⋆ 1. The Chow ring of $\mathbb{G}_1\mathbb{P}^4$.

   (a) List all the Schubert classes for $\mathbb{G}_1\mathbb{P}^4$. For each class, state the codimension, give both the a-notation and λ-notation, and describe the associated Schubert condition. For example, one line of your list will be

codim	class	condition
3	$(1,3) = \{2,1\}$	meet a line lying in a solid

   a "solid" being a 3-dimensional linear subspace.

   (b) Make the 10×10 multiplication table for the Chow ring of $A^*(\mathbb{G}_1\mathbb{P}^4)$.

   (c) Find m so that there will be a finite number of lines meeting m given lines, in general. Then find the number of lines meeting m general lines.

   (d) How many lines will meet 6 planes, in general?

   (e) What is the degree of $\mathbb{G}_1\mathbb{P}^4 \subset \mathbb{P}^9$? (Hint: Since $\mathbb{G}_1\mathbb{P}^4$ has dimension $(r+1)(n-r) = 6$, i.e., codimension 3, you need to see how many times a general solid (3-dimensional subspace, codimension 6) in $\mathbb{P}^9$ meets the Grassmannian. The Schubert class of codimension 1 is given by intersecting the Grassmannian with a hyperplane.)

   (f) Create your own enumerative problem that can be answered using this Chow ring.

⋆ 2. Give geometric explanations for the following calculations in $A^*(\mathbb{G}_1\mathbb{P}^3)$:

   (a) $(0,3)^2 = (1,2)^2 = (0,1)$;

   (b) $(1,3)(0,3) = (1,3)(1,2) = (0,2)$;

   (c) $(0,3)(1,2) = 0$;

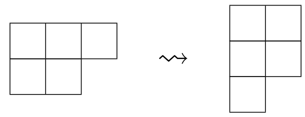
   (d) $(1,3)^2 = (0,3) + (1,2)$.

3. **Linear duality.** In lecture 10, we saw that $G(r,n)$ is isomorphic to $G(n-r,n)$, or in the language of projective geometry,

$$\mathbb{G}_r\mathbb{P}^n = G(r+1,n+1) \approx G(n-r,n+1) = \mathbb{G}_{n-r-1}\mathbb{P}^n.$$

For instance, $\mathbb{G}_1\mathbb{P}^4 \approx \mathbb{G}_2\mathbb{P}^4$, i.e., there is a duality between lines and planes in $\mathbb{P}^4$. (Equivalently, each 2-dimensional subspace V in $\mathbb{A}^5$ is dual to a 3-dimensional subspace, W. Fixing an inner product, you can just take W to be the 3-dimensional subspace orthogonal to V.) There is a corresponding duality with the Chow rings for Grassmannians. The dual for the Schubert class $\{\lambda_0,\ldots,\lambda_r\}$ in the Chow ring for

$\mathbb{G}_r\mathbb{P}^n$ having Young diagram Y_λ is the class in the Chow ring for $\mathbb{G}_{n-r-1}\mathbb{P}^n$ having as its Young diagram the transpose of Y_λ. For instance, the class $\{3,2\} \in A^5(\mathbb{G}_1\mathbb{P}^4)$ is dual to $\{2,2,1\} \in A^5(\mathbb{G}_2\mathbb{P}^4)$:



Each of the enumerative problems we have considered using the Schubert calculus has an associated dual problem. For instance, in the plane, there is one line containing two given points and there is one point contained in two lines.

(a) List all the Schubert classes for $\mathbb{G}_2\mathbb{P}^4$, their codimensions, and the relevant Schubert conditions. Match each class with its dual class in the Chow ring for $\mathbb{G}_1\mathbb{P}^4$. Compare the Schubert conditions for each class and its dual. Note how the word "contains" is dual to "contained in" but the word "meeting" is self-dual.

(b) What are the duals of the enumerative problems in 1 (c), (d), and (e)?

4. **Poincaré duality.** The Chow ring $A^*(\mathbb{G}_r\mathbb{P}^n)$ is a free abelian group on the Schubert classes, $(a_0, \dots, a_r)$, where $0 \le a_0 < \cdots < a_r \le n$. This means that every element of the Chow ring can be written uniquely as an integer combination of the Schubert classes. If there are d classes in codimension ℓ, we have that $A^\ell(\mathbb{G}_r\mathbb{P}^n) \approx \mathbb{Z}^d$. The number d is called the *rank* of $A^\ell(\mathbb{G}_r\mathbb{P}^n)$ and is known as the codimension 2ℓ-th *Betti number* for $\mathbb{G}_r\mathbb{P}^n$ (the odd Betti numbers are all zero for $\mathbb{G}_r\mathbb{P}^n$ over the complex numbers). The *Poincaré dual* of the class $(a_0, \dots, a_r)$ is the class $(n-a_r, \dots, n-a_0)$. When we speak of the dual of a class in this problem, we will mean the Poincaré dual.

(a) Show that a class and its dual sit in complementary codimensions, i.e., the sum of their codimensions is $\dim \mathbb{G}_r\mathbb{P}^n$.

(b) What does duality say about the list of Betti numbers?

(c) Compute a few specific examples to show that the product of a class and its dual is the class of a point in $\mathbb{G}_r\mathbb{P}^n$, i.e., the class of an r-plane in $\mathbb{P}^n$. Besides the formal calculation, confirm using geometric intuition that there is only one r-plane satisfying the two Schubert conditions.

(d) Prove that the product of a class and its dual is always the class of a point in $\mathbb{G}_r\mathbb{P}^n$.

i. What is the class $(a_0, \dots, a_r)$ of a point in $\mathbb{G}_r\mathbb{P}^n$, i.e., $[r\text{-plane}]$? What is its Young diagram, $\{\lambda_0, \dots, \lambda_r\}$?

ii. What is the maximum width and the maximum height of a nonzero class in $A^*(\mathbb{G}_r\mathbb{P}^n)$?

iii. How many total squares are in the Young diagrams for $(a_0, \dots, a_r)$ and its dual? What is the relation between the Young diagrams of a class, its dual, and the class of a point?

iv. Complete the proof.

(e) Show that the product of $a = (a_0, \ldots, a_r)$ with any Schubert class b in the same codimension as its dual is 0 unless b is the dual of a.

Here is a nice property of duality: writing the class of a subvariety $X \subset \mathbb{G}_r\mathbb{P}^n$ as

$$[X] = \sum_a c_a \, (a_0, \ldots, a_r),$$

the integers c_a are given by $[X] \cdot (n - a_r, \ldots, n - a_0)$ (just multiply both sides of the equation by $(n - a_r, \ldots, n - a_0)$).