⋆ 1. Consider $\mathbb{P}^5$ as parametrizing plane conics with $(a_0, \ldots, a_5) \in \mathbb{P}^5$ corresponding to

$$a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2.$$

In problem set 5, you calculated that the set of conics tangent to $\{x = 0\}$ is given by the quadric $a_4^2 - 4a_3a_5 = 0$. You also saw that the set of conics passing through a given point forms a hyperplane in $\mathbb{P}^5$. What is the number m such that there is a finite number of conics tangent to $\{x = 0\}$ and passing through m general points? Fixing m such points, how many conics will there be meeting the given conditions? **Challenge:** give a concrete example of m points so that over the real numbers you get the expected number of conics meeting the conditions.

2. In the lecture, we saw that the Chow variety for cubics in $\mathbb{P}^3$ is a subset of $\mathbb{P}^{49}$, and we essentially calculated the point on the Chow variety corresponding to the twisted cubic. Use CoCoA to answer the following questions.

   (a) What is the point on the Chow variety for the cubic that is the projective closure of image of the mapping $t \mapsto (t^2 - 1, t(t^2 - 1), 0) \in \mathbb{A}^3 \subset \mathbb{P}^3$?

   (b) In what size projective space does the Chow variety for curves of degree 2 in $\mathbb{P}^3$ sit, according to our construction? (This problem is easily done by hand, without the help of a computer.)

   (c) In what size projective space does the Chow variety for cubic surfaces in $\mathbb{P}^4$ sit? (Note: "surface" means "2-dimensional variety".)

⋆ 3. Here is another version of Bezout's theorem. Let X and Y be subvarieties of $\mathbb{P}^n$ such that $\dim X + \dim Y = n$. If $X \cap Y$ is a finite set, then the number of points in $X \cap Y$, counting multiplicities (at points of tangency) is the product $(\deg X)(\deg Y)$.

   (a) Use Bezout's theorem to show that a curve of degree 2 in $\mathbb{P}^3$ must lie in a plane.

   (b) Generalize the previous result: what is the largest degree d such that a curve of degree d in $\mathbb{P}^n$ must sit in a hyperplane?

   (c) Why should the Chow variety of curves of degree 2 in $\mathbb{P}^3$ be 8-dimensional?

4. A variety in $\mathbb{P}^n$ with Hilbert polynomial $P(t) = 1$ is a variety of degree 1 (from the leading coefficient of the Hilbert polynomial) and dimension $0 = \deg P$. Varieties in $\mathbb{P}^n$ with this Hilbert polynomial are exactly single points of $\mathbb{P}^n$. It seems that the Hilbert variety parametrizing the collection of single points should just be $\mathbb{P}^n$, itself. Follow the construction of the Hilbert variety given in the lecture notes to show in what sense this turns out to be true. Some hints: (1) the ideal for a point $p = (1, a_1, \ldots, a_n) \in \mathbb{P}^n$ is

gotten by homogenizing the ideal for the corresponding point in affine space; (2) using the notation from the lecture, the smallest value for d_0 is 1 (it can't be zero because the ideals $I(X)_{d \geq d_0}$ will not be generated by $I(X)_0 = \{0\}$).

5. Let V be a vector space. Given $v \in V$, we get a linear function

$$\begin{aligned} v^*\colon V^* &\to k \\ L &\mapsto L(v) \end{aligned}$$

This defines a linear mapping of V to its double-dual (the dual of its dual space)

$$\begin{aligned} \iota\colon V &\to V^{**} \\ v &\mapsto v^* \end{aligned}$$

Show ι is one-to-one. Show that ι is an isomorphism if $\dim V < \infty$.

$\star$ 6. Show that if

$$0 \to V' \to V \to V'' \to 0$$

is an exact sequence of vector spaces, then the induced sequence

$$0 \to (V'')^* \to V^* \to (V')^* \to 0$$

is exact.

7. In $\Lambda^r V$, show that the relation

$$v_1 \wedge \cdots \wedge v_r = 0, \qquad \text{if } v_i = v_j \text{ for some } i \neq j$$

implies that swapping v_i for v_j introduces a minus sign:

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \ldots v_r = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \ldots v_r.$$

(Hint: in $v_1 \wedge \cdots \wedge v_r$, replace both v_i and v_j by $v_i + v_j$ to get a wedge product that is zero. Expand.) Conversely, show that the second relation implies the first when working over a field of characteristic not equal to 2.

8. Generalization of the cross product.

   (a) Consider the vector space $\Lambda^2 k^3$ with ordered basis $e_2 \wedge e_3, -e_1 \wedge e_3, e_1 \wedge e_2$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ is the standard basis for k^3. Let $v = (v_1, v_2, v_2)$ and $w = (w_1, w_2, w_3)$ be elements of k^3. Show that $v \wedge w$ written with respect to the given coordinates is the cross-product

$$v \times w = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

(b) To generalize the cross-product, choose the ordered basis

$$(-1)^i \, e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n, \quad \text{for } i = 1, \ldots, n$$

for $\Lambda^{n-1}k^n$ where $\widehat{e_i}$ denotes omitting e_i. Define the cross-product of vectors $v_1, \ldots, v_{n-1}$ as the vector whose coordinates are those of $v_1 \wedge \cdots \wedge v_{n-1}$ with respect to the given basis for $\Lambda^{n-1}k$. Note that these coordinates are, up to sign, the $(n-1) \times (n-1)$ minors of the matrix whose rows are the v_i. Show that this cross-product is perpendicular to each v_i (and hence to the linear space spanned by $v_1, \ldots, v_{n-1}$). (Hint: consider the $n \times n$ matrix whose rows are $v_1, \ldots, v_{n-1}$ and v_i. Having a repeated row, its determinant is 0. Compute the determinant be expanding along the last row.)

$\star$ 9. Let $e_1, \ldots, e_n$ be the standard basis for k^n, and let $v_1, \ldots, v_n \in k^n$. Show that

$$v_1 \wedge \cdots \wedge v_n = \det(v_1, \ldots, v_n) \, e_1 \wedge \cdots \wedge e_n.$$

(Hint: how does one characterize the determinant?)