by a set of monomials $M$. Show that $f \in I$ iff each term of $f$ is divisible by some monomial in $M$.

2. Calculate the Hilbert function of $I = (x_1x_3, x_1x_4, x_2x_4)$ by hand using the algorithm presented in Lecture 13.

3. Order the terms in the following polynomials using lex, deglex, and revlex ordering, in turn. What is the initial term in each case?
   
   (a) $f = x + 3x - x^2 + z^2 - y^3$.
   
   (b) $g = x^2yz + xy^6 + 2xy^3 - 4x^2y^3z^2$.

4. Give a simple example of an ideal $I = (f_1, \ldots, f_s)$ such that $\text{in}_{>}(I) \neq (\text{in}_{>}(f_1), \ldots, \text{in}_{>}(f_s))$.

5. Macaulay’s theorem. Let $S = k[x_1, \ldots, x_n]$ with monomial ordering $>$, and let $I \subseteq S$ be an ideal. Let $B$ be the set of monomials of $S$ that are not in $\text{in}_{>}(I)$. Prove that $B$ is a $k$-vector space basis for $S/I$.

   Hints:
   
   (a) To show linear independence, let $f = \sum \alpha_i x^{a_i} \in S$ with $\alpha_i \neq 0$ and $x^{a_i} \in B$. Suppose that $f = 0 \in S/I$, i.e., $f \in I$. Now think about the initial term of $f$.
   
   (b) To show $B$ spans, suppose it does not. Among all elements of $S/I$ not in the span of $B$, choose one, $f$, with a smallest initial term. There are two cases to consider depending on whether $\text{in}_{>}(f) \in \text{in}_{>}(I)$. In either case, argue there is an element of $S/I$ not in the span of $B$ but with an even smaller initial term.

6. With $I \subset S$ as in the previous problem, Macaulay’s theorem says that $S/I$ and $S/\text{in}_{>}(I)$ are isomorphic as $k$-vector spaces but not necessarily as rings. Now let $S = k[x, y]$ with deglex monomial ordering, and let $I = (y - x^2)$.

   (a) Use Macaulay’s theorem to exhibit $k$-bases of $S/I$ and of $S/\text{in}_{>}(I)$ consisting of the same set of monomials.
   
   (b) Show that $S/I$ and $S/\text{in}_{>}(I)$ are not isomorphic as rings.

7. Let $x^{a_1}$ be a monomial and $I' = (x^{a_2}, \ldots, x^{a_s})$ be a monomial ideal in $k[x_1, \ldots, x_n]$. In Lecture 13, we considered the mapping which is multiplication by $x^{a_1}$:

   $$S \xrightarrow{x^{a_1}} S/I'.$$
Show that
\[
\ker(x^{a_1}) = \left( \frac{x^{a_2}}{\gcd(x^{a_1}, x^{a_2})}, \ldots, \frac{x^{a_s}}{\gcd(x^{a_1}, x^{a_s})} \right)
\]
where \(\gcd(x^a, x^b) = x^{\min\{a_1, b_1\}} \cdots x^{\min\{a_n, b_n\}}\).