Gröbner Bases

Gröbner bases are the central tool of computational algebraic geometry.

Examples of computations for which they are useful:

- the ideal membership problem: \( f \in I \);
- Hilbert functions;
- resolutions;
- elimination theory;
- finding solutions to systems of equations;
- intersections of ideals.
Main Idea

Reduce all problems in polynomial rings to problems concerning monomials.
Notation

\[ S = k[x_1, \ldots, x_n]. \]

- **monomial**: \( x^a = x_1^{a_1} \cdots x_n^{a_n} \)
- **exponent vector** for \( x^a \): \( a = (a_1, \ldots, a_n) \)
- **degree**: \( \text{deg } x^a = |a| = \sum_i a_i \)
- **term**: \( \alpha x^a \) where \( \alpha \in k \)
  
  - Every polynomial is a sum of terms.
- **monomial ideal**: an ideal generated by monomials
- **division of monomials**: \( x^a | x^b \) if \( x^b = f x^a \) for some \( f \in S \).
  
  - \( x^a | x^b \) iff \( b \geq a \), i.e., \( b_i \geq a_i \) for all \( i \).
Membership problem

1 \in (x^2 + y - 3, xy^2 + 2x, y^3)

Yes!

\begin{align*}
1 &= \frac{-1}{27} (y^2 + 3y + 9)(x^2 + y - 3) \\
    &\quad - \frac{1}{108} (xy^4 + 3xy^3 + 7xy^2 - 6xy - 18x)(xy^2 + 2x) \\
    &\quad + \frac{1}{108} (x^2 y^3 + 3x^2 y^2 + 9x^2 y + 4)y^3
\end{align*}

The problem is easier for monomial ideals...
Proposition

Let $I \subseteq S$ be a monomial ideal generated by a set of monomials $M$. Then $f \in I$ iff each term of $f$ is divisible by some monomial in $M$.

Proof.

HW.

Corollary

Every monomial ideal is generated by a finite set of monomials.

Proof.

Hilbert basis theorem and the above Proposition.

Challenge: Prove this without recourse to the Hilbert basis theorem. (Consider the exponents of any monomial generating set. Which are necessary?)
Definition

The **Hilbert function** for a homogeneous ideal \( I \subseteq S = k[x_1, \ldots, x_n] \) is the function

\[
H_{S/I}(d) = \dim_k S_d/I_d.
\]

First Goal

Calculate the Hilbert polynomial of the monomial ideal

\[
I = (x^{a_1}, \ldots, x^{a_s}).
\]
Write

\[ I = (x^{a_1}) + I' \]

where \( I' = (x^{a_2}, \ldots, x^{a_s}) \), and consider the sequence

\[ S(-|a_1|) \xrightarrow{\cdot x^{a_1}} S/I' \xrightarrow{\pi} S/I \longrightarrow 0 \]

where \( |a_1| = \sum_i a_{1i} = \deg x^{a_1} \).

Claim

The sequence is exact: \( \text{image}(\cdot x^{a_1}) = \ker \pi \).
\[ l = (x^{a_1}, \ldots, x^{a_s}) = (x^{a_1}) + l' \]

\[ S(-|a_1|) \xrightarrow{\cdot x^{a_1}} S/l' \xrightarrow{\pi} S/l \longrightarrow 0 \]

**Claim:** the sequence is exact: \( \text{image}(\cdot x^{a_1}) = \ker \pi. \)

**Proof.**

- Say \( \pi(f) = 0. \)
- Pick a representative for \( f \) in \( S. \) Call it \( f. \)
- We may assume \( f \) has no terms divisible by \( x^{a_2}, \ldots, x^{a_s}. \)
- \( \pi(f) = 0 \implies f \in (x^{a_1}, \ldots, x^{a_s}). \)
- Earlier Proposition implies each term of \( f \) is divisible by some \( x^{a_i}. \)
- Thus, \( x^{a_1} \mid f, \) so \( f \in \text{image}(\cdot x^{a_1}). \)
\[ S(-|a_1|) \xrightarrow{\cdot x^{a_1}} S/I' \xrightarrow{\pi} S/I \rightarrow 0 \]

**HW**

\[
\ker(\cdot x^{a_1}) = \left( \frac{x^{a_2}}{\gcd(x^{a_1}, x^{a_2})}, \ldots, \frac{x^{a_s}}{\gcd(x^{a_1}, x^{a_s})} \right)
\]

where \( \gcd(x^a, x^b) = x_1^{\min\{a_1, b_1\}} \ldots x_n^{\min\{a_n, b_n\}} \).

Let \( J = \ker(\cdot x^{a_1}) \) to get the short exact sequence

\[ 0 \rightarrow S/J(-|a_1|) \xrightarrow{\cdot x^{a_1}} S/I' \rightarrow S/I \rightarrow 0 \]
Calculating the Hilbert function of $S/I$

\[ 0 \rightarrow S/J(-|a_1|) \rightarrow S/I' \rightarrow S/I \rightarrow 0 \]

Take degrees:

\[ 0 \rightarrow (S/J)_{d-|a_1|} \rightarrow (S/I')_{d} \rightarrow (S/I)_{d} \rightarrow 0 \]

Hilbert function

\[ H_{S/I}(d) = H_{S/I'}(d) - H_{S/J}(d - |a_1|). \]

$I'$ and $J$ are monomial ideals with fewer generators. Repeat.
Next Goal

Reduce the problem of calculating the Hilbert function of an arbitrary ideal to the problem of calculating the Hilbert function of a monomial ideal.
Monomial Orderings

Definition

A monomial ordering on $S = k[x_1, \ldots, x_n]$ is a total ordering on the monomials of $S$ such that

1. $x^b > x^a$ $\implies$ $x^c x^b > x^c x^a$ for all $x^c$;
2. 1 is the smallest monomial.
**lex: Lexicographical Ordering**

$x^b >_{\text{lex}} x^a$ if the left-most nonzero entry of $b - a$ is positive.

*(Mantra: more of the early variables)*

$$x^2 > xy > xz > x > y^2 > yz > y > z^2 > z > 1$$

**deglex: Degree Lexicographical Ordering**

$x^b >_{\text{deglex}} x^a$ if $|b| > |a|$ or if $|b| = |a|$ and $x^b >_{\text{lex}} x^a$. *(Mantra: By degree, breaking ties with lex)*

$$x^2 > xy > xz > y^2 > yz > z^2 > x > y > z > 1$$

**revlex: Reverse Lexicographical Ordering**

$x^b >_{\text{revlex}} x^a$ if $|b| > |a|$ or if $|b| = |a|$ and the right-most nonzero entry of $b - a$ is negative. *(Mantra: fewer of the late variables)*

$$x^2 > xy > y^2 > xz > yz > z^2 > x > y > z > 1$$
Notes

- From now on, fix a monomial ordering, $>$, on $S = k[x_1, \ldots, x_n]$.
- We will also compare terms: for nonzero $\alpha, \beta \in k$,
  \[ \alpha x^b > \beta x^a \text{ if } x^b > x^a. \]

Definition

- The **initial term** of $f \in S$, denoted $\text{in}_>(f)$, is the largest term of $f$ with respect to $>$. 
- The **initial ideal** of an ideal $I$ is the monomial ideal
  \[ \text{in}_>(I) = (\text{in}_>(f) : f \in I). \]
Macaulay’s Theorem

A preliminary

Lemma

Every nonempty set of monomials \( \{x^{a_i}\} \) has a least element.

Proof.

Since \( S \) is Noetherian the ideal generated by the monomials is generated by a finite subset. Take a least element of this subset.
Theorem (Macaulay)

Let $I \subseteq S$ be an ideal and $>$ a monomial ordering. Let $B$ be the set of monomials of $S$ not contained in $\text{in}_>(I)$. Then $B$ is a $k$-vector space basis for $S/I$.

Proof.

HW (minimal criminal argument).

Corollary

$$H_{S/I} = H_{S/\text{in}_>(I)}$$

Important Point: We have reduced the problem of computing the Hilbert function of an ideal to that of computing the Hilbert function of a monomial ideal.