PCMI 2008 Undergraduate Summer School

Lecture 10: Grassmannians II

David Perkinson

Reed College
Portland, OR

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Examples of Parameter Spaces

$$\{\text{plane conics}\} \longleftrightarrow \mathbb{P}^5$$

$$a_0x^2 + a_1xy + \cdots + a_5z^2 \leftrightarrow (a_0, a_1, \ldots, a_5)$$

$$\left\{ \begin{array}{l} \text{hypersurfaces of} \\ \text{degree } d \text{ in } \mathbb{P}^n \end{array} \right\} \longleftrightarrow \mathbb{P}^N, \qquad N = \binom{n+d}{d} - 1$$

$$\{r\text{-subspaces of } \mathbb{A}^n\} \longleftrightarrow G(r,n) \overset{\text{Plücker}}{\hookrightarrow} \mathbb{P}^{\binom{n}{r}-1}$$

Chow Variety

Goal

Parametrize the set of algebraic sets of degree d and whose components all have dimension r in $\mathbb{P}^n$.

For an algebraic set $X \subset \mathbb{P}^n$ with $\deg X = d$ and all components of dimension r, define

$$\mathcal{P}_X = \{L \in G : X \cap L \neq \emptyset\}$$

where $G = G(n-r, n+1) = \mathbb{G}_{n-r-1}\mathbb{P}^n$.

- $\mathcal{P}_X$ is the intersection of a hypersurface $Z(f_X) \subset \mathbb{P}^N$ of degree d with $G \subset \mathbb{P}^N$ (Plücker embedding).
- $f_X \in S(G)_d = k[x_0, \ldots, x_N]_d / I(G)_d \approx k^M, \qquad M = H_G(d).$
- The mapping $\phi\colon X \mapsto f_X \in \mathbb{P}^{M-1}$ is one-to-one.

The Chow variety is the closure of the image of ϕ.

Example

Let C be the twisted cubic in $\mathbb{P}^3$, the image of the mapping

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^3 \\ (s,t) & \mapsto & (s^3, s^2t, st^2, t^3) \end{array}$$

$\dim C = 1$, $\deg C = 3$

$$\mathcal{P}_C = \{L \in \mathbb{G}_1\mathbb{P}^3 : L \cap C \neq \emptyset\} \subset \mathbb{P}^5$$

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Use R::=Q[a[1..4],s,t,x[1..6]];
M:=Mat([[s^2t,st^2,t^3,s^3],a]);
M;
Mat([
  [s^2t, st^2, t^3, s^3],
  [a[1], a[2], a[3], a[4]]
])
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C:=Minors(2,M);
J:=Minimalized(Elim([a[1],a[2],a[3],a[4],s,t],Ideal(C-x)));
J;
Ideal(-2x[3]x[4] + 2x[2]x[5] - 2x[1]x[6], -x[1]^3 + x[2]^2x[3]
- 3x[1]x[2]x[5] - x[4]x[5]^2 + 2x[1]^2x[6] + x[2]x[5]x[6] - x[1]x[6]^2)
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Use S::=Q[x[1..6]];
K:=Ideal(BringIn(J.Gens));
K;
Ideal(-2x[3]x[4] + 2x[2]x[5] - 2x[1]x[6], -x[1]^3 + x[2]^2x[3]
- 3x[1]x[2]x[5] - x[4]x[5]^2 + 2x[1]^2x[6] + x[2]x[5]x[6] - x[1]x[6]^2)
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K.Gens[1];
-2x[3]x[4] + 2x[2]x[5] - 2x[1]x[6]
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Hilbert(S/Ideal(K.Gens[1]));
H(t) = 1/12t^4 + 2/3t^3 + 23/12t^2 + 7/3t + 1   for t >= 0
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H:=Hilbert(S/Ideal(K.Gens[1]));
EvalHilbertFn(H,3);
50
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Recap

$$\{\text{lines meeting } C\} = \mathbb{G}_1\mathbb{P}^3 \cap Z(f) \subset \mathbb{P}^5$$

where

$$f = -x_1^3 + x_2^2 x_3 - 3x_1x_2x_5 - x_4x_5^2 + 2x_1^2x_6 + x_2x_5x_6 - x_1x_6^2$$

$$f \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]_3 / I(\mathbb{G}_1\mathbb{P}^3)_3 \approx \mathbb{C}^{50}$$

The Chow variety sits in $\mathbb{P}^{49}$, and the twisted cubic corresponds to the point determined by f.

The Hilbert Variety

Goal

Parametrize all algebraic sets in $\mathbb{P}^n$ having the same Hilbert polynomial.

Recall: the Hilbert polynomial encodes the degree and dimension.

Theorem

Given a polynomial $P \in \mathbb{Q}[t]$, there exists an integer d_0 such that for any algebraic set X with Hilbert polynomial $P_X = P$,

1. *the Hilbert function $H_X(d) = P(d)$ for $d \geq d_0$.*
2. *$I(X)_{d \geq d_0}$ is generated by $I(X)_{d_0}$.*

For X as above, fix $d \geq d_0$ and let $S = k[x_0, \ldots, x_n]$. Then dim $I(X)_d$ is determined by the Hilbert polynomial:

$$P(d) = \dim_k S_d / I(X)_d \Longrightarrow \dim I(X)_d = \binom{n+d}{n} - P(d).$$

Letting $d \geq d_0$ and $r = \dim_k I(X)_d$ and $N = \dim S_d$,

$$I(X)_d \in G(r, N).$$

We get a mapping

$$\phi\colon \left\{ \begin{array}{c} \text{algebraic sets with} \\ \text{Hilbert polynomial } P \end{array} \right\} \longrightarrow G(r, N)$$
$$X \mapsto I(X)_d$$

The closure of the image of ϕ is the Hilbert variety.

Wedge products

V a vector space/k

$\Lambda^r V$ is a vector space generated by the symbols

$$v_1 \wedge \cdots \wedge v_r, \qquad v_i \in V$$

Relations

- $v_1 \wedge \cdots \wedge v_r = 0$ if $v_i = v_j$ for some $i \neq j$.
- $v_1 \wedge \cdots \wedge (av_i + bv_i') \wedge \cdots \wedge v_r =$
  $a\, v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r + b\, v_1 \wedge \cdots \wedge v_i' \wedge \cdots \wedge v_r$

HW

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_r = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r$$

The Plücker embedding without coordinates

Definition

- $G(r, V)$ is the Grassmannian of r-dimensional subspaces of V.
- $\mathbb{P}(W) = G(1, W)$ is projective space on W.

Plücker embedding

$$\begin{array}{rcl} G(r, V) & \longrightarrow & \mathbb{P}(\Lambda^r V) \\ \mathrm{Span}\{v_1, \ldots, v_r\} & \mapsto & v_1 \wedge \cdots \wedge v_r \end{array}$$

Duality

Definition

The dual of a vector space V over k is the vector space of linear maps from V to k, denoted V^*.

- For $\dim V = n < \infty$, choosing a basis, any linear function $L\colon V \to k$ becomes dot product by a fixed vector in k^n. Thus, $V^* \approx k^n \approx V$.
- $V \hookrightarrow V^{**}$, isomorphism if $\dim V < \infty$.

A linear function between vector spaces

$$V \xrightarrow{\phi} W$$

induces a linear mapping

$$\begin{array}{rcl} W^* & \to & V^* \\ L & \mapsto & L \circ \phi \end{array}$$

Proposition

$$0 \to W \to V \to V/W \to 0 \quad \textit{exact}$$
$$\Downarrow$$
$$0 \to (V/W)^* \to V^* \to W^* \to 0 \quad \textit{exact}$$

$$\begin{array}{ccc} G(r, V) & \approx & G(n-r, V^*) \\ W & \mapsto & (V/W)^* \end{array}$$

- Choosing a basis: $G(r, n) \approx G(n-r, n)$.
- Special case: $\mathbb{P}^n = G(1, n+1) \approx G(n, n+1) = (\mathbb{P}^n)^*$.