A Category of Polytopes

A Thesis
Presented to
The Division of Mathematics and Natural Sciences
Reed College

In Partial Fulfillment
of the Requirements for the Degree
Bachelor of Arts

Lawrence Vincent Valby

May 2006

Approved for the Division
(Mathematics)

David Perkinson

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Abstract

We consider a category called **POLY** whose objects are polytopes and whose arrows are affine mappings. In Chapter 1 we introduce this category and another category consisting of "cones" and linear mappings, which proves to be useful in studying **POLY**. In Chapter 2 we give a construction providing a tensor product for polytopes. Additionally, we present a right adjoint to this tensor product, which gives us a natural way of indentifying the hom-sets of **POLY** with polytopes. We then proceed to describe the facets and some of the vertices of these "internal homs". In Chapter 3 the limits and colimits of **POLY** are discussed. The main thrust of Chapter 4 is towards examining the categorical properties of simplices. After a detour involving the characterization of when monos and epis split in **POLY**, we prove that the simplices are the regular projectives of **POLY**. From there we use simplices to define the notion of a "kernel-polytope", and we examine its properties. Chapter 5 presents a couple of negative results. We find that taking the polar of a polytope does not yield a functor "in a natural way". We also show that the polar is not a dual in the categorical sense, at least with respect to the tensor product.

Chapter 1

Introduction to POLY and CONE

1.1 Categories

A category consists of the following things:

- a collection of objects
- a collection of arrows
- two operations, dom and cod, that assign to each arrow an object, called the domain and codomain, respectively
- an operation, $\circ$, that assigns to each pair of arrows (f, g) for which $\mathrm{dom}(f) = \mathrm{cod}(g)$ an arrow $f \circ g$, whose domain is $\mathrm{dom}(g)$ and whose codomain is $\mathrm{cod}(f)$

Further, these things must satisfy the following two properties to be considered a category:

- **Associativity.** For any three arrows f, g, h (which have appropriate domains and codomains), $f \circ (g \circ h) = (f \circ g) \circ h$.
- **Identity arrows.** For each object A there is an arrow 1_A with domain and codomain both A such that 1) $f \circ 1_A = f$ for any arrow f with $\mathrm{dom}(f) = A$, and 2) $1_A \circ g = g$ for any arrow g with $\mathrm{cod}(g) = A$.

Notation. We often write $A \in \mathbf{C}$ if A is an object in some category $\mathbf{C}$, while we write f in $\mathbf{C}$ if f is an arrow in $\mathbf{C}$.

Examples. An example of a category is **SET** whose objects are sets and whose arrows are just mappings between them. The composition of the arrows is the usual composition of mappings, and the identity arrows are the usual identity mappings.

Another example is the category whose objects are again sets, but whose arrows are relations. In detail, if A and B are sets, then an arrow between them would be a subset of $A \times B$. Composition can then be defined as follows. Given $R \subseteq A \times B$ and $S \subseteq B \times C$,

$$S \circ R := \{(a,c) \mid \exists\, b \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}.$$

Associativity does indeed follow, and identity arrows are given by the diagonal, $\{(a,a) \mid a \in A\}$.

A third example is one whose objects are finite-dimensional vector spaces over $\mathbf{R}$, and whose arrows are linear maps. We shall call this category **VEC**.

1.2 A category of polytopes

1.2.1 Affine mappings

An element $x \in \mathbf{R}^n$ is said to be an *affine combination of* $p_1, \ldots, p_k \in \mathbf{R}^n$ if there exist real numbers $r_1, \ldots, r_k$ such that

$$\sum_{i=1}^{k} r_i = 1 \text{ and } \sum_{i=1}^{k} r_i p_i = x$$

A subset $P \subseteq \mathbf{R}^n$ is called an *affine subspace* if all affine combinations of elements from P are actually in P. It can easily be seen that P is an affine subspace iff $P - p$ is a linear subspace of $\mathbf{R}^n$ for some (hence, any) $p \in P$.

The *affine span* of a subset P of $\mathbf{R}^n$, denoted affspan(P), is the smallest affine subspace that contains P. For example, the affine span of a line segment in $\mathbf{R}^2$ is given by extending the line segment into a line (which does not have to go through the origin). The dimension of affspan(P) is given by the dimension of the linear subspace affspan$(P) - a$ for any $a \in$ affspan(P) (this is well-defined).

Let P and Q be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let $f\colon P \to Q$ be a mapping. A mapping f is called an *affine mapping* if it preserves affine combinations, i.e., for every affine combination $\sum_{i=1}^{k} r_i p_i$ of elements of P that is in P, we have

$$f(\sum_{i=1}^{k} r_i p_i) = \sum_{i=1}^{k} r_i f(p_i)$$

It can be shown that a mapping $f\colon P \to Q$ is affine iff there exists a linear mapping $\ell\colon \mathbf{R}^n \to \mathbf{R}^m$ and an element $y \in \mathbf{R}^m$ such that $f(p) = \ell(p) + y$ for $p \in P$.

A set of vectors v_i is said to be *affinely independent* if for any scalars β_i with $\sum \beta_i = 0$ and $\sum \beta_i v_i = 0$, it follows that $\beta_i = 0$. It can be shown that v_i are affinely independent iff $\{v_i - v_1 \mid i \geq 2\}$ is linearly independent. Thus, for any affine subspace P, the largest affinely independent subset of P has size $s+1$ iff P has dimension s. Also, an affine map $f\colon P \to Q$ is determined by where it sends any largest affinely independent set.

A *convex combination* is the same thing as an affine combination, except that we require the scalars to be non-negative. I.e., if $\sum_{i=1}^{k} r_i = 1$ and $r_i \geq 0$ for each i, then $\sum_{i=1}^{k} r_i p_i$ is a convex combination of P. A subset of a real coordinate space is called *convex* if it is closed under taking convex combinations. The *convex hull* of A, denoted conv(A), is the smallest convex set containing A. In fact, conv$(A) =$

$\{\sum r_i a_i \mid \sum r_i = 1,\ r_i \geq 0,\ a_i \in A\}$. We may speak of a map preserving convex combinations, as we did with affine combinations.

It turns out that for any convex set P, a mapping $f\colon P \to Q$, where Q is a subset of some real coordinate space, preserves convex combinations iff it preserves affine combinations. One direction of this is trivial. The other direction, in which we are given a map f that preserves convex combinations, can be proven by showing that there exists a (unique) affine extension of f to the affine span of P. Further, one can prove by induction that a map f with a convex domain P preserves convex combinations iff it preserves "binary convex combinations" — i.e., for any $r \in [0,1]$ and $p, p' \in P$ we have $f(rp + (1-r)p') = rf(p) + (1-r)f(p')$. As we shall mainly be dealing with convex sets, these results simplify the calculations in proving a map is affine, and are used implicitly in what follows.

1.2.2 Polytopes

A subset $P \subseteq \mathbf{R}^n$ is called a *polytope* if there exists a finite subset A of $\mathbf{R}^n$ such that $\mathrm{conv}(A) = P$. The dimension of a polytope is given by the dimension of the affine span of the polytope.

Another characterization of the polytope involves considering intersections of halfspaces. A subset S of $\mathbf{R}^n$ is called a *closed halfspace* if there exists an element y in $\mathbf{R}^n$ and a real number r such that $S = \{x \in \mathbf{R}^n \mid y \cdot x \leq r\}$. A finite intersection of closed halfspaces is described by a system of inequalities. Such systems will be denoted by using matrices. E.g.,

$$\begin{cases} (a_1, b_1) \cdot \vec{x} \leq r_1 \\ (a_2, b_2) \cdot \vec{x} \leq r_2 \end{cases}$$

becomes $B\vec{x} \leq \vec{r}$ where

$$B = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

and

$$\vec{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

Theorem 1. *A subset $P \subseteq \mathbf{R}^n$ is a polytope iff P is a bounded finite intersection of closed halfspaces.*

PROOF. See Ziegler pp. 29–32. □

Let $y \in \mathbf{R}^n$ and let r be a real number. Then the inequality $y \cdot x \leq r$ is called *valid* for a polytope $P \subseteq \mathbf{R}^n$ if all $x \in P$ satisfy the inequality. The *faces* of a polytope P are then defined as $P \cap \{x \in \mathbf{R}^n \mid y \cdot x = r\}$ where $y \cdot x \leq r$ is any valid inequality. It follows immediately that P and $\emptyset$ are faces of any polytope P. Also, a face is necessarily a subpolytope, since it is a bounded finite intersection of closed halfspaces. The faces of dimension 0 are called *vertices*, those of dimension 1, *edges*, and those of dimension $\dim(P) - 1$, *facets*. The vertices are particularly useful, as one can prove the following. Let V be the set of vertices of some polytope P. Then $\mathrm{conv}(V) = P$, and if $\mathrm{conv}(A) = P$, then $V \subseteq A$ (see Ziegler p. 52). The *barycenter* of a polytope is the average of its vertices: $\frac{v_1+\cdots+v_k}{k}$. The barycenter is always in the polytope, because it is a convex combination of points in the polytope. The *relative interior* of a polytope P is defined as the points in P that are not on any of the faces of P, except for the face P itself. The n-simplex, Δ_n, is defined as follows:

$$\Delta_n := \mathrm{conv}(\{e_1, \ldots, e_{n+1}\}) \subseteq \mathbf{R}^{n+1}$$

where e_i denotes the i^{th} standard basis vector. It is called the "n"-simplex because it is n-dimensional.

Finally we define a category of polytopes called **POLY**. The objects are polytopes in any real coordinate space, and the arrows are affine mappings between the polytopes. Verifying that this is a category is straightforward. E.g., since identity mappings are affine, we have identity arrows. Though this is the category we shall be most concerned with in what follows, other categories of polytopes are possible. For instance we could require that each polytope's barycenter be at the origin, and that the arrows be linear maps (maps that preserve barycenters). Although this category seems to be too limited in the number of possible maps, there is a benefit in the maps being linear. However, we can in a way get the best of both worlds by considering a new category called **CONE**. Although its arrows are linear maps, **CONE**'s structure closely resembles **POLY**'s. The relationship between these two categories provides much of the substance for this thesis.

1.3 A category of cones

A *cone* is a set C supplied with addition and a scalar multiplication by elements of $\mathbf{R}_{\geq 0}$, satisfying the following axioms:

- Addition is associative, commutative, and there is an additive identity
- Scalar multiplication is associative and 1 is a multiplicative identity
- Two distributive laws hold: $(r + s)c = rc + sc$ and $r(c + d) = rc + rd$ for all $r, s \in \mathbf{R}_{\geq 0}$ and $c, d \in C$
- There is a cancellation law: if $a + c = b + c$ for some $a, b, c \in C$, then $a = b$.
- There is a finite subset $A \subseteq C$ with $\mathrm{span}(A) = C$.

One should note that there are just two differences from the axioms for a finite-dimensional vector space. First, for cones we drop the requirement that there be additive inverses. And second, we add the cancellation law. This cancellation law indeed does not follow from the other axioms, because one can form a structure that satisfies all the regular vector space axioms except the existence of additive inverses, yet still does not abide by this cancellation law.

Example: The Quasicone. Let D and D' be two distinct copies of $\mathbf{R}_{\geq 0}$. Let us agree that $a, b, 5, \ldots$ designate elements of D, while $a', b', 5', \ldots$ designate elements of D'. Now we define a binary operation $\tilde{+}$ on $D \cup D'$. For $a, b \in D$ and $a', b' \in D'$, we say

$$\begin{array}{lcl} a \mathbin{\tilde{+}} b & := & a + b \\ a' \mathbin{\tilde{+}} b & := & a + b \\ a \mathbin{\tilde{+}} b' & := & a + b \\ a' \mathbin{\tilde{+}} b' & := & (a+b)' \end{array}$$

We define scalar multiplication, $\cdot$, as follows. For $a \in D$, $a' \in D'$ and $r \in \mathbf{R}_{\geq 0}$, we say

$$\begin{array}{lcl} r \cdot a & := & ra \\ r \cdot a' & := & (ra)' \end{array}$$

It is easily verified that these operations satisfy all the axioms for a cone except the cancellation law. E.g., if x and y are any elements in $D \cup D'$ and r is an element of $\mathbf{R}_{\geq 0}$, then

$$r \cdot (x \mathbin{\tilde{+}} y) = r \cdot x \mathbin{\tilde{+}} r \cdot y$$

since both sides are clearly "numerically" equal, and the LHS is in D iff one of x or y is in D iff the RHS is in D. Finally, here is an example that shows that this structure fails the cancellation law: $5 + 3 = 5' + 3$, but $5 \neq 5'$.

The objects of the category **CONE** are cones, while the arrows are the linear maps between the cones. The next section provides a way of relating **POLY** and **CONE**.

1.4 A functor from poly to cone.

Let $\mathbf{A}$ and $\mathbf{B}$ be categories. A *functor* F from $\mathbf{A}$ to $\mathbf{B}$ consists of an operation that takes each object $A \in \mathbf{A}$ to an object $F(A) \in \mathbf{B}$, and an operation that takes each arrow $f\colon A \to A'$ in $\mathbf{A}$ to an arrow $F(f)\colon F(A) \to F(A')$ in $\mathbf{B}$, which satisfy the following two properties:

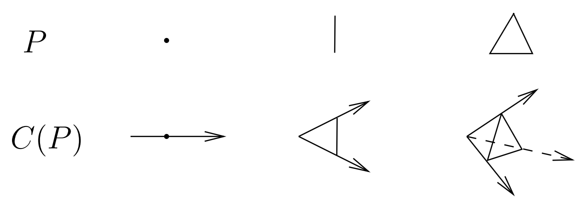
- For each object $A \in \mathbf{A}$, $F(1_A) = 1_{F(A)}$.
- For each composable pair of arrows f and g in $\mathbf{A}$, $F(f \circ g) = F(f) \circ F(g)$.

We now define a functor C from **POLY** to **CONE**: for any non-empty polytopes $P \subseteq \mathbf{R}^n$ and $Q \subseteq \mathbf{R}^n$, and for any arrow $f\colon P \to Q$ in **POLY**, we define

$$\begin{array}{lcl} \mathrm{C}(P) & := & \mathbf{R}_{\geq 0}\{(p, 1) \in \mathbf{R}^{n+1} \mid p \in P\} \\ \mathrm{C}(f)(\alpha(p, 1)) & := & \alpha(f(p), 1) \end{array}$$

(For $P = \emptyset$ we must stipulate that $\mathrm{C}(P) := \{\vec{0}\}$.) $\mathrm{C}(P)$ is indeed a cone. This is all straightforward, but I will verify here that $\mathrm{C}(P)$ is closed under addition. Let $\alpha(p_1, 1)$ and $\beta(p_2, 1)$ be elements of $\mathrm{C}(P)$. If $\alpha + \beta = 0$, then $\alpha = \beta = 0$, and so $\alpha(p_1, 1) + \beta(p_2, 1) = (\vec{0}, 0) \in \mathrm{C}(P)$. If $\alpha + \beta \neq 0$, then $\frac{\alpha p_1 + \beta p_2}{\alpha + \beta} \in P$, so $\alpha(p_1, 1) + \beta(p_2, 1) = (\alpha + \beta)(\frac{\alpha p_1 + \beta p_2}{\alpha + \beta}, 1) \in \mathrm{C}(P)$. $\mathrm{C}(f)$ is in fact the unique linear extension of $(f \times 1)$ to $\mathrm{C}(P)$.

Here are some examples of the effect of this functor on objects:



Chapter 2

Tensor product, Internal hom

2.1 Hom-sets

A hom-set is a set whose elements are all the arrows between two objects of a category. For instance, for polytopes P and Q, $\hom(P, Q)$ is the set of all affine mappings from P to Q. In some categories this set can be given structure in a natural way so as to make it an object of the category. This is in particular true for **POLY**, **CONE**, and **VEC**. In this chapter, as we outline how an "internal hom" can be defined as a right adjoint to a monoidal product, we specifically exhibit the constructions that work in our categories.

Let $\mathbf{C}$ be a category. We define the *representable functors*: for each object $C \in \mathbf{C}$ we define a functor $\hom(C, \bullet)\colon \mathbf{C} \to \mathbf{SET}$. Explicitly, for $D \in \mathbf{C}$ and $f\colon D \to D'$ in $\mathbf{C}$

$$\begin{aligned} \hom(C,\bullet)(D) &:= \hom(C,D) \\ \hom(C,\bullet)(f) &:= f_*\colon \hom(C,D) \to \hom(C,D') \end{aligned}$$

Similarly, we can define the *contravariant representable functors* by $\hom(\bullet, C)$. Contravariant functors are the same as *covariant* (regular) functors except that they reverse the direction of the arrows. For example, for $f\colon D \to D'$, $\hom(\bullet, C)(f)$ is an arrow from $\hom(\bullet, C)(D')$ to $\hom(\bullet, C)(D)$. A contravariant functor $F\colon \mathbf{C} \to \mathbf{D}$ is a covariant functor from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{D}$, where $\mathbf{C}^{\mathrm{op}}$ is the category obtained by reversing all the arrows in $\mathbf{C}$ (this results in a well-defined category).

Given categories $\mathbf{C}$ and $\mathbf{D}$, one may form the product category $\mathbf{C} \times \mathbf{D}$, whose objects are pairs (c, d) where $c \in \mathbf{C}$ and $d \in \mathbf{D}$, and whose arrows are pairs (f, g) where f is an arrow in $\mathbf{C}$ and g is an arrow in $\mathbf{D}$. Composition is defined componentwise. In particular, we may now speak of the representable bifunctor for any category $\mathbf{C}$: $\hom(\bullet,\bullet)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{SET}$.

2.2 Isomorphism and adjunction

Two objects A and B in a category are *isomorphic* if there exist arrows $f\colon A \to B$ and $g\colon B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. In this case f and g are called *isomorphisms*. With category theory we may additionally capture how two different constructions can yield "naturally" isomorphic objects.

Let $F, G\colon \mathbf{C} \to \mathbf{D}$ be functors. A *natural transformation* from F to G is a collection of arrows $\alpha_C\colon F(C) \to G(C)$ for each object $C \in \mathbf{C}$, with the property that for each arrow $f\colon C \to C'$ in $\mathbf{C}$, the following diagram commutes:

$$\begin{tikzcd} F(C) \arrow[r, "\alpha_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "\alpha_{C'}"'] & G(C') \end{tikzcd}$$

A natural transformation α is called a *natural isomorphism* if each arrow α_C is an isomorphism. Given any categories $\mathbf{C}$ and $\mathbf{D}$ one may form a new category $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ whose objects are functors from $\mathbf{C}$ to $\mathbf{D}$ and whose arrows are natural transformations. The isomorphisms of objects in this category are just the natural isomorphisms.

Example. In $\mathbf{VEC}$, there is a natural isomorphism between $\mathrm{Id}_{\mathbf{VEC}}$, the identity functor on $\mathbf{VEC}$, and $\bullet^{**}$, the double-dual functor. However, $\bullet^*$ is not naturally isomorphic to $\mathrm{Id}_{\mathbf{VEC}}$.[1]

Two categories $\mathbf{C}$ and $\mathbf{D}$ are said to be *isomorphic* if there are some functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ such that $F \circ G = \mathrm{Id}_{\mathbf{D}}$ and $G \circ F = \mathrm{Id}_{\mathbf{C}}$. The categories $\mathbf{C}$ and $\mathbf{D}$ are said to be *equivalent* if the weaker condition holds that $F \circ G \cong \mathrm{Id}_{\mathbf{D}}$ and $G \circ F \cong \mathrm{Id}_{\mathbf{C}}$ (i.e., these functors are naturally isomorphic). A still weaker condition is that F and G are adjoints of each other. Although any two functors that form an equivalence of categories will be adjoints of each other, an adjunction in general says more about the nature of the functors than the underlying categories.

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is a *left adjoint* of a functor $G\colon \mathbf{D} \to \mathbf{C}$ (equivalently, G is a *right adjoint* of F) if there exists a natural isomorphism between the functors

$$\hom(F(\bullet), \bullet)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{SET}$$

and

$$\hom(\bullet, G(\bullet))\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{SET}.$$

Equivalently we can say: there is a collection of bijections

$$\varphi_{C,D}\colon \hom(F(C), D) \to \hom(C, G(D))$$

that is "natural in C and D". This means that for any two arrows $f\colon C \to C'$ in $\mathbf{C}$ and $g\colon D \to D'$ in $\mathbf{D}$, the following diagrams commute:

[1] Strictly speaking it does not make sense to speak of a natural isomorphism between a contravariant functor ($\bullet^*$) and a covariant one ($\mathrm{Id}_{\mathbf{VEC}}$), but a dinatural isomorphism — a more general notion — makes sense, but still is not satisfied here. See MacLane pp. 214-6.

$$\begin{tikzcd}
\hom(F(C'),D) \arrow[r, "\varphi_{C,D}"] \arrow[d, "\hom(F(f),\mathrm{Id}_D)"'] & \hom(C',G(D)) \arrow[d, "\hom(f,\mathrm{Id}_{G(D)})"] \\
\hom(F(C),D) \arrow[r, "\varphi_{C',D}"'] & \hom(C,G(D))
\end{tikzcd}$$

$$\begin{tikzcd}
\hom(F(C),D) \arrow[r, "\varphi_{C,D}"] \arrow[d, "\hom(\mathrm{Id}_{F(C)},g)"'] & \hom(C,G(D)) \arrow[d, "\hom(\mathrm{Id}_C,G(g))"] \\
\hom(F(C),D') \arrow[r, "\varphi_{C,D'}"'] & \hom(C,G(D'))
\end{tikzcd}$$

One can show that any two left adjoints of the same functor are naturally isomorphic, and similarly for right adjoints.

Example. Many "forgetful" functors have left adjoints. For instance, let **Grp** be the category whose objects are groups and whose arrows are homomorphisms, and let $G\colon \mathbf{Grp} \to \textbf{SET}$ be the functor that takes a group to the set consisting of its elements: G forgets about the structure of the group. Then define a functor $F\colon \textbf{SET} \to \mathbf{Grp}$ that takes a set X to the free group generated by X. It follows that F is a left adjoint of G, because of the correspondence between maps $F(A) \to B$ and $A \to G(B)$, where A is a set and B is a group.

2.3 Monoidal products

A monoidal product on a category $\mathbf{C}$ consists of the following data:

1. a functor $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$
2. a natural isomorphism $\alpha_{ABC}\colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)$
3. a designated "unit object" $I \in \mathbf{C}$, with two natural isomorphisms $\lambda_A\colon I \otimes A \to A$ and $\rho_A\colon A \otimes I \to A$

This data must satisfy the following:

- **Associativity axiom.** For any objects $C_1, \ldots, C_n$, if we are given two objects E_1 and E_2 obtained by adding parentheses and (possibly) I's to the expression $C_1 \otimes \cdots \otimes C_n$, then any two isomorphisms of E_1 and E_2 composed of α's, λ's, ρ's, and their inverses, are equal.

The associativity axiom ensures that there is a canonical isomorphism between any such objects E_1 and E_2. The following theorem provides a way of checking whether this axiom is satisfied.

Theorem 2. [**Coherence**] *Let a category be supplied with the data 1–3 above. Then $\otimes$ is a monoidal product, i.e. it satisfies the associativity axiom, iff the following three properties hold:*

1. *$\lambda_I = \rho_I \colon I \otimes I \to I$*
2. *For all objects A and B, this "triangle diagram" commutes:*

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho_A \otimes \mathrm{Id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{Id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

3. *For all objects A, B, C, and D, this "pentagon diagram" commutes:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha"] \arrow[d, "\alpha \otimes \mathrm{Id}_D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "\mathrm{Id}_A \otimes \alpha"']
\end{tikzcd}$$

PROOF. See MacLane pp. 161-166. □

Examples. Define a functor $\times \colon \mathbf{SET} \times \mathbf{SET} \to \mathbf{SET}$ by saying that for sets A, B and mappings $f \colon A \to B$ and $g \colon A' \to B'$

$$\begin{array}{lcl} A \times B & := & \text{the regular cartesian product} \\ (f \times g)(a, a') & := & (f(a), g(a')) \end{array}$$

Then $\times$, supplied with a one-element set I and the obvious natural isomorphisms α, λ, and ρ, is a monoidal product for **SET**. The disjoint union of sets, with $I := \emptyset$, gives another monoidal product for **SET**. The usual tensor product $\otimes$ on **VEC** is another example.

A monoidal category (a category with a given monoidal product) is called *symmetric* if it is supplied with isomorphisms $\gamma_{AB} \colon A \otimes B \to B \otimes A$ so that the associativity axiom, when enhanced with γ and its inverse, is still satisfied. The relevant coherence theorem asserts that only the following three diagrams are required to commute for all objects A, B, and C:

$$\begin{tikzcd}
A \otimes B \arrow[r, "\gamma"] \arrow[rr, bend right, "\mathrm{Id}_{A \otimes B}"'] & B \otimes A \arrow[r, "\gamma"] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes A \arrow[rr, "\gamma"] \arrow[dr, "\lambda_A"'] & & A \otimes I \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha"] \arrow[d, "\gamma \otimes \mathrm{Id}_C"'] & A \otimes (B \otimes C) \arrow[r, "\gamma"] & (B \otimes C) \otimes A \arrow[d, "\alpha"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha"'] & B \otimes (A \otimes C) \arrow[r, "\mathrm{Id}_B \otimes \gamma"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

For a proof of this coherence theorem see MacLane "Natural associativity and commutativity" pp. 28-46.

2.4 Tensor products for cones

In this section we develop the notion of the tensor product for cones, and we show that it gives us a monoidal product on **CONE**.

2.4.1 Congruence relations on cones

Let C be a cone. A *congruence relation* on C is an equivalence relation $\sim$ on C with two additional properties:

1. For any $a, b \in C$ and $r \in \mathbf{R}_{\geq 0}$, $a \sim b \Longrightarrow ra \sim rb$, and
2. For any $a, b, c \in C$, $a \sim b \Longleftrightarrow a + c \sim b + c$.

Let $\sim$ be a congruence relation on C. Now we define a new cone $C/\!\sim$ that consists of the equivalence classes of $\sim$. Addition and scalar multiplication are defined as expected using representative elements. That this works is ensured by 1 and the right implication-direction of 2. For instance, we must verify that if $a \sim a'$ and $b \sim b'$, then $a + b \sim a' + b'$. Here we apply 2 twice to obtain $a + b \sim a' + b \sim a' + b'$. All of the cone axioms are satisfied by inheritance except the cancellation law, which in this case says that for all $\bar{a}, \bar{b}, \bar{c} \in C/\!\sim$, $\bar{a} + \bar{c} = \bar{b} + \bar{c} \Rightarrow \bar{a} = \bar{b}$. But this statement is exactly the left implication-direction of 2. This additional assumption on the equivalence relation is indeed necessary, for one can otherwise obtain a quotient that is not actually a cone, as the following example illustrates.

Example. Define an equivalence relation on $\mathbf{R}^2_{\geq 0}$ by saying that for every element (x, y) and (x', y') in $\mathbf{R}^2_{\geq 0}$,

$$(x, y) \sim (x', y') \Longleftrightarrow \begin{cases} (x, y) = (x', y'), \text{ or} \\ y, y' \neq 0 \text{ and } x + y = x' + y' \end{cases}$$

It is easily verified that $\sim$ is an equivalence relation, and that it satisfies 1 and the right implication-direction of 2. However, we have $(1, 0) + (0, 1) = (1, 1) \sim (0, 2) = (0, 1) + (0, 1)$, but $(1, 0) \not\sim (0, 1)$, so it doesn't satisfy the left implication-direction. It still makes sense to speak of $\mathbf{R}^2_{\geq 0}/\!\sim$, but it is not a cone. In fact, it is isomorphic to the Quasicone defined earlier.

Let C' be a subcone of C. Define a cone equivalence relation $\mathrm{R}C'$ on C by saying that for all $a, b \in C$,

$$\mathrm{R}C'(a, b) \Longleftrightarrow \text{there exist } c, d \in C' \text{ such that } a + c = b + d.$$

That this is a congruence relation on C is easily verified. Thus, for any subcone C' of a cone C, we may form the quotient $C/C' := C/\mathrm{R}C'$. There is an obvious mapping $\phi\colon C \to C/C'$ that takes each element to its equivalence class. This quotient has the following property: given a linear mapping $f\colon C \to X$ with $f(C') = 0$, there is a unique mapping $u\colon C/C' \to X$ such that $u \circ \phi = f$, as suggested by the following diagram:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\phi"'] & X \\ C/C' \arrow[ur, dotted, "\exists! u"'] & \end{tikzcd}$$

Given any relation $X \subseteq C^2$ on a cone C, one may form the congruence relation $\mathrm{E}(X)$ generated by X, which is defined by

$$E(X) \ := \ \bigcap \{Y \subseteq C^2 \mid X \subseteq Y \text{ and } Y \text{ is a congruence relation on } C\}$$

This is well-defined, since the intersection of congruence relations is a congruence relation, and the intersection will never be empty (let $Y = C^2$).

2.4.2 Definition of tensor product for cones

Let C and D be cones. We now define $C \otimes D$, the tensor product of C and D. Let $A := \{f \in (\mathbf{R}_{\geq 0})^{C\times D} \mid f = 0 \text{ almost everywhere}\}$. I.e., A is the set of mappings from the cartesian product of C and D to $\mathbf{R}_{\geq 0}$ that send all but a finite number of points to zero. There is an obvious way to supply A with the structure of a cone: for $f, g \in A$ and $r \in \mathbf{R}_{\geq 0}$, $(f+g)(c,d) := f(c,d) + g(c,d)$, and $(rf)(c,d) := rf(c,d)$. Temporarily let us write $\sum \alpha_i(c_i \odot d_i)$ for the mapping $f \in A$ that takes (c_i, d_i) to α_i and is zero elsewhere.

Let X be the subset of A^2 that contains, for each $c, c' \in C$, $d, d' \in D$, and $\alpha \in \mathbf{R}_{\geq 0}$, all of the following pairs:

$$((c+c') \odot d, (c \odot d) + (c' \odot d)) \tag{1}$$
$$(c \odot (d+d'), (c \odot d) + (c \odot d')) \tag{2}$$
$$(\alpha(c \odot d), (\alpha c) \odot d) \tag{3}$$
$$(\alpha(c \odot d), c \odot (\alpha d)) \tag{4}$$

Finally, $C \otimes D := A/\mathrm{E}(X)$. We use $\sum \alpha_i(c_i \otimes d_i)$ to denote the equivalence class of $\sum \alpha_i(c_i \odot d_i)$.

A mapping of the form $f\colon C \times D \to H$ between cones is called *bilinear* if for all $c, c' \in C$, $d, d' \in D$, and $\alpha \in \mathbf{R}_{\geq 0}$,

$$\begin{aligned} f(c+c', d) &= f(c,d) + f(c',d) \\ f(c, d+d') &= f(c,d) + f(c,d') \\ f(\alpha c, d) &= \alpha f(c,d) \\ &= f(c, \alpha d) \end{aligned}$$

There is a standard bilinear mapping $\phi\colon C \times D \to C \otimes D$ defined by $\phi(c,d) = c \otimes d$.

Proposition 3. *For any bilinear mapping $f\colon C \times D \to H$ between cones, there is a unique linear mapping $g\colon C \otimes D \to H$ such that $f = g \circ \phi$.*

PROOF. Define $g\colon C \otimes D \to H$ by $g(c \otimes d) = f(c,d)$. Is g well-defined? Let

$$\sum_{i=1}^{n} \alpha_i (c_i \otimes d_i) = \sum_{i=1}^{k} \beta_i (e_i \otimes f_i)$$

I wish to show that

$$g(\sum_{i=1}^{n} \alpha_i (c_i \otimes d_i)) = g(\sum_{i=1}^{k} \beta_i (e_i \otimes f_i))$$

To this end, define

$$\begin{array}{cccl} \tilde{f} & : & A & \rightarrow & H \\ & & a & \mapsto & \displaystyle\sum_{(c,d)} a(c,d) f(c,d) \end{array}$$

Define a congruence relation Y on A by saying that $(a, a') \in Y$ iff $\tilde{f}(a) = \tilde{f}(a')$. It is easily checked that Y contains all of the pairs designated by items (1) through (4) above due to f being bilinear. This means that $Y \supseteq X$. Thus, $Y \supseteq E(X)$. In other words, if $a \approx_{E(X)} a'$, then $a \approx_Y a'$. Returning to our original problem, then:

$$\begin{aligned} g(\sum_{i=1}^{n} \alpha_i (c_i \otimes d_i)) &= \sum_{i=1}^{n} \alpha_i g(c_i \otimes d_i) \\ &= \sum_{i=1}^{n} \alpha_i f(c_i, d_i) \\ &= \tilde{f}(\sum_{i=1}^{n} \alpha_i (c_i \odot d_i)) \\ &= \tilde{f}(\sum_{i=1}^{k} \beta_i (e_i \odot f_i)) \quad [\sum_{i=1}^{n} \alpha_i (c_i, d_i) \approx_{E(X)} \sum_{i=1}^{k} \beta_i (e_i, f_i)] \\ &= \sum_{i=1}^{k} \beta_i f(e_i, f_i) \\ &= g(\sum_{i=1}^{k} \beta_i (e_i \otimes f_i)) \end{aligned}$$

So g is well-defined. Clearly $g \circ \phi = f$, and there can be no other linear mappings with this property. □

In fact, the property of $C \otimes D$ stated in the proposition completely describes $C \otimes D$. That is, for any cone L, if there is a bilinear map $\lambda\colon C \times D \to L$ such that for any cone H and bilinear map $f\colon C \times D \to H$ there is a unique linear map $g\colon L \to H$ for which $g \circ \lambda = f$, then $L \cong C \otimes D$. Indeed, we could have used this property to define the tensor product of cones. Also, since a bilinear map $C \times D \to H$ gives rise to a unique linear map $C \otimes D \to H$ in this way, I will often "define" a map from $C \otimes D \to H$ by what it does to elements $c \otimes d$, omitting the obvious argument that the map extends uniquely to all of $C \otimes D$.

2.4.3 Tensor products for cones and vector spaces related

A tensor product of two vector spaces V and W is a vector space L for which there exists a bilinear map $\lambda\colon V \times W \to L$ (bilinear here means the obvious thing) such that for any vector space H and bilinear map $f\colon V \times W \to H$ there is a unique linear map $g\colon L \to H$ for which $g \circ \lambda = f$. It is easily shown that any two tensor products are isomorphic, so *the* tensor product is denoted $V \otimes W$. We can construct $V \otimes W$ in the same way that we constructed $C \otimes D$; the only difference is that we tensor over $\mathbf{R}$ instead of $\mathbf{R}_{\geq 0}$. One can easily show that if δ_i is a basis for V and ϵ_i is a basis for W, then $\delta_i \otimes \epsilon_j$ is a basis for $V \otimes W$.

Cones are sometimes vector spaces. Thus there may seem to be some ambiguity when we consider $C \otimes D$ as to whether we mean the tensor product as cones or as vector spaces. However this is not the case, as for any vector spaces C and D, $C \otimes_{\mathbf{R}_{\geq 0}} D$ and $C \otimes_{\mathbf{R}} D$ are essentially the same thing. The difference lies only in what scalar multiplication technically means in each case. If one of C or D is not a vector space, it does not make sense to speak of $C \otimes_{\mathbf{R}} D$.

The tensor product of two cones is sometimes a vector space. For instance, $C \otimes \mathbf{R}$, all of whose elements can be expressed as $c \otimes 1 + c' \otimes -1$ for some $c, c' \in C$, is indeed a vector space (the inverse of $c \otimes 1 + c' \otimes -1$ is $c' \otimes 1 + c \otimes -1$). In fact, this construction provides the substance for a useful functor V between **CONE** and **VEC**. But before we define V it will be helpful to note some of the structure of $C \otimes \mathbf{R}$ first.

Lemma 4. *Let C be a cone. Let $\Delta := \{(c,c) \in C \times C \mid c \in C\}$. Then $(C \times C)/\Delta \cong C \otimes \mathbf{R}$.*

PROOF. Define

$$\begin{array}{cccl} \phi & : & C \times \mathbf{R} & \to & (C \times C)/\Delta \\ & & (c,r) & \mapsto & \begin{cases} (rc,0) & \text{if } r \geq 0 \\ (0,-rc) & \text{otherwise} \end{cases} \end{array}$$

Now we show by cases that ϕ is bilinear. I will only consider one case, since the rest are similar. Let $r \geq 0$, $s < 0$, and $r + s \geq 0$. We need that $\phi(c, r+s) = \phi(c,r) + \phi(c,s)$. It suffices to show that $(rc + sc, 0) = (rc, -sc)$ in $(C \times C)/\Delta$. This, in turn, is true because $(0,0), (-sc,-sc) \in \Delta$, and $(rc+sc,0) + (-sc,-sc) = (rc,-sc) + (0,0)$ in $C \times C$. Now, since we know ϕ is bilinear we have a linear map $\rho\colon C \otimes \mathbf{R} \to (C \times C)/\Delta$ with $\rho(c \otimes 1) = (c, 0)$.

Now define

$$\begin{array}{cccc} g & : & C \times C & \to & C \otimes \mathbf{R} \\ & & (c,d) & \mapsto & c \otimes 1 + d \otimes -1 \end{array}$$

Since g is linear and $g(\Delta) = 0$, we get a linear map $h\colon (C \times C)/\Delta \to C \otimes \mathbf{R}$ with $h(c,d) = c \otimes 1 + d \otimes -1$. Since h and ρ are clearly inverses of each other, we have that $(C \times C)/\Delta \cong C \otimes \mathbf{R}$. □

Proposition 5. *The map $i\colon C \to C \otimes \mathbf{R}$ defined by $i(c) = c \otimes 1$ is a linear injection. Hence, C is a subcone of $C \otimes \mathbf{R}$.*

PROOF. Define $j\colon C \to (C \times C)/\Delta$ by $j(c) := (c, 0)$. Recalling the isomorphism $h\colon (C \times C)/\Delta \to C \otimes \mathbf{R}$ defined in the proof of Lemma 4, I note that $i = h \circ j$, so it suffices to show that j is a linear injection. Clearly it is linear. Assume that $(c, 0) = (d, 0)$ in $(C \times C)/\Delta$. Then there are $e, f \in C$ with $(c, 0) + (e, e) = (d, 0) + (f, f)$ holding in $C \times C$. Thus, $e = f$ and $c + e = d + f$. By the cancellation law, $c = d$. □

As a corollary, I note that $C \otimes \mathbf{R}$ is the smallest vector space containing C. By this I mean that given any vector space V with an injection $C \hookrightarrow V$, there exists a linear injection of $C \otimes \mathbf{R}$ into V so that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, hook, "i"] \arrow[dr, hook] & C \otimes \mathbf{R} \arrow[d, hook, dotted, "\exists!"] \\ & V \end{tikzcd}$$

The map $\varphi\colon C \otimes \mathbf{R} \to V$ defined by $\varphi(c \otimes 1 + d \otimes -1) := c - d$ can be shown to be well-defined using Proposition 5. That φ is linear and injective is clear.

Definition. [V: **CONE** $\to$ **VEC**] For any cones C and C', and for any linear map $f\colon C \to C'$ define

$$\begin{aligned} \mathrm{V}(C) &:= C \otimes \mathbf{R} \\ \mathrm{V}(f)(c_1 \otimes 1 + c_2 \otimes -1) &:= f(c_1) \otimes 1 + f(c_2) \otimes -1 \end{aligned}$$

I will often denote $\mathrm{V}(C)$ by $C_{\mathbf{R}}$. The map $\mathrm{V}(f)$ is in fact the unique linear extension of f to $C_{\mathbf{R}}$.

A nice property of V is that it preserves injections. Let $f\colon C \to D$ be an injection between cones. Now suppose that $\mathrm{V}(f)(c \otimes 1 + c' \otimes -1) = 0$. By the linearity and definition of $\mathrm{V}(f)$ we have $f(c) \otimes 1 = f(c') \otimes 1$. Thus, $f(c) = f(c')$ by Proposition 5. Since f is injective we have that $c = c'$, from which it follows that $c \otimes 1 + c' \otimes -1 = 0$. Thus, $\mathrm{V}(f)$ is injective.

This functor V is actually just one example of a number of functors given by tensoring by a cone. Let D be cone. Then given any arrow $f\colon C \to C'$ in **CONE**, there exists a unique arrow $\tilde{f}\colon C \otimes D \to C' \otimes D$ such that $\tilde{f}(c \otimes d) = f(c) \otimes d$ for each $c \in C$ and $d \in D$. Thus we may define a functor $\bullet \otimes D\colon$ **CONE** $\to$ **CONE** as follows:

$$\begin{aligned} (\bullet \otimes D)(C) &:= C \otimes D \\ (\bullet \otimes D)(f) &:= \tilde{f} \end{aligned}$$

Proposition 6. *Let C and D be cones, and V and W be vector spaces such that $C \subseteq V$ and $D \subseteq W$. Then $C \otimes D \subseteq V \otimes W$. Specifically,*

$$C \otimes D \cong \{\sum_i \alpha_i (c_i \otimes d_i) \in V \otimes W \mid \alpha_i \in \mathbf{R}_{\geq 0},\ c_i \in C,\ \text{and } d_i \in D\}.$$

PROOF. Since $C_{\mathbf{R}}$ and $D_{\mathbf{R}}$ are the smallest vector spaces containing C and D, we have injections $v\colon C_{\mathbf{R}} \hookrightarrow V$ and $w\colon D_{\mathbf{R}} \hookrightarrow W$. It is easily shown that tensoring vector spaces by a vector space preserves injections. So we have injections $\tilde{v}\colon C_{\mathbf{R}} \otimes D_{\mathbf{R}} \hookrightarrow V \otimes D_{\mathbf{R}}$ and $\tilde{w}\colon D_{\mathbf{R}} \otimes V \hookrightarrow W \otimes V$. Since tensoring is commutative we get the injection $\tilde{w} \circ \tilde{v}\colon C_{\mathbf{R}} \otimes D_{\mathbf{R}} \hookrightarrow V \otimes W$.

By Proposition 5 we have injections $i_C\colon C \hookrightarrow C_{\mathbf{R}}$ and $i_D\colon D \hookrightarrow D_{\mathbf{R}}$. We may form the map $i_C \otimes i_D\colon C \otimes D \to C_{\mathbf{R}} \otimes D_{\mathbf{R}}$ that takes $c \otimes d$ to $(c \otimes 1) \otimes (d \otimes 1)$. There is an obvious isomorphism $\chi\colon (C \otimes D)_{\mathbf{R}} \to C_{\mathbf{R}} \otimes D_{\mathbf{R}}$ defined by $\chi(c \otimes d \otimes 1)\colon = (c \otimes 1) \otimes (d \otimes 1)$. In fact, the following diagram commutes:

$$\begin{tikzcd} C \otimes D \arrow[r, hook, "i_{C\otimes D}"] \arrow[dr, "i_C \otimes i_D"'] & (C \otimes D)_{\mathbf{R}} \arrow[d, "\chi"] \\ & C_{\mathbf{R}} \otimes D_{\mathbf{R}} \end{tikzcd}$$

So, $i_C \otimes i_D$ is an injection. A trivial calculation shows that $\tilde{w} \circ \tilde{v} \circ (i_C \otimes i_D)$ is the desired injection. □

2.4.4 Tensor product is a monoidal product for cones

We've seen how $\bullet \otimes D$ is a functor for any cone D. In fact, $\bullet \otimes \bullet$ is a functor from **CONE** $\times$ **CONE** to **CONE**. It takes mappings $f\colon C \to C'$ and $g\colon D \to D'$ to the mapping $f \otimes g$ defined by $(f \otimes g)(c \otimes d) := f(c) \otimes g(d)$. This bifunctor can be made into a monoidal product. Define $I := \mathbf{R}_{\geq 0}$. Let the associative isomorphism $\alpha_{CDE}\colon (C \otimes D) \otimes E \to C \otimes (D \otimes E)$ be the map defined by $\alpha_{CDE}((c \otimes d) \otimes e) := c \otimes (d \otimes e)$. Trivially this is a natural isomorphism. Define $\lambda_C\colon I \otimes C \to C$ by $\lambda_C(r \otimes c) := rc$. This map has as inverse $c \mapsto 1 \otimes c$, and is natural in C. Likewise, define $\rho_C\colon C \otimes I \to C$ by $\rho_C(c \otimes r) := rc$. One can easily verify that the three coherence conditions of Theorem 2 are satisfied, so $\otimes$, supplied with I, α, λ, and ρ, is indeed a monoidal product. Further, $\otimes$ can be made into a symmetric monoidal product by specifying the natural isomorphism $\gamma_{CD}\colon C \otimes D \to D \otimes C$ defined by $\gamma_{CD}(c \otimes d) := d \otimes c$. The relevant coherence conditions are satisfied.

2.5 The tensor product for polytopes

It makes sense to speak of biaffine maps (maps that preserve affine combinations in both coordinates) because for any two polytopes P and Q, $P \times Q$ is indeed a polytope. Thus, it makes sense to speak of a tensor product of polytopes. In this section we give the construction and show it satisfies the regular tensor product property. Also, we note how it gives a monoidal product for **POLY**.

Let $P \subseteq \mathbf{R}^n$ and $Q \subseteq \mathbf{R}^m$ be polytopes. Then $P \otimes Q$ is defined to be

$$\{\sum_i \alpha_i (p_i, 1) \otimes (q_i, 1) \in \mathbf{R}^{n+1} \otimes \mathbf{R}^{m+1} \mid \alpha_i \in \mathbf{R}_{\geq 0} \text{ and } \sum_i \alpha_i = 1\}$$

Given the isomorphism $\mathbf{R}^{n+1} \otimes \mathbf{R}^{m+1} \cong \mathbf{R}^{nm+n+m+1}$, this is equivalent to the set of convex combinations of elements of the form

$$(p_1q_1, \ldots, p_iq_j, \ldots, p_nq_m, p_1, \ldots, p_n, q_1, \ldots, q_m, 1)$$

where $(p_1, \ldots, p_n) \in P$ and $(q_1, \ldots, q_m) \in Q$.

By Proposition 6, $\mathrm{C}(P) \otimes \mathrm{C}(Q)$ is given by

$$\{\sum \alpha_i(p_i, 1) \otimes (q_i, 1) \in \mathbf{R}^{n+1} \otimes \mathbf{R}^{m+1} \mid \alpha_i \in \mathbf{R}_{\geq 0}\}$$

Thus, it follows that $P \otimes Q$ is a subset of this cone. Further, $P \otimes Q$ is a "slice" of $\mathrm{C}(P) \otimes \mathrm{C}(Q)$ in the following sense. Let e_i be the standard basis for $\mathbf{R}^{n+1}$ and let f_i be the standard basis for $\mathbf{R}^{m+1}$. For any $x \in \mathbf{R}^{n+1} \otimes \mathbf{R}^{m+1}$ let us use $x_{(n+1)(m+1)}$ to denote the coefficient of $e_{n+1} \otimes f_{m+1}$ in the representation of x by the basis $e_i \otimes f_j$. Then

$$P \otimes Q = \{x \in \mathrm{C}(P) \otimes \mathrm{C}(Q) \mid x_{(n+1)(m+1)} = 1\}.$$

Clearly, $P \otimes Q$ is a polytope in $\mathbf{R}^{n+1} \otimes \mathbf{R}^{m+1}$ since it is generated by a finite set, namely $\{(p, 1) \otimes (q, 1) \mid p \in \mathrm{vert}(P) \text{ and } q \in \mathrm{vert}(Q)\}$. However, it remains to be seen that $P \otimes Q$ is indeed a tensor product. To this end, define the obvious biaffine map $\lambda\colon P \times Q \to P \otimes Q$ by $\lambda(p, q) := (p, 1) \otimes (q, 1)$, and let $f\colon P \times Q \to H$ be a biaffine mapping. We need to show that there exists a unique affine map $g\colon P \otimes Q \to H$ such that $g \circ \lambda = f$. Define a map $\tilde{f}\colon \mathrm{C}(P) \times \mathrm{C}(Q) \to \mathrm{C}(H)$ by saying $\tilde{f}(r(p, 1), s(q, 1)) := rs(f(p, q), 1)$. This is indeed well-defined and, in fact, bilinear. Thus, by Proposition 3 we get a linear map $g'\colon \mathrm{C}(P) \otimes \mathrm{C}(Q) \to \mathrm{C}(H)$ with $g' \circ \phi = \tilde{f}$, where ϕ is the standard map from $\mathrm{C}(P) \times \mathrm{C}(Q)$ to $\mathrm{C}(P) \otimes \mathrm{C}(Q)$. In particular, we have

$$g'((p, 1) \otimes (q, 1)) = (f(p, q), 1)$$

Using the isomorphism $\{(h, 1) \in \mathrm{C}(H)\} \cong H$, and restricting g' to $P \otimes Q$, we obtain an affine map $g\colon P \otimes Q \to H$ with $g(\lambda(p, q)) = f(p, q)$, as desired. Clearly g is the unique affine map with this property.

This tensor product provides a symmetric monoidal product for **POLY**. We may define the functor $\bullet \otimes \bullet\colon \mathbf{POLY} \times \mathbf{POLY} \to \mathbf{POLY}$ on arrows by letting $f \otimes g$ be defined by $(p, 1) \otimes (q, 1) \mapsto (f(p), 1) \otimes (g(q), 1)$ for affine maps $f\colon P \to P'$ and $g\colon Q \to Q'$. We let $I := \mathbf{R}^0$, a polytope of one element. And we let α, λ, ρ, and γ be the obvious natural isomorphisms. The coherence conditions are indeed satisfied. Thus we have

Proposition 7. **POLY**, *supplied with* $\otimes$, *is a symmetric monoidal category.*

2.6 Internal hom

Let $\mathbf{C}$ be a category with a monoidal product $\otimes$. Then for each object $B \in \mathbf{C}$ there is a functor $\bullet \otimes B$. An *internal hom* for $\mathbf{C}$ is a collection of functors $[B, \bullet]$, one for each object B, such that $[B, \bullet]$ is a right adjoint of $\bullet \otimes B$. A symmetric monoidal category is called *closed* if such an internal hom exists. Given an internal hom $[B, \bullet]$, there is a unique way to make $[\bullet, \bullet]$ into a functor from $\mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ to $\mathbf{C}$ such that the isomorphism $\varphi\colon \hom(A\otimes B, C) \to \hom(A, [B, C])$ becomes natural in A, C, *and* B. This is the "adjunctions with a parameter" theorem in MacLane p. 100.

We now define a functor $[\bullet, \bullet]\colon \mathbf{CONE}^{\mathrm{op}} \times \mathbf{CONE} \to \mathbf{CONE}$. For cones C and D, $[C, D]$ is defined as $\hom(C, D)$ supplied with cone structure in the obvious way. For instance, for $f, g\colon C \to D$, $f + g\colon C \to D$ is defined by $(f + g)(c) := f(c) + g(c)$. All the axioms for a cone are satisfied, including the cancellation law. For maps $f\colon C' \to C$ and $g\colon D \to D'$, $[f, g] := f^* \circ g_*$. One can easily verify that $f^* \circ g_*$ is linear, $[\mathrm{Id}_C, \mathrm{Id}_D] = \mathrm{Id}_{[C,D]}$, and that $[f', g'] \circ [f, g] = [f \circ f', g' \circ g]$ for appropriate maps f, f', g, and g'. Thus, $[\bullet, \bullet]$ is indeed a functor.

Now we verify that for any cone D, the functor $[D, \bullet]$ is a right adjoint of $\bullet \otimes D$. We need a collection of bijections $\varphi_{CE}\colon \hom(C \otimes D, E) \to \hom(C, [D, E])$ natural in C and E. Define φ by $\varphi_{CE}(x)(c)(d) = x(c \otimes d)$. We wish to show that φ has an inverse. Let y be a linear map from C to $[D, E]$. Since the map $\eta\colon C\times D \to E$ defined by $\eta(c, d) := y(c)(d)$ is bilinear, there is a map $x\colon C\otimes D \to E$ with $x(c\otimes d) = y(c)(d)$. Clearly, this mapping $y \mapsto x$ is an inverse of φ. Let $f\colon C' \to C$ be linear. To verify that φ is natural in C, one must verify that the following diagram commutes.

$$\begin{tikzcd}
\hom(C \otimes D, E) \arrow[r, "\varphi_{CE}"] \arrow[d, "(f\otimes \mathrm{Id}_D)^*"'] & \hom(C, [D, E]) \arrow[d, "f^*"] \\
\hom(C' \otimes D, E) \arrow[r, "\varphi_{C'E}"'] & \hom(C', [D, E])
\end{tikzcd}$$

I omit the easy diagram chase. Now let $f\colon E \to E'$. Naturality in E amounts to this diagram commuting:

$$\begin{tikzcd}
\hom(C \otimes D, E) \arrow[r, "\varphi_{CE}"] \arrow[d, "f_*"'] & \hom(C, [D, E]) \arrow[d, "{[\mathrm{Id}_D, f]_*}"] \\
\hom(C \otimes D, E') \arrow[r, "\varphi_{CE'}"'] & \hom(C, [D, E'])
\end{tikzcd}$$

Again, I omit the diagram chase. One may verify that φ is also natural in D, demonstrating that the functor $[\bullet, \bullet]$ we defined is the functor we would get if we applied the "adjunctions with a parameter" thereom to $[D, \bullet]$.

To form an internal hom for **POLY**, we use the same procedure as we did for cones, except here we must make the identification of affine maps with matrices and hence real coordinates. Let P be a polytope of dimension s, and let $Q \subseteq \mathbf{R}^m$ be a polytope defined by the l inequalities $Bx \leq z$. First let $\phi_P\colon \mathrm{affspan}(P) \to \mathbf{R}^s$ be an affine isomorphism. The affine maps from $\mathbf{R}^s$ to $\mathbf{R}^m$ can be identified with matrices in the usual way. Explicitly, an $m \times (s + 1)$ matrix A takes a point $\vec{x} = (x_1, \ldots, x_s) \in \mathbf{R}^s$ to

$$A\begin{pmatrix} \vec{x} \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1s} & c_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{ms} & c_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_s \\ 1 \end{pmatrix}$$

Further, via ϕ_P, each affine map $f\colon P \to Q$ may be identified with a unique map $\mathbf{R}^s \to \mathbf{R}^m$. To summarize this identification, let $M_{PQ}\colon \hom(P,Q) \to [P,Q] \subseteq \mathbf{R}^{m(s+1)}$ be the bijective map that takes each affine map to its matrix representation; $[P,Q]$ is defined simply as the image of this map. Explicitly,

$$[P,Q] := \{A \in \mathbf{R}^{m(s+1)} \mid A \begin{pmatrix} \phi_P(p) \\ 1 \end{pmatrix} \in Q,\ \forall p \in P\}$$

We wish to show that $[P,Q]$ is indeed a polytope. Let $v_1, \ldots, v_{s+1}$ be $s+1$ affinely independent vertices of P. Then

$$\begin{aligned} [P,Q] &= \{A \in \mathbf{R}^{m(s+1)} \mid A \begin{pmatrix} \phi_P(v) \\ 1 \end{pmatrix} \in Q,\ \forall v \in \mathrm{vert}(P)\} \\ &\subseteq \{A \in \mathbf{R}^{m(s+1)} \mid A \begin{pmatrix} \phi_P(v_i) \\ 1 \end{pmatrix} \in Q,\ \forall i \text{ with } 1 \le i \le s+1\} \\ &\cong Q^{s+1} \end{aligned}$$

So $[P,Q]$ is bounded. Further, we shall presently show that the inequalities in

$$[P,Q] = \{A \in \mathbf{R}^{m(s+1)} \mid BA \begin{pmatrix} \phi_P(v) \\ 1 \end{pmatrix} \le z,\ \forall v \in \mathrm{vert}(P)\}$$

can be rearranged to obtain an intersection of half-spaces, thereby ensuring that $[P,Q]$ is a polytope. To this end, let us make the following definitions. Assume P has k vertices, and let us consider them (via ϕ_P) as elements $\vec{v_i}$ of $\mathbf{R}^s$. Let us use $(v_{i1}, \ldots, v_{is})$ to denote the representation of $\vec{v_i}$ by the standard basis. Define

$$\bar{V} := \begin{pmatrix} v_{11} & \cdots & v_{1s} & 1 \\ \vdots & & \vdots & \vdots \\ v_{k1} & \cdots & v_{ks} & 1 \end{pmatrix} = \begin{pmatrix} \vec{v_1} & 1 \\ \vdots & \vdots \\ \vec{v_k} & 1 \end{pmatrix}$$

$$B \otimes \bar{V} := \begin{pmatrix} b_{11}\bar{V} & \cdots & b_{1m}\bar{V} \\ \vdots & & \vdots \\ b_{l1}\bar{V} & \cdots & b_{lm}\bar{V} \end{pmatrix}$$

$$\vec{z} := z \otimes \left(\left. \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \right\} k \text{ times} \right) := \begin{pmatrix} z_1 \\ \vdots \\ z_1 \\ \\ \vdots \\ \\ z_l \\ \vdots \\ z_l \end{pmatrix}$$

and note that an element

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1s} & c_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{ms} & c_m \end{pmatrix} \in \mathbf{R}^{m(s+1)}$$

may be considered a column vector

$$\begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ c_1 \\ \\ \vdots \\ \\ a_{m1} \\ \vdots \\ c_1 \end{pmatrix}$$

Then it is easily verified that

$$[P, Q] = \{A \in \mathbf{R}^{m(s+1)} \mid (B \otimes \bar{V})A \leq \vec{z}\}$$

Given affine mappings $f\colon P' \to P$ and $g\colon Q \to Q'$, we may form a mapping $[f, g]\colon [P, Q] \to [P', Q']$ by $A \mapsto M_{P'Q'}(g \circ M_{PQ}^{-1}(A) \circ f)$. This mapping can be shown to be affine because M and its inverse are themselves "affine". I.e.,

$$M_{PQ}(rf + (1-r)f') = rM_{PQ}(f) + (1-r)M_{PQ}(f')$$

and similarly for the inverse. Since it can be shown that $[\mathrm{Id}_P, \mathrm{Id}_Q] = \mathrm{Id}_{[P,Q]}$, and that $[f', g'] \circ [f, g] = [f \circ f', g' \circ g]$ for appropriate maps f, f', g, and g', we have a functor $[\bullet, \bullet]\colon \mathbf{POLY}^{\mathrm{op}} \times \mathbf{POLY} \to \mathbf{POLY}$.

The verification that $[Q, \bullet]$ is a right adjoint of $\bullet \otimes Q$ for any polytope Q, is very similar to how we did it for cones. We need a collection of bijections φ_{PR} natural in P and R: define

$$\begin{array}{rcrcl} \varphi_{PR} & : & \hom(P \otimes Q, R) & \to & \hom(P, [Q, R]) \\ & & x\colon P \otimes Q \to R & \mapsto & p \mapsto M_{QR}(q \mapsto x(p \otimes q)) \end{array}$$

This mapping is well-defined in the sense that $\varphi_{PR}(x)$ is indeed affine. Further, φ_{PR} has a well-defined inverse, namely

$$\begin{array}{rcrcl} \eta_{PR} & : & \hom(P, [Q, R]) & \to & \hom(P \otimes Q, R) \\ & & y\colon P \to [Q, R] & \mapsto & p \otimes q \mapsto M_{QR}^{-1}(y(p))(q) \end{array}$$

The argument for naturality in P and R resembles closely the one given for the naturality of C and E above, and is omitted. Also one can show that φ is natural in Q as well. The above discussion gives us the following proposition:

Proposition 8. **POLY**, *supplied with* $\otimes$, *is a symmetric monoidal closed category.*

Let $\mathbf{C}$ be a category with monoidal product $\otimes$. Intuitively, it makes sense to call a right adjoint $[B, \bullet]$ an "internal hom" because in this case there is a functor $F\colon \mathbf{C} \to \mathbf{SET}$ such that there is an isomorphism $F[B, C] \cong \hom(B, C)$ natural in B and C. Define $F := \hom(I, \bullet)$. Then by adjointness there is an isomorphism $\varphi\colon \hom(I \otimes B, C) \to \hom(I, [B, C])$ natural in C and, as per the adjunction with a parameter theorem, in B. Since we have the natural isomorphism $\lambda\colon I \otimes B \to B$, we can obtain a natural isomorphism $\hom(B, C) \cong \hom(I \otimes B, C)$. Putting these two natural isomorphisms together we get

$$\hom(B, C) \cong \hom(I, [B, C])$$

In the case of **CONE** and **POLY** this returns the isomorphisms $\hom(\mathbf{R}_{\geq 0}, [C, D]) \cong \hom(C, D)$ and $\hom(\mathbf{R}^0, [P, Q]) \cong \hom(P, Q)$.

Now that we have been explicit about how $\hom(\bullet, \bullet)$ and $[\bullet, \bullet]$ are related, I will be more care-free with notation by saying, for instance, "$\hom(P, Q)$" when I mean "$[P, Q]$".

2.7 Some properties of $\hom(P, Q)$

2.7.1 Dimension of $\hom(P, Q)$

Proposition 9. *Let P and Q be polytopes of dimension s and t, respectively. Then* $\dim(\hom(P, Q)) = (s + 1)t$.

PROOF. Since the dimension of P is s, there is an injection $P \hookrightarrow \Delta_s$. By way of this injection, we may identify $\hom(\Delta_s, Q)$ with a subset of $\hom(P, Q)$. Since $\hom(\Delta_s, Q) \cong Q^{s+1}$ and so $\hom(\Delta_s, Q)$ has dimension $(s + 1)t$, we conclude that the dimension of $\hom(P, Q)$ is at least this amount. Yet it cannot be more, since the linear space of all affine mappings from $\mathbf{R}^s$ to $\mathbf{R}^t$ has dimension $(s + 1)t$ and $\hom(P, Q)$ can be identified with a subset of this space. $\square$

2.7.2 Facets of $\hom(P, Q)$

In this section we prove that the facets of $\hom(P, Q)$ are given by selecting a vertex of P and a facet of Q. Explicitly, if v is a vertex of P and L is a facet of Q, then $\{f \in \hom(P, Q) \mid f(v) \in L\}$ is a facet of $\hom(P, Q)$, and each facet of $\hom(P, Q)$ arises in this way. And, in particular, if there are k vertices of P and l facets of Q, then there are kl facets of $\hom(P, Q)$.

Recall that a polytope $Q \in \mathbf{R}^m$ may be written as a bounded, finite intersection of closed half-spaces, where each half-space is represented by an inequality $b_i \cdot x \leq z_i$. It can be shown that such a collection of inequalities can always be reduced to obtain a collection that gives the same intersection but is not *redundant*. (A collection of inequalities is *redundant* if there is some inequality that is satisfied whenever the other inequalities are satisfied, i.e., this inequality is unnecessary, or redundant.) Further, the facets of a full-dimensional polytope are given by the inequalities of any non-redundant collection of inequalities defining the polytope. I.e., if $Q = \{x \in \mathbf{R}^m \mid B \cdot x \leq z\}$ is full-dimensional and $B \cdot x \leq z$ is not redundant, then for each row b_i of B, we get a unique facet $\{x \in \mathbf{R}^m \mid b_i \cdot x = z_i\}$, and all the facets of Q arise in this way. (For justification cf. Brøndsted pp. 52-53)

Let $P \subseteq \mathbf{R}^n$ and $Q \subseteq \mathbf{R}^m$ be full-dimensional polytopes, and let $Q = \{x \in \mathbf{R}^m \mid B \cdot x \leq z\}$ for some non-redundant collection of inequalities $B \cdot x \leq z$. It follows from Proposition 9 that $\hom(P, Q)$ is full-dimensional. In section 2.6 we presented $\hom(P, Q)$ as an intersection of half-spaces:

$$\hom(P, Q) = \{A \in \mathbf{R}^{m(n+1)} \mid (B \otimes \bar{V})A \leq \vec{z}\}.$$

Note that each inequality of $(B \otimes \bar{V})A \leq \vec{z}$ may be rearranged to obtain an inequality of the form

$$b_i \cdot A \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ 1 \end{pmatrix} \leq z_i$$

where $b_i \cdot x \leq z_i$ represents a facet of Q, and v is a vertex of P. Thus, if we show that $(B \otimes \bar{V})A \leq \vec{z}$ is not redundant, then it follows that the facets of $\hom(P, Q)$ are exactly given by $\{A \in \mathbf{R}^{m(s+1)} \mid b_i \cdot (Av) = z_i\} = \{f \in \hom(P, Q) \mid f(v) \in L\}$ where the L is the facet defined by an inequality $b_i \cdot x \leq z_i$, and v is a vertex of P.

We will prove that the collection $(B \otimes \bar{V})A \leq \vec{z}$ of inequalities is not redundant, but first we need to make a couple of observations. First, even though we are assuming that P and Q are full-dimensional, our results actually apply to all hom-polytopes, because the properties we are interested in are preserved by affine isomorphism. Likewise, we may assume that 0 is in the relative interior of Q. In this case, the column vector z consists only of positive numbers, since for each inequality $b_i \cdot x \leq z_i$ we must have $0 = b_i \cdot 0 < z_i$ (otherwise 0 would be in some facet). Thus, we may further assume that z consists only of 1's, as the inequality $b_i \cdot x \leq z_i$ is equivalent to $\frac{b_i}{z_i} \cdot x \leq 1$.

Second, we will make use of the Farkas Lemma. One of the forms of this lemma states that a collection $C \cdot x \leq y$ of w inequalities is redundant iff one of these inequalities, say $c_i \cdot x \leq y_i$, has the property that

1. there is a non-negative row vector $d = (d_1, \ldots, d_{w-1})$ with $d \cdot \tilde{C} = c_i$ and $d \cdot \tilde{y} \leq y_i$, or

2. there is a non-negative row vector $d = (d_1, \ldots, d_{w-1})$ with $d \cdot \tilde{C} = 0$ and $d \cdot \tilde{y} < 0$

where $\tilde{C}$ is the matrix C with the row c_i removed, and $\tilde{y}$ is y with y_i removed (see Ziegler p. 41). We will in fact use this lemma twice in our proof, once for $B \cdot x \leq z$ and once for $(B \otimes \bar{V})A \leq \vec{z}$.

Lemma 10. *Let $P \subseteq \mathbf{R}^n$ be a full-dimensional polytope with k vertices $\vec{v_1}, \ldots, \vec{v_k}$. Define the matrix*

$$\bar{V} := \begin{pmatrix} \vec{v_1} & 1 \\ \vdots & \vdots \\ \vec{v_k} & 1 \end{pmatrix}$$

Let $Q \subseteq \mathbf{R}^m$ be a full-dimensional polytope with l facets. Let $Q = \{x \in \mathbf{R}^m \mid B \cdot x \leq z\}$ for some non-redundant collection of inequalities $B \cdot x \leq z$, with z consisting only of 1's. Let $\vec{z}$ be the matrix consisting of kl 1's in one column. Then the collection of inequalities $(B \otimes \bar{V})A \leq \vec{z}$ is not redundant.

PROOF. Suppose, to get a contradiction that $(B \otimes \bar{V})A \leq \vec{z}$ is redundant. Let $C = B \otimes \bar{V}$. The Farkas Lemma says that either condition 1 or 2 above is met. Since $\vec{z}$ consists only of 1's, condition 2 would imply that $\sum d_i < 0$, so it cannot be met. Thus there is a non-negative row vector $d = (d_2, \ldots, d_{lk})$ with $d \cdot \tilde{C} = c$ and $\sum_{i=2}^{lk} d_i \leq 1$, for some row c of C. However, we might as well assume that c is the first row of C. Here is a representation of C with the first row removed (the k vertices of P are written as $(v_{11}, \ldots, v_{1n}), \ldots, (v_{k1}, \ldots, v_{kn})$):

$$\begin{array}{ccccccccccc}
b_{11}v_{11} & b_{11}v_{12} & \cdots & b_{11}v_{1n} & b_{11} & \cdots & b_{1m}v_{11} & b_{1m}v_{12} & \cdots & b_{1m}v_{1n} & b_{1m} \\
\hline
b_{11}v_{21} & b_{11}v_{22} & \cdots & b_{11}v_{2n} & b_{11} & & b_{1m}v_{21} & b_{1m}v_{22} & \cdots & b_{1m}v_{2n} & b_{1m} \\
\vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots & \vdots \\
b_{11}v_{k1} & b_{11}v_{k2} & \cdots & b_{11}v_{kn} & b_{11} & & b_{1m}v_{k1} & b_{1m}v_{k2} & \cdots & b_{1m}v_{kn} & b_{1m} \\
\hline
\multicolumn{5}{c}{b_{21}\bar{V}} & \cdots & \multicolumn{5}{c}{b_{2m}\bar{V}} \\
\hline
\multicolumn{5}{c}{\vdots} & & \multicolumn{5}{c}{\vdots} \\
\hline
\multicolumn{5}{c}{b_{l1}\bar{V}} & \cdots & \multicolumn{5}{c}{b_{lm}\bar{V}}
\end{array}$$

Notice how we organized the rows of $\tilde{C}$ into l blocks. For each $2 \leq i \leq l$, define

$$e_i := \sum_{j=(i-1)k+1}^{ik} d_j$$

and define $e_1 := \sum_{j=2}^{k} d_j$. Let $e := (e_1, e_2, ..., e_l)$. We have grouped the elements of d based on what block they multiply in $d \cdot \tilde{C}$, and have taken the sum of each group. Now restrict your attention to the $i(n+1)^{\text{th}}$ columns for $1 \leq i \leq m$. Since

$d \cdot \tilde{C} = c$, it follows that $e_1 b_{1i} + e_2 b_{2i} + \cdots + e_l b_{li} = b_{1i}$ for each $1 \leq i \leq m$. I.e., $e_1 b_1 + e_2 b_2 + \cdots + e_m b_m = b_1$, where b_i denotes the i^{th} row of B. I wish to show that $e_1 = 1$ (and hence $e_i = 0$ for $i \geq 2$). Suppose, to get a contradiction, that $e_1 \neq 1$. Then $e_1 < 1$ since $\sum_{i=1}^{l} e_i = \sum_{i=2}^{lk} d_i \leq 1$, and e consists of non-negative numbers. It follows that

$$E := \left(\frac{e_2}{1-e_1}, \ldots, \frac{e_m}{1-e_1}\right)$$

is a non-negative row vector with the property that $E \cdot \tilde{B} = b_1$ and $E \cdot \tilde{z} = \frac{\sum_{i \geq 2} e_i}{1-e_1} \leq 1$. Thus, by the Farkas Lemma, B is redundant — a contradiction. So we have established that $e_1 = 1$ and $e_i = 0$ for $i \geq 2$. In particular we learn that $d_i = 0$ for $i \geq k+1$, and $\sum_{i=2}^{k} d_i = 1$. Since B is not redundant, it contains no row consisting entirely of zeroes. Hence, we may assume that $b_{11} \neq 0$. Now restrict your attention to the first n columns of $\tilde{C}$. Since, $d \cdot \tilde{C} = c$, it follows that $d_2 b_{11} v_{2i} + d_3 b_{11} v_{3i} + \cdots + d_k b_{11} v_{ki} = b_{11} v_{1i}$ for $1 \leq i \leq n$. Since $b_{11} \neq 0$, we have $d_2 v_{2i} + d_3 v_{3i} + \cdots + d_k v_{ki} = v_{1i}$ for $1 \leq i \leq n$. In other words, $\sum_{i=2}^{k} d_i \vec{v_i} = \vec{v_1}$. Thus, we have a convex combination of vertices giving another, different vertex. This is a contradiction. $\square$

Theorem 11. *Let P and Q be polytopes. If v is a vertex of P and L is a facet of Q, then $\{f \in \hom(P,Q) \mid f(v) \in L\}$ is a facet of $\hom(P,Q)$, and each facet of $\hom(P,Q)$ arises in this way. In particular, if there are k vertices of P and l facets of Q, then there are kl facets of $\hom(P,Q)$.*

2.7.3 Some vertices of $\hom(P,Q)$

In this section we prove that a mapping that sends all of P to a vertex $q_0 \in Q$ is a vertex of $\hom(P,Q)$.

Lemma 12. *Let $P \subseteq \mathbf{R}^n$ be a full-dimensional polytope with $\vec{0}$ in the relative interior of P. Let $Q \subseteq \mathbf{R}^m$ be a polytope with a vertex q_0. Then the mapping $\pi\colon P \to Q$ defined by $\pi(p) := q_0$ is a vertex of $\hom(P,Q)$.*

PROOF. Let $v_0 \cdot x \leq r_0$ be a face-defining inequality for the vertex $q_0 \in Q$. Now let $\bar{0}$ be the $m \times n$ matrix consisting entirely of zeroes. We may adjoin v_0 to $\bar{0}$ to obtain an $m \times (n+1)$ matrix, which we will denote $\bar{0} \vee v_0$. Now consider the inequality $\bar{0} \vee v_0 \cdot x \leq r_0$. I claim that it is a valid inequality for $\hom(P,Q) \subseteq \mathbf{R}^{m(n+1)}$. Let $\rho \in \hom(P,Q)$. Since $\vec{0} \in P$, ρ may be written in the form $A \vee q$ for some $q \in Q$ and some $m \times n$ matrix A. Now, $\bar{0} \vee v_0 \cdot A \vee q = v_0 \cdot q \leq r_0$, since $v_0 \cdot x \leq r_0$ is valid for all $q \in Q$. Thus, $\bar{0} \vee v_0 \cdot x \leq r_0$ is valid for $\hom(P,Q)$. So it defines a face of $\hom(P,Q)$, but we need make sure that it defines a vertex. I.e., we need to show that $\pi = \bar{0} \vee q_0$ is the only element of $\{A \vee q \in \hom(P,Q) \mid \bar{0} \vee v_0 \cdot A \vee q = r_0\}$. Since $\bar{0} \vee v_0 \cdot A \vee q = r_0$ implies that $q = q_0$, we know that $\vec{0} \in P$ maps to q_0 under any element of this face. Yet, since $\vec{0}$ is in the relative interior of P, $\vec{0}$ is a strictly positive convex combination of $n+1$ affinely independent points in P; i.e., there exists $\lambda_i > 0$ and $x_i \in P$ with $\sum \lambda_i = 1$ and $\vec{0} = \sum_{i=0}^{n+1} \lambda_i x_i$ (see Ziegler p. 60). So for a map ρ in this face, $\rho(\vec{0}) = \rho(\sum_{i=0}^{n+1} \lambda_i x_i) = \sum_{i=0}^{n+1} \lambda_i \rho(x_i) = q_0$. But there are only trivial convex combinations yielding q_0 since q_0 is a vertex. Thus, $\rho(x_i) = q_0$ for each i, and hence $\rho = \pi$. $\square$

Theorem 13. *Let $P \subseteq \mathbf{R}^n$ and $Q \subseteq \mathbf{R}^m$ be polytopes. Then the map $\pi\colon P \to Q$ defined by $\pi(p) := q_0$, where q_0 is a vertex of Q, is itself a vertex in* $\hom(P, Q)$.

PROOF. Let $\dim(P) = s$, and let b denote the barycenter of P. Since b can be represented by a strictly positive convex combination of $s + 1$ affinely independent points in P, b is in the relative interior of P (see Ziegler p. 60). Hence, $\tilde{P} := P - b$ has $\vec{0}$ in its relative interior. We may also assume that $\tilde{P}$ is full-dimensional. Using the obvious isomorphism $\theta\colon P - b \to P$ defined by addition by b, we obtain an isomorphism $\hom(\theta, \mathrm{Id}_Q)$ between $\hom(P, Q)$ and $\hom(\tilde{P}, Q)$. Since polytope isomorphisms take vertices to vertices, and $\hom(\theta, \mathrm{Id}_Q)(\pi) = \pi \circ \theta$ is a vertex in $\hom(\tilde{P}, Q)$ (by Lemma 8), π is a vertex in $\hom(P, Q)$. $\square$

Chapter 3

Limits and Colimits

In this chapter we present categorical constructions called finite limits and the corresponding colimits obtained by reversing the arrows involved. The preservation of these (co)limits by a (co)continuous functor is also discussed.

3.1 Limits

3.1.1 Products

In any category, a product of two objects A and B is an object $A \times B$ supplied with arrows $p_A \colon A \times B \to A$ and $p_B \colon A \times B \to B$ such that for any object X and arrows $f \colon X \to A$ and $g \colon X \to B$ there exists a unique arrow h from X to $A \times B$ such that $p_A \circ h = f$ and $p_B \circ h = g$. This condition can be presented graphically as:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, dotted, "\exists! h"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}$$

In any category, any two products are isomorphic, so one speaks of "the" product. A category is said to have binary products if for any two objects there is a product. The categories **SET**, **VEC**, **CONE**, and **POLY** all have binary products. The regular cartesian product works in each case.

A *terminal* object in a category is an object 1 such that for any object A there exists exactly one arrow from A to 1. A terminal object can be thought of as a product of no objects. A category is said to have all finite products if it has binary products and a terminal object. **SET**, **VEC**, **CONE**, and **POLY** have all finite products. In **SET** and **POLY** any singleton is a terminal object. In **VEC** and **CONE**, $\{\vec{0}\}$ is a terminal object.

3.1.2 Equalizers

In any category, an equalizer of two arrows $f, g \colon A \to B$ is an object E supplied with an arrow $e \colon E \to A$ such that $f \circ e = g \circ e$, and for any $x \colon X \to A$ with

$f \circ x = g \circ x$ there exists a unique arrow $l\colon X \to E$ such that $e \circ l = x$. Here is a picture summarizing this situation:

$$
\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
X \arrow[u, dotted, "\exists! l"] \arrow[ur, "x"'] & &
\end{tikzcd}
$$

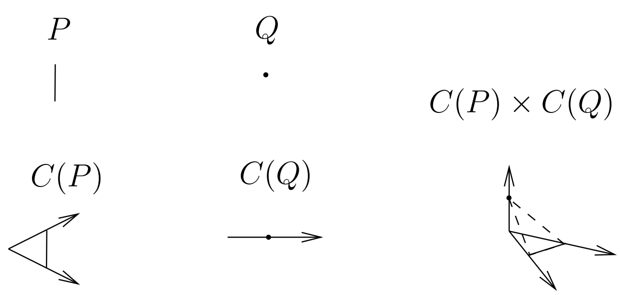
It is clear that if E_1 and E_2 are both equalizers for the same pair of arrows, then $E_1 \cong E_2$. In **SET** an equalizer for any two functions $f, g\colon A \to B$ is the inclusion of $\{a \in A \mid f(a) = g(a)\}$ into A. In **POLY** the same construction works because this set is indeed a polytope. In **VEC** and **CONE** as well the same construction yields an equalizer.

3.1.3 Continuity

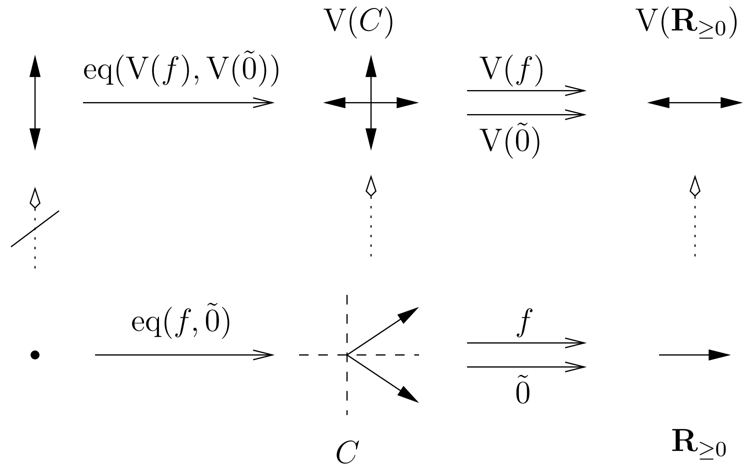
A functor $F\colon \mathbf{C} \to \mathbf{D}$ is said to be *finitely continuous* if it preserves all finite products and all equalizers. This means three things:

1. Given a terminal object $1 \in \mathbf{C}$, $F(1)$ is a terminal object in $\mathbf{D}$
2. Given a product $C \times C' \in \mathbf{C}$, $F(C \times C') \cong F(C) \times F(C')$
3. Given an equalizer of any two arrows $f, g\colon C \to C'$, $F(\text{equalizer}(f, g)) \cong \text{equalizer}(F(f), F(g))$.

Neither C nor V is a continuous functor. To see this, let P be a line segment and let Q be a point. Then $P \times Q \cong P$, but $\mathrm{C}(P) \times \mathrm{C}(Q)$ is not isomorphic to $\mathrm{C}(P)$, as can be seen in the following picture:



Thus, C does not preserve products. Even easier, $\mathrm{C}(\mathbf{R}^0) \cong \mathbf{R}_{\geq 0}$ is not isomorphic to $\{\vec{0}\}$, so C does not preserve terminal objects. Meanwhile, V does not preserve equalizers. Let $C := \{(x, y) \in \mathbf{R}^2 \mid x \geq y \geq -x\}$. Define $f\colon C \to \mathbf{R}_{\geq 0}$ by $f(x, y) = x$. Then the equalizer of f and $\tilde{0}$ is $\{\vec{0}\}$, and $\mathrm{V}(\{\vec{0}\}) \cong \{\vec{0}\}$. However, the equalizer of $\mathrm{V}(f)$ and $\mathrm{V}(\tilde{0})$ is a line.



3.2 Colimits

3.2.1 Coproducts

The coproduct is defined in the same way as the product except that we reverse the direction of the arrows involved. A *coproduct* of two objects A and B is an object $A \oplus B$ supplied with arrows $p_A \colon A \to A \oplus B$ and $p_B \colon B \to A \oplus B$ such that for any object X and arrows $f \colon A \to X$ and $g \colon B \to X$ there exists a unique arrow h from $A \oplus B$ to X such that $h \circ p_A = f$ and $h \circ p_B = g$.

$$\begin{tikzcd} & X & \\ A \arrow[ur, "f"] \arrow[r, "p_A"'] & A \oplus B \arrow[u, dotted, "\exists! h"] & B \arrow[ul, "g"'] \arrow[l, "p_B"] \end{tikzcd}$$

In **SET** the disjoint union gives the coproduct. In **VEC** and **CONE** the coproduct is actually the same as the product. In **POLY**, a construction called the "join" provides a coproduct. Given two polytopes $P \subseteq \mathbf{R}^n$ and $Q \subseteq \mathbf{R}^m$ we may form a new polytope called the *join* of P and Q, which I will denote by $P \oplus Q$ (for obvious reasons), though it is sometimes denoted $P \bowtie Q$, or $P * Q$. Let $P' := \{(p, \vec{0}_{\mathbf{R}^m}, 0) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \mid p \in P\}$, and let $Q' := \{(\vec{0}_{\mathbf{R}^n}, q, 1) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \mid q \in Q\}$. Then

$$P \oplus Q := \mathrm{conv}(P' \cup Q').$$

This construction is a coproduct because each element of $P \oplus Q$ is uniquely expressible in the form $s(p, \vec{0}_W, 0) + (1 - s)(\vec{0}_V, q, 1)$ for some $s \in [0, 1]$.

An *initial* object is an object 0 such that for any object A there is a unique arrow from 0 to A. An initial object can be thought of as a coproduct of no objects. Initial objects for **POLY**, **SET**, **CONE**, and **VEC** are $\emptyset$, $\emptyset$, $\{\vec{0}\}$, and $\{\vec{0}\}$, respectively.

3.2.2 Coequalizers

A coequalizer of two arrows $f, g \colon A \to B$ is an object E supplied with an arrow $e \colon B \to E$ such that $e \circ f = e \circ g$ and for any $x \colon B \to X$ with $x \circ f = x \circ g$ there exists a unique arrow $l\colon E \to X$ such that $l \circ e = x$.

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] \arrow[dr, "x"'] & E \arrow[d, dotted, "\exists! l"] \\
& & X
\end{tikzcd}
$$

In **SET** a coequalizer of two arrows $f, g\colon A \to B$ is given by the smallest equivalence relation containing $\{(f(a), g(a)) \in B^2 \mid a \in A\}$. I.e., we let D be the set of equivalence classes of this equivalence relation and we let $d\colon B \to D$ be the mapping that assigns each element of B to its equivalence class. In the category of vector spaces a coequalizer of two linear maps $f, g\colon V \to W$ is obtained in a similar way. In this case $W/\mathrm{im}(f-g)$ supplied with the representative mapping is the coequalizer.

To obtain a coequalizer of two arrows in **CONE**, we make use of the functor V. Let $f, g\colon C \to D$ be linear maps between cones. Then let $v\colon D_{\mathbf{R}} \to X$ be a coequalizer of $\mathrm{V}(f), \mathrm{V}(g)\colon C_{\mathbf{R}} \to D_{\mathbf{R}}$. Now I claim that $v|_D\colon D \to v(D)$ is a coequalizer of f and g. First, $v(D)$ is certainly a cone. Second, since $v \circ f = v \circ g$, we have $v|_D \circ f = v|_D \circ g$. Third, consider an arbitrary arrow $h\colon D \to H$. There certainly exists a map $u\colon X \to H_{\mathbf{R}}$ such that $u \circ v = \mathrm{V}(h)$. I claim that $u(v(D)) \subseteq H$, so that $u|_{v(D)}$ is a map from $v(D)$ to H such that $u|_{v(D)} \circ v|_D = h$. Well, let $v(d) \in v(D)$. Then $u(v(d)) = \mathrm{V}(h)(d) = h(d) \in H$. Further, this map u is unique in its commuting property since $v|_D$ is surjective. I've now shown that $v|_D\colon D \to v(D)$ is the desired coequalizer.

$$
\begin{tikzcd}
C_{\mathbf{R}} \arrow[r, shift left, "\mathrm{V}(f)"] \arrow[r, shift right, "\mathrm{V}(g)"'] & D_{\mathbf{R}} \arrow[r, "v"] \arrow[dr, "\mathrm{V}(h)"'] & X \arrow[d, dotted, "\exists! u"] \\
& & H_{\mathbf{R}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D \arrow[r, "v|_D"] \arrow[dr, "h"'] & v(D) \arrow[d, dotted, "u|_{v(D)}"] \\
& & H
\end{tikzcd}
$$

The same process using C obtains a coequalizer of any two affine maps $f, g\colon P \to Q$ between polytopes. Let $v\colon \mathrm{C}(Q) \to E$ be a coequalizer of $\mathrm{C}(f)$ and $\mathrm{C}(g)$. I claim that $v|_Q\colon Q \to v(Q)$ is a coequalizer of f and g (we identify, of course, Q and $\{(q,1) \in \mathrm{C}(Q) \mid q \in Q\}$.) So we ask the question: Is $v(Q)$ a polytope? Note first that $v(Q)$ may be identified with a subset of the vector space $E_{\mathbf{R}}$ and hence with some $\mathbf{R}^l$. Now, is $v(Q)$ convex? Well, since v is linear it is affine, so it preserves all affine combinations, including convex combinations. It follows that $v(Q)$ is convex. Is it finitely generated? Take vertices q_i for Q. I claim that $v(q_i, 1)$ generates $v(Q)$. This follows easily since v is affine. The rest of the properties of the coequalizer can be verified just as they were for cones.

Example. We present an example of a coequalizer in **POLY**. Let $P := [0,1]$, let $Q := \Delta_3$, and let $E := \operatorname{conv}\{(0,0),(1,0),(0,1),(1,1)\}$. Thus, P is a line segment, Q is a tetrahedron, and E is a square. Now define the following affine maps:

$$\begin{array}{rcrcl} f & : & P & \to & Q \\ & & 0 & \mapsto & (\frac{1}{2},\frac{1}{2},0,0) \\ & & 1 & \mapsto & (0,0,\frac{1}{2},\frac{1}{2}) \end{array}$$

$$\begin{array}{rcrcl} g & : & P & \to & Q \\ & & 0 & \mapsto & (0,0,\frac{1}{2},\frac{1}{2}) \\ & & 1 & \mapsto & (\frac{1}{2},\frac{1}{2},0,0) \end{array}$$

$$\begin{array}{rcccc} e & : & Q & \to & E \\ & & (1,0,0,0) & \mapsto & (0,0) \\ & & (0,1,0,0) & \mapsto & (1,1) \\ & & (0,0,1,0) & \mapsto & (1,0) \\ & & (0,0,0,1) & \mapsto & (0,1) \end{array}$$

Results from section 4.3 can be used to verify that e is indeed the coequalizer of f and g.

3.2.3 Cocontinuity

A functor $F\colon \mathbf{C} \to \mathbf{D}$ is said to be *finitely cocontinuous* if it preserves all finite coproducts (this means initial objects and binary coproducts) and all coequalizers. The functors C and V are both cocontinuous. Let us now see that C is cocontinuous. First note that trivially $\mathrm{C}(\emptyset) = \{\vec{0}\}$, so C takes initial objects to initial objects. Next, let $P \oplus Q$ be the join of two polytopes. We need to show that $\mathrm{C}(P \oplus Q) \cong \mathrm{C}(P) \times \mathrm{C}(Q)$. First I note that

$$\mathrm{C}(P) \times \mathrm{C}(Q) = \mathbf{R}_{\geq 0}\{(\alpha p, \alpha, (1-\alpha)q, 1-\alpha) \mid \alpha \in [0,1],\, p \in P, \text{ and } q \in Q\}$$

Then we may define the mapping $\phi\colon \mathrm{C}(P) \times \mathrm{C}(Q) \to \mathrm{C}(P \oplus Q)$ by stating that

$$\phi(r(\alpha p, \alpha, (1-\alpha)q, 1-\alpha)) = r(\alpha(p, \vec{0}_W, 0) + (1-\alpha)(\vec{0}_V, q, 1), 1)$$

One can easily verify that this map is well-defined, bijective, and linear. Thus, C takes binary coproducts to binary coproducts. Finally, consider a coequalizer $h\colon Q \to S$ of a pair of maps $f\colon P \to Q$ and $g\colon P \to Q$. We want to show that $\mathrm{C}(S)$ is a coequalizer of $\mathrm{C}(f)$ and $\mathrm{C}(g)$. Well, let $e\colon \mathrm{C}(Q) \to E$ be a coequalizer of $\mathrm{C}(f)$ and $\mathrm{C}(g)$. It suffices to show that $E \cong \mathrm{C}(S)$. From section 3.5 we know that $e\mid_Q\colon Q \to e(Q)$ is a coequalizer of f and g. Thus, $e(Q) \cong S$. Further, we have $\mathrm{C}(e(Q)) \cong E$ since $r(e(q),1) \mapsto re(q)$ is a linear bijection. Finally, since functors respect isomorphisms, we get $E \cong \mathrm{C}(S)$.

Now we show that V is cocontinuous. Clearly, $\{\vec{0}\}_{\mathbf{R}} = \{\vec{0}\}$, so initial objects get sent to initial objects. Further we may define a linear isomorphism from $(C \times D)_{\mathbf{R}}$ to $C_{\mathbf{R}} \times D_{\mathbf{R}}$ by sending $(c,d) \times 1$ to $(c \otimes 1, d \otimes 1)$. Thus, coproducts are sent to coproducts. Now let $e\colon D \to E$ be a coequalizer of $f\colon C \to D$ and $g\colon C \to D$. We need to show that $E_{\mathbf{R}}$ is a coequalizer of $\mathrm{V}(f)$ and $\mathrm{V}(g)$. Let $h\colon D_{\mathbf{R}} \to H$ be a coequalizer of $\mathrm{V}(f)$ and $\mathrm{V}(g)$. It is obvious that $h(D) \cong E$, since they are both coequalizers of f and g. So we just need to show that $h(D)_{\mathbf{R}} \cong H$. Since $h(D)_{\mathbf{R}}$ is the "smallest vector space containing $h(D)$", as discussed in Section 2.4.3, we obtain an injective linear mapping $\zeta\colon h(D)_{\mathbf{R}} \to H$ with $h(d) \otimes 1 \mapsto h(d)$. Since $h(D_{\mathbf{R}}) = \mathrm{span}(h(D))$, and since h is surjective, as any coequalizer in **VEC** is surjective, we obtain that $H = \mathrm{span}(h(D))$. Since the image of ζ is $\mathrm{span}(h(D))$, we know that ζ is surjective. Thus, ζ is an isomorphism, and so $h(D)_{\mathbf{R}} \cong H$. Hence, $\mathrm{V}(E) \cong \mathrm{V}(h(D)) \cong H$, so $\mathrm{V}(E)$ is a coequalizer of $\mathrm{V}(f)$ and $\mathrm{V}(g)$.

3.3 Pullbacks

There is a rigorous definition of what "limits" (and "colimits") are, but for our purposes here it is sufficient just to mention that any finite limit (colimit) may expressed by using combinations of finite products and equalizers (coproducts and coequalizers).[1] Thus, it is natural to just concern ourselves with (co)products and (co)equalizers. However, in Chapter 4, we make use of the pullback limit, and it is convenient to give a thorough treatment now.

A *pullback* of arrows

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is an object P supplied with arrows $p_1\colon P \to A$ and $p_2\colon P \to B$ such that the square

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

commutes, and, further, given any arrows $x_1\colon X \to A$ and $x_2\colon X \to B$ that make this diagram below commute

$$\begin{tikzcd} X \arrow[r, "x_2"] \arrow[d, "x_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

[1] For a discussion of limits, see, e.g., MacLane pp. 62-74.

there is a unique arrow $u\colon X \to P$ that makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[dr, dotted, "\exists! u"] \arrow[drr, bend left, "x_2"] \arrow[ddr, bend right, "x_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Any category that has products and equalizers also has pullbacks. In detail, to form the pullback of arrows

$$\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

in a category with products and equalizers, first let $\pi_1\colon A \times B \to A$ and $\pi_2\colon A \times B \to B$ be the product projections, and then let $e\colon E \to A \times B$ be the equalizer of the arrows $f \circ \pi_1$ and $g \circ \pi_2$. Then E supplied with $\pi_1 \circ e\colon E \to A$ and $\pi_2 \circ e\colon E \to B$ is the desired pullback.

$$\begin{tikzcd}
E \arrow[rr, "\pi_2 \circ e"] \arrow[dr, "e"] \arrow[dd, "\pi_1 \circ e"'] & & B \arrow[dd, "g"] \\
& A \times B \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
A \arrow[rr, "f"'] & & C
\end{tikzcd}$$

This argument shows us that in particular **POLY** has pullbacks. Explicitly, a pullback of affine maps $f\colon P \to S$ and $g\colon Q \to S$ is $\{(p,q) \in P \times Q \mid f(p) = g(q)\}$ supplied with the evident inclusions $(p,q) \mapsto p$ and $(p,q) \mapsto q$. In the case where $f = g$, we call the pullback $K_f := \{(p,p') \in P^2 \mid f(p) = f(p')\}$ the *kernel pair* of f.

Chapter 4

Splitting, Simplices, the Kernel-polytope

4.1 Monos, epis, splitting

Let $\mathbf{C}$ be a category. Let $f\colon A \to B$ be in $\mathbf{C}$. The arrow f is a *monomorphism* (*mono*) if for any object $X \in \mathbf{C}$ and arrows $g_1, g_2\colon X \to A$ we have

$$f \circ g_1 = f \circ g_2 \quad \Longrightarrow \quad g_1 = g_2$$

In **SET** an arrow is mono iff it is injective. In **POLY** the same characterization holds.

An arrow $f\colon A \to B$ in $\mathbf{C}$ is called an *epimorphism* (*epi*) if for any object $X \in \mathbf{C}$ and arrows $g_1, g_2\colon B \to X$ we have

$$g_1 \circ f = g_2 \circ f \quad \Longrightarrow \quad g_1 = g_2$$

We note that epi is the dual of mono in the sense that an arrow f is a mono in $\mathbf{C}$ iff f is an epi in $\mathbf{C}^{\mathrm{op}}$. In **SET** an arrow is epi iff it is surjective. In **POLY**, on the other hand, the same characterization does not hold. Only the weaker condition that the affine span of the image of f should cover its codomain is required. In symbols,

$$f\colon A \to B \text{ in } \mathbf{POLY} \text{ is epi} \quad \Longleftrightarrow \quad \operatorname{affspan}(\operatorname{im}(f)) \supseteq B$$

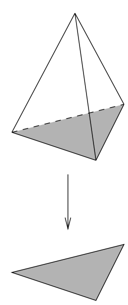
A mono $f\colon A \to B$ is called *regular* if it is an equalizer of some arrows, i.e., there exists arrows $g, h\colon B \to X$ such that f is an equalizer of g and h. An epi is *regular* if it is a coequalizer of some arrows.

A mono $f\colon A \to B$ in $\mathbf{C}$ is said to *split* if there exists an arrow $g\colon B \to A$ in $\mathbf{C}$ such that $g \circ f = 1_A$. Similarly, an epi $f\colon A \to B$ is said to *split* if there exists an arrow $g\colon B \to A$ such that $f \circ g = 1_B$. In either case, g is called a *splitting*. One may note that for two objects $A, B \in \mathbf{C}$, there exists a split mono from A to B iff there exists a split epi from B to A. This follows from the fact that each statement is the dual of the other. In **SET** every mono and every epi splits. However, this is not the case in **POLY**; we now proceed to present a characterization of when monos and epis split in **POLY**.

First note that the property of whether a mono splits is preserved by isomorphisms. By this we mean that if $s\colon \tilde{A} \to A$ and $t\colon B \to \tilde{B}$ are isomorphisms, then any arrow $f\colon A \to B$ in **POLY** is a mono that splits iff $\tilde{f}\colon \tilde{A} \to \tilde{B}$ is a mono that splits, where $\tilde{f}$ is defined by $\tilde{f}(x) = (t \circ f \circ s)(x)$ for all $x \in \tilde{A}$.

Let $f\colon A \to B$ be mono. We are interested in whether f splits. Since f is injective we have that $A \cong \mathrm{im}(f)$, so, by the foregoing observation, we might as well assume that $A \subseteq B$ and that f is the inclusion of A into B. Furthermore, we may assume that $\vec{0} \in A$ (and hence that f is linear) since translations are isomorphisms.

In Proposition 13 below we present a characterization of when such an inclusion splits. Intuitively, it states that there is a splitting iff there is some way in which we may flatten B to get A. For example, the inclusion of a triangle as a face of a triangular prism splits, because we may squash the prism down onto the triangle:



Proposition 13. *Let A and B be polytopes. Assume $\vec{0} \in A$, $A \subseteq B$. Let $f\colon A \to B$ be the inclusion of A into B. Then f splits iff there exists a linear subspace K such that $B \subseteq K + A$ and $\mathrm{span}(B) = K \oplus \mathrm{span}(A)$, where a splitting is given by the restriction to B of the projection span$(A) \oplus K \to \mathrm{span}(A)$ (here "$\oplus$" means direct sum).*

PROOF. Suppose f splits. Let $g\colon B \to A$ be an affine mapping such that $g \circ f = \mathrm{id}_A$ (g is in fact linear). Let $\tilde{g}\colon \mathrm{span}(B) \to \mathrm{span}(A)$ be the unique linear extension of g to $\mathrm{span}(B)$. Then let $K = \ker(\tilde{g})$. To the end of showing that $\mathrm{span}(B) = K \oplus \mathrm{span}(A)$, let $\sum_i \lambda_i b_i \in \mathrm{span}(B)$. Since for any $b \in B$, $g^2(b) = g(b)$, we have $b - g(b) \in \ker(\tilde{g})$. Thus,

$$\begin{aligned} \sum_i \lambda_i b_i &= \sum_i \lambda_i[(b_i - g(b_i)) + g(b_i)] \\ &= \sum_i \lambda_i(b_i - g(b_i)) \ + \ \sum_i \lambda_i g(b_i) \\ &\in K + \mathrm{span}(A) \end{aligned}$$

Now I must show that this is the unique decomposition of an element of $\mathrm{span}(B)$ into its K- and $\mathrm{span}(A)$-parts. Let $\sum_i \lambda_i b_i = k + \sum_i \kappa_i a_i$ for some $k \in K$ and some

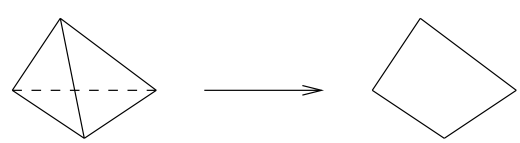
$\sum_i \kappa_i a_i \in \text{span}(A)$. Then

$$\begin{aligned}
\tilde{g}(\sum_i \lambda_i b_i) = \sum_i \lambda_i g(b_i) &= \tilde{g}(k + \sum_i \kappa_i a_i) \\
&= \sum_i \kappa_i g(a_i) \\
&= \sum_i \kappa_i a_i
\end{aligned}$$

Thus, the span(A)-part must be $\sum_i \lambda_i g(b_i)$. Immediately it follows that the K-part must be $\sum_i \lambda_i (b_i - g(b_i))$. To show that $B \subseteq K + A$, let $b \in B$. Since $b - g(b) \in ker(\tilde{g})$ and $g(b) \in A$, we have $b = (b - g(b)) + g(b) \in K + A$.

Now suppose that there exists a linear subspace K with the properties described in the statement of the proposition. Then for each element $b \in B$ there is a unique element in A, call it $g(b)$, for which there exists an element $k \in K$ such that $b = k + g(b)$. In fact, this mapping $g \colon B \to A$ is the desired left inverse of f. First, this mapping g is clearly affine. Second, for any $a \in A$, $(g \circ f)(a) = g(a) = a$. $\square$

Now we turn to when epis split in **POLY**. It is a necessary condition that an epi be surjective in order to split, but this is not a sufficient condition, as demonstrated by the following diagram. It depicts the projection of a triangular prism onto a plane, so that the image is a quadrilateral. To say that this projection splits would be to say that there exists a plane on which all the vertices of the prism lie. This is indeed not the case.



Proposition 14. *Let P and Q be polytopes. A surjective epi $f \colon P \to Q$ splits iff there is a selection A of one element from each pre-image $f^{-1}(v)$ where $v \in \text{vert}(Q)$ such that* $\dim(\text{conv}(A)) = \dim(Q)$.

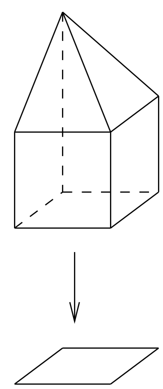
PROOF. Suppose that f splits. Then let $g \colon Q \to P$ be an affine map, with $f \circ g = \text{Id}_Q$. Define the selection A by choosing $g(v) \in f^{-1}(v)$ for each vertex $v \in Q$. Since Id_Q is injective, g must be injective, so we have $\text{im}(g) \cong Q$. Clearly,

$$\text{im}(g) = \text{conv}(\{g(v) \mid v \in \text{vert}(Q)\}) = \text{conv}(A),$$

so in particular we may conclude that the dimensions of $Q \cong \text{im}(g)$ and $\text{conv}(A)$ are the same.

Suppose that we have a selection A with the required property. Let $h \colon \text{conv}(A) \to Q$ be the restriction of f to $\text{conv}(A)$. Clearly h is surjective, since h maps onto all the vertices of Q. If we can show that h is also injective, then it will follow that h's inverse is a splitting for f. Let $a \in A$, and define linear subspaces $L_{\mathrm{conv}(A)} :=$ affspan(conv(A)) $- a$ and $L_Q :=$ affspan(Q) $- h(a)$. Then let $\tilde{h} \colon L_{\mathrm{conv}(A)} \to L_Q$ be the unique linear extension of h. It follows that $\tilde{h}$ is surjective, and that h is injective iff $\tilde{h}$ is injective. Since $\dim(\mathrm{conv}(A)) = \dim(Q)$, we know, by the definition of the dimension of a polytope, that $\dim(L_{\mathrm{conv}(A)}) = \dim(L_Q)$. Thus, the linear map $\tilde{h}$ must be injective (by the rank-nullity theorem). Therefore, h is injective. □

One would like to be able to say something about this selection, but there seems to be no easy selection that always works. For instance selecting the barycenters of the pre-images does not work in the following case:



Although this epi — the projection straight downwards — does split, the "barycenter selection" does not give us a splitting.

4.2 Simplices

Recall that the n-simplex, Δ_n, is given by:

$$\Delta_n := \mathrm{conv}(\{e_1, \ldots, e_{n+1}\}) \subseteq \mathbf{R}^{n+1}$$

An object P in a category $\mathbf{C}$ is called a *projective* if for all objects E, X and for every arrow $f \colon P \to X$ and epi $e \colon E \to X$ there exists an arrow $g \colon P \to E$ such that $e \circ g = f$, as suggested by the diagram:

$$\begin{tikzcd} & E \arrow[d, "e"] \\ P \arrow[ur, dotted, "\exists g"] \arrow[r, "f"'] & X \end{tikzcd}$$

Let P be a non-empty polytope. Then P is not a projective in **POLY**. To see this, note that we can ensure that $\mathrm{im}(f)$ is not a subset of $\mathrm{im}(e)$.

Regular projectives are defined in the same way as projectives except that we replace "epi" by "regular epi". Thus, an object P is called a *regular projective* if for all objects E, X and for every arrow $f: P \to X$ and regular epi $e: E \to X$ there exists an arrow $g: C \to E$ such that $e \circ g = f$. In fact, **POLY** has a number of regular projectives, since, as we prove below, the regular projectives are exactly the simplices (up to isomorphism).

Lemma 15. *The regular epis in* **POLY** *are exactly the surjections.*

PROOF. Let $f: P \to Q$ be an affine surjection of polytopes. Clearly f is epi. We show that f is the coequalizer of its kernel pair, $K_f = \{(p, p') \in P^2 \mid f(p) = f(p')\}$, supplied with the evident projections $\pi_1: (p, p') \mapsto p$ and $\pi_2: (p, p') \mapsto p'$. Clearly we have $f \circ \pi_1 = f \circ \pi_2$. Let $x: P \to X$ be an affine map with $x \circ \pi_1 = x \circ \pi_2$. Then if $f(p) = f(p')$, certainly $x(p) = x(p')$. Thus we may define a mapping $u: Q \to X$ by $u(f(p)) : = x(p)$. This map is easily shown to be affine since f and x are affine. Thus, f is a coequalizer.

Now suppose that $f: P \to Q$ is an epi that is also the coequalizer of some arrows g and h. Then the mapping $\tilde{f}: P \to \mathrm{im}(f)$ obtained by restricting the codomain of f has the property $\tilde{f} \circ g = \tilde{f} \circ h$, so since f is a coequalizer, there is a (unique) map $u: Q \to \mathrm{im}(f)$ with $u \circ f = \tilde{f}$. Let x be a point in Q. We wish to show that $x \in \mathrm{im}(f)$. Since f is epi, we may express x as an affine combination of points in $\mathrm{im}(f)$: let $x = \sum \alpha_i f(p_i)$ for some $p_i \in P$ and some $\alpha_i \in \mathbf{R}$ with $\sum \alpha_i = 1$. Since $u \circ f = \tilde{f}$, we have $u(x) = u(\sum \alpha_i f(p_i)) = \sum \alpha_i u(f(p_i)) = \sum \alpha_i \tilde{f}(p_i) = \sum \alpha_i f(p_i)$. I.e., $u(x) = x$, so $x \in \mathrm{im}(f)$. $\square$

Proposition 16. *Any regular projective in* **POLY** *is isomorphic to some simplex, and every simplex is a regular projective.*

PROOF. First we shall show that Δ_n is a regular projective. Let $f: \Delta_n \to X$ be affine, and let $e: E \to X$ be a regular epi. By the lemma we know that e is surjective. Thus, each of $e^{-1}(f(e_i))$ is non-empty, so from each pick a point v_i. Then we may define an affine map $g: \Delta_n \to E$ by $g(e_i) := v_i$. Clearly we have $e \circ g = f$.

Now we show that any regular projective is isomorphic to some Δ_n. Let P be a regular projective with $n + 1$ vertices $v_1, \ldots, v_{n+1}$. Let $e: \Delta_n \to P$ be the affine surjection that takes e_i to v_i. Then by the lemma, e is a coequalizer, so let $g: P \to \Delta_n$ be an affine map that commutes in

$$\begin{tikzcd} & \Delta_n \arrow[d, "e"] \\ P \arrow[ur, dotted, "g"] \arrow[r, "\mathrm{Id}_P"'] & P \end{tikzcd}$$

We obviously have $e \circ g = \mathrm{Id}_P$; moreover, since $e^{-1}(v_i) = \{e_i\}$ for each i, we have that $g(e(e_i)) = g(v_i) = e_i$, so $g \circ e = \mathrm{Id}_{\Delta_n}$. Thus, $P \cong \Delta_n$. $\square$

Let P be a polytope with t vertices $v_1, \ldots, v_t$. Let us agree that $\Delta(P)$ denotes the simplex that has the same number of vertices as P. We may then obtain an affine surjection $e: \Delta(P) \to P$ that sends e_i to v_i for each i (e is determined up to the ordering of the vertices of P). Now, let us in general denote the kernel pair

$K_e = \{(x, y) \in \Delta(P)^2 \mid e(x) = e(y)\}$ by $K(P)$, and let us call it the *kernel-polytope*. Let $f\colon K(P) \to \Delta(P)$ and $g\colon K(P) \to \Delta(P)$ be the evident projections. Since e is surjective, e is in fact the coequalizer of f and g (cf. the argument of Lemma 11).

$$\begin{tikzcd} K(P) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \Delta(P) \arrow[r, "e"] & P \end{tikzcd}$$

We will call this sequence the *standard presentation of* P. It is determined up to the ordering of the vertices of P. A reordering gives the same sequence up to a linear change of coordinates.

Let P and Q be polytopes and let

$$\begin{tikzcd} K(Q) \arrow[r, shift left, "f_Q"] \arrow[r, shift right, "g_Q"'] & \Delta(Q) \arrow[r, "e_Q"] & Q \end{tikzcd}$$

and

$$\begin{tikzcd} K(P) \arrow[r, shift left, "f_P"] \arrow[r, shift right, "g_P"'] & \Delta(P) \arrow[r, "e_P"] & P \end{tikzcd}$$

be their respective standard presentations. There is a close relationship between the arrows $a\colon P \to Q$ and the arrows $v\colon \Delta(P) \to \Delta(Q)$ that "respect the kernel-polytopes". We will say an arrow $v\colon \Delta(P) \to \Delta(Q)$ *respects the kernel-polytopes* if there exists an arrow $w\colon K(P) \to K(Q)$ such that the following diagram commutes:

$$\begin{tikzcd} \Delta(Q) & K(Q) \arrow[l, "g_Q"'] \arrow[r, "f_Q"] & \Delta(Q) \\ \Delta(P) \arrow[u, "v"] & K(P) \arrow[l, "g_P"'] \arrow[r, "f_P"] \arrow[u, "w"'] & \Delta(P) \arrow[u, "v"'] \end{tikzcd}$$

First note that given any arrow $v\colon \Delta(P) \to \Delta(Q)$ that respects the kernel-polytopes, we may obtain a unique arrow $a\colon P \to Q$ that commutes in this diagram:

$$\begin{tikzcd} \Delta(Q) \arrow[r, "e_Q"] & Q \\ \Delta(P) \arrow[u, "v"'] \arrow[r, "e_P"] & P \arrow[u, dotted, "a"'] \end{tikzcd}$$

This property follows just from the fact that the standard presentation is a coequalizer diagram. Indeed, any "coequalizer presentations" $A \rightrightarrows B \longrightarrow P$ and $A' \rightrightarrows B' \longrightarrow Q$ of P and Q will share this property that any arrow between B and B' that respects A and A' will yield a unique arrow a from P to Q. The special property that the standard presentation has is that, conversely, any arrow $a\colon P \to Q$ gives rise to some arrow between the presentations. I.e., for any arrow $a\colon P \to Q$, there is some arrow $v\colon \Delta(P) \to \Delta(Q)$ that respects the kernel-polytopes, and that commutes in the following diagram:

$$\begin{tikzcd} \Delta(Q) \arrow[r, "e_Q"] & Q \\ \Delta(P) \arrow[u, dotted, "v"'] \arrow[r, "e_P"] & P \arrow[u, "a"'] \end{tikzcd}$$

To see this, note that since $\Delta(P)$ is a regular projective, and since e_Q is a regular epi, there is some arrow $v\colon \Delta(P) \to \Delta(Q)$ that commutes in the above diagram. Since $e_P \circ f_P = e_P \circ g_P$, it follows that $e_Q \circ v \circ f_P = e_Q \circ v \circ g_P$. Thus, since $K(Q)$ is a pullback, we have a (unique) arrow $w\colon K(P) \to K(Q)$ that commutes in the following diagram:

$$\begin{tikzcd}
K(P) \arrow[dr, dotted, "\exists! w"] \arrow[drr, bend left, "v\circ f_P"] \arrow[ddr, bend right, "v\circ g_P"'] & & \\
& K(Q) \arrow[r, "f_Q"] \arrow[d, "g_Q"'] & \Delta(Q) \arrow[d, "e_Q"] \\
& \Delta(Q) \arrow[r, "e_Q"'] & Q
\end{tikzcd}$$

This w ensures that v respects the kernel-polytopes.

In section 4.3 we study the kernel-polytope in more detail, but first we conclude this section by noting a special property of simplices involving the tensor product and the join.

Proposition 17. *For any polytope P, $P \otimes \Delta_n \cong P^{\oplus(n+1)}$.*

PROOF. I will prove that there is a bijection between the hom-sets $\hom(P \otimes \Delta_n, X)$ and $\hom(P^{\oplus(n+1)}, X)$ that is natural in X. The Yoneda Lemma from category theory then implies that $P \otimes \Delta_n \cong P^{\oplus(n+1)}$ in **POLY**. The argument runs as follows:

$$\begin{array}{rcl}
\hom(P \otimes \Delta_n, X) & \cong & \hom(P, [\Delta_n, X]) \\
& \cong & \hom(P, X^{n+1}) \\
& \cong & \hom(P^{\oplus(n+1)}, X)
\end{array}$$

where each $\cong$ represents a set of bijections that is natural in X. The first bijection is given simply by the fact that the tensor product is the left adjoint of the internal hom (cf. section 2.6). The second natural bijection can be verified using the bijection $\varphi\colon \hom(P, [\Delta_n, X]) \to \hom(P, X^{n+1})$ defined by

$$f\colon P \to [\Delta_n, X] \quad \mapsto \quad p \mapsto (f(p)(e_1), \ldots, f(p)(e_{n+1}))$$

The third natural bijection can in fact easily be shown to hold in any category that has products and coproducts. □

Remark. In particular, this proposition shows that $\Delta_n \cong (\mathbf{R}^0)^{\oplus(n+1)}$. Actually, if one showed this statement independently, one would obtain an alternative proof of the proposition. A basic fact from category theory is that right adjoints preserve limits, and left adjoints preserve colimits (see MacLane pp. 114-115). Thus, since $P \otimes \bullet$ is a left adjoint (to the internal hom functor), it preserves coproducts. The alternative proof would then be:

$$\begin{array}{rcl}
P \otimes \Delta_n & \cong & P \otimes (\mathbf{R}^0)^{\oplus(n+1)} \\
& \cong & (P \otimes \mathbf{R}^0)^{\oplus(n+1)} \\
& \cong & P^{\oplus(n+1)}
\end{array}$$

4.3 The kernel-polytope

Let $P \subseteq \mathbf{R}^n$ be a polytope with t vertices $v_1, \ldots, v_t$. Recall from section 4.2 that the $K(P)$ is the kernel-pair of the mapping $e\colon \Delta(P) \to P$ that takes e_i to v_i. In Theorem 18 we present the dimension, facets, and vertices of the kernel-polytope $K(P)$ in terms of the dimension and vertices of P. It turns out that there is a close connection between the structure of $K(P)$ and the affine dependencies among the vertices of P.

An *affine dependency* among the vertices of P is a vector $a = (a_1, \ldots, a_t) \in \mathbf{R}^t$ such that

$$\sum_{i=1}^{t} a_i v_i = 0 \quad \text{and} \quad \sum_{i=1}^{t} a_i = 0.$$

For any vector $w \in \mathbf{R}^n$, let $\tilde{w} := (w_1, \ldots, w_n, 1)$. Then the affine dependencies on the vertices of P are elements of the kernel of the matrix whose columns are $\tilde{v}_1, \ldots, \tilde{v}_t$.

For any vector x in a real coordinate space, the *support* of x, $\mathrm{supp}(x)$, is defined as the non-zero coordinates of x. I.e., if $x = (1, 1, -1, -1, 0)$, then $\mathrm{supp}(x) = \{1, 2, 3, 4\}$. A *minimal affine dependency* on the vertices of P is a non-zero affine dependency a with minimal support, meaning that there is no affine dependency b on the vertices of P with $\mathrm{supp}(b) \subsetneq \mathrm{supp}(a)$. A *normalized affine dependency* is a non-zero affine dependency a with $\sum_{i:a_i>0} a_i = 1$. Every affine dependency is given by scaling some normalized affine dependency. Moreover, one can show that every affine dependency is a sum of minimal dependencies, and hence every normalized affine dependency is given by a convex combination of normalized minimal affine dependencies. If you specify a support, there are in fact exactly two normalized minimal affine dependencies (m and $-m$) with that given support, hence there is a finite number of such dependencies.

Given any nonzero affine dependency a, let $c = \sum_{i:a_i>0} a_i$ and define vectors a^+ and a^- with respective coordinates

$$a_i^+ = \begin{cases} \frac{a_i}{c} & \text{if } a_i \geq 0 \\ 0 & \text{if } a_i < 0 \end{cases} \quad \text{and} \quad a_i^- = \begin{cases} -\frac{a_i}{c} & \text{if } a_i \leq 0 \\ 0 & \text{if } a_i > 0. \end{cases}$$

It follows that $\sum_i a_i^+ e_i$ and $\sum_i a_i^- e_i$ are elements of $\Delta(P)$ and

$$e(\textstyle\sum_i a_i^+ e_i) = \sum_i a_i^+ v_i = \sum_i a_i^- v_i = e(\sum_i a_i^- e_i),$$

the common value being a point $p \in P$. We have written p as a convex combination of vertices of P in two different ways. The supports of a^+ and a^- are disjoint, and so we have found disjoint sets of vertices of P such that p is in the convex hull of each. Furthermore, we see that (a^+, a^-) is an element of $K(P)$. We will call (a^+, a^-) the point of $K(P)$ associated with the affine dependency a. However, not all points in $K(P)$ arise in this way. Given a point $(x, y) \in K(P)$, we have $\sum_i x_i v_i = \sum_i y_i v_i$ and $\sum_i x_i = \sum_i y_i = 1$, so we are representing a point $p := \sum_i x_i v_i \in P$ as a convex combination of vertices of P in two (possibly) different ways. But the supports of x and y need not be disjoint, so x and y may be reusing vertices. Regardless of whether the supports of x and y are in fact disjoint, we may obtain an affine dependency $x - y$ on the vertices of P. But (x, y) is the point in $K(P)$ associated with the affine dependency $x - y$ iff x and y have disjoint supports.

Theorem 18. *The kernel-polytope $K(P)$ has dimension $2(t-1)-d$ where t is the number of vertices of P and $d = \dim P$. There are $2t$ facets for $K(P)$, each defined by setting a coordinate equal to zero. For $i = 1, \ldots, t$, the point (e_i, e_i) is a vertex of $K(P)$. The remaining vertices are (m^+, m^-), where m is a normalized minimal affine dependency on the vertices of P.*

PROOF. Let V be the matrix whose columns are the vertices of P, and let $\tilde{V}$ be the same matrix with a final row of 1s appended. Then $d + 1 = \mathrm{rk}(\tilde{V})$. Let $\mathbf{1}$ be the vector of 1s in R^t, and define the matrix

$$M := \begin{pmatrix} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & -1 \\ V & -V & 0 \end{pmatrix}$$

It follows that

$$K(P) = \{(x, y) \in (\mathbf{R}^t)^2 \mid M \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \text{ and } x_i \geq 0, y_i \geq 0 \text{ for } i = 1, \ldots, t\}.$$

Consider the elements of the kernel of M with last coordinate equal to 1. Since $K(P)$ is nonempty, these elements form an affine space of dimension one less than the dimension of the kernel of M. Further, this affine space meets the set $\{(x, y, 1) \in \mathrm{R}^{2t+1} \mid x_i \geq 0, y_i \geq 0 \text{ for } i = 1, \ldots, t\}$ in its interior, for instance at $(\frac{1}{t}\mathbf{1}, \frac{1}{t}\mathbf{1}, 1)$. Therefore,

$$\begin{aligned} \dim K(P) + 1 &= \dim\ker \begin{pmatrix} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & -1 \\ V & -V & 0 \end{pmatrix} \\ &= 2t + 1 - \dim\mathrm{rk} \begin{pmatrix} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & -1 \\ V & -V & 0 \end{pmatrix} \\ &= 2t + 1 - \dim\mathrm{rk} \begin{pmatrix} 0 & 1 \\ \tilde{V} & 0 \end{pmatrix} = 2t + 1 - (d + 2). \end{aligned}$$

Hence, $\dim K(P) = 2(t-1) - d$, as claimed.

The inequalities defining $K(P)$ are $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, \ldots, t$ (additionally there are a number of equalities helping to define $K(P)$, but these cannot be facets). These inequalities are affinely independent, hence each defines a facet of $K(P)$.

Now to describe the vertices of $K(P)$. Let (x, y) be an arbitrary point in $\Delta(P)^2$. Thus, $x = \sum_i x_i e_i$ and $y = \sum_i y_i e_i$ with $x_i, y_i \geq 0$ for $i = 1, \ldots, t$ and $\sum_i x_i = \sum_i y_i = 1$. For (x, y) to lie in $K(P)$ requires that $\sum_i x_i v_i = \sum_i y_i v_i$.

For each i, the dot product of the vector $(e_i, e_i) \in \mathbf{R}^{2t}$ with $(x, y) \in K(P)$ is $x_i + y_i$. The maximum value of the dot product as (x, y) ranges over the points of

$K(P)$ is 2, occurring precisely at $(x,y)=(e_i,e_i)$. Thus, (e_i,e_i) is a vertex of $K(P)$. Note that each of these vertices lies on all but two of the facets.

Since all the points $(x,y)\in K(P)$ with $x=y$ are contained in $\mathrm{conv}\{(e_i,e_i)\}$, any other vertices (x,y) must have $x\neq y$. Let (x,y) be any element of $K(P)$ with $x\neq y$ and suppose that x and y do not have disjoint supports. Let $x_j,y_j>0$. Then $(1-t)(x,y)+t(e_j,e_j)\in K(P)$ for $|t|$ sufficiently small (the requirement is that $|t|\leq\min\{x_j,y_j\}$). Thus, (x,y) is contained in the interior of a line segment contained in $K(P)$ and hence is not a vertex. So if (x,y) is a vertex with $x\neq y$, (x,y) is the point in $K(P)$ associated with the affine dependency $x-y$. Suppose, to get a contradiction, that $x-y$ is not minimal. Then let m be a minimal affine dependency with $\mathrm{supp}(m)\subsetneq\mathrm{supp}(x-y)$. Then (m^+,m^-) is a point in $K(P)$ that lies on all the facets that (x,y) lies on and then some, contradicting the fact that (x,y) is a vertex.

Now we show that each normalized minimal affine dependency m does indeed give rise to a vertex (m^+,m^-) by giving a face-defining direction c. Let

$$c \;=\; \sum_{i\in\mathrm{supp}(m^+)}(e_i,0)\;+\;\sum_{j\in\mathrm{supp}(m^-)}(0,e_j).$$

The dot product of c with an arbitrary $(x,y)\in K(P)$ is

$$c\cdot(x,y)\;=\;\sum_{i\in\mathrm{supp}(m^+)}x_i\;+\;\sum_{j\in\mathrm{supp}(m^-)}y_j\leq 2$$

with equality iff $\mathrm{supp}(x)=\mathrm{supp}(m^+)$ and $\mathrm{supp}(y)=\mathrm{supp}(m^-)$. Thus, equality implies (x,y) is the point in $K(P)$ associated with $x-y$. Also, since m is minimal, any affine dependency with the same support as m must be $\pm m$. Thus, equality means $x-y=m$; so $(x,y)=(m^+,m^-)$. □

Remark 1. In the theorem we proved that the vertices of $K(P)$ are given by (e_i,e_i) for each $1\leq i\leq t$, and (m_j^+,m_j^-) for each normalized minimal affine dependency m_j. Thus, for any non-zero affine dependency a there is some convex combination of these vertices yielding (a^+,a^-). Since all of the vertices consist only of positive coordinates, and the supports of a^+ and a^- are disjoint, the vertices (e_i,e_i) cannot contribute to this convex combination. Thus, we can express (a^+,a^-) as a convex combination $\sum_j\alpha_j(m_j^+,m_j^-)$ where each m_j is a normalized minimal affine dependency with $\mathrm{supp}(m_j)\subseteq\mathrm{supp}(a)$. Let $\tilde{K}:=\mathrm{conv}\{(m_j^+,m_j^-)\}\subseteq K(P)$. Although $K(P)$ is the kernel-pair, $\tilde{K}$ also has some noteworthy properties. As we have just seen, $\tilde{K}$ in a sense contains all of the non-zero affine dependencies among the vertices of P. Also, if we restrict the maps f_P and g_P to $\tilde{K}$, e_P remains a coequalizer:

$$\tilde{K}\underset{g_P}{\overset{f_P}{\rightrightarrows}}\Delta(P)\xrightarrow{e_P}P$$

Thirdly, there is a close connection between affine dependencies among the vertices of $\tilde{K}$ and linear dependencies among the affine dependencies m_i. Notice that the vertices of $\tilde{K}$ come in pairs $(m_i^+, m_i^-), (m_i^-, m_i^+)$ which correspond to pairs $m_i, -m_i$ of normalized minimal affine dependencies among the vertices of P. Given any affine dependency $\sum \alpha_i(m_i^+, m_i^-) + \beta_i(m_i^-, m_i^+) = \vec{0}$ among the vertices of $\tilde{K}$, we obtain a linear dependence $\sum(\alpha_i - \beta_i)m_i = \vec{0}$ on (half of) the normalized minimal affine dependencies m_i of P. Conversely, given a linear dependence $\sum \gamma_i m_i$, we obtain an affine dependence $\sum \gamma_i(m_i^+, m_i^-) + (-\gamma_i)(m_i^-, m_i^+) = \vec{0}$.

Remark 2. From the standard presentation of P, we obtain $K(P)$. One may then, in turn, find the standard presentation of $K(P)$, and obtain $K(K(P))$. One may continue this process and obtain a sequence of simplices

$$(\Delta(P), \Delta(K(P)), \Delta(K(K(P))), \ldots)$$

associated with P. Alternatively, one could use $\tilde{K}$ to obtain a sequence:

$$(\Delta(P), \Delta(\tilde{K}(P)), \Delta(\tilde{K}(\tilde{K}(P))), \ldots).$$

These sequences should be seen as attempts to mimic the free resolutions of modules in commutative algebra. Let us write $\Delta(n)$ for the simplex with n vertices. The $\tilde{K}$-sequence for the square is $(\Delta(4), \Delta(2))$, because the square has four vertices, and there are two normalized minimal affine dependencies among the vertices of the square (m and $-m$). The sequence terminates where it does because $\tilde{K}(P)$ is a simplex. The $\tilde{K}$-sequence for the symmetric octahedron is $(\Delta(6), \Delta(6), \Delta(2))$.

Chapter 5

The Polar in Categorical Terms

In this chapter we show that taking the polar of a polytope cannot be made into a functor in a natural way. Also, we show that the polar is not a dual in the categorical sense. Nor, indeed, does the "dual" of a cone – a generalization of the polar of a polytope – yield a dual in the categorical sense.

5.1 Duality in a monoidal category

A symmetric monoidal category $(\mathbf{C}, \otimes, I)$ is called *rigid* if for every object V there is an object V^* called the *dual* of V supplied with two arrows $e_V \colon V^* \otimes V \to I$ and $i_V \colon I \to V \otimes V^*$ such that the following two diagrams commute:

$$\begin{tikzcd} V \arrow[r, "i_V \otimes \mathrm{Id}_V"] \arrow[rr, bend right, "\mathrm{Id}_V"'] & V \otimes V^* \otimes V \arrow[r, "\mathrm{Id}_V \otimes e_V"] & V \end{tikzcd}$$

$$\begin{tikzcd} V^* \arrow[r, "\mathrm{Id}_{V^*} \otimes i_V"] \arrow[rr, bend right, "\mathrm{Id}_{V^*}"'] & V^* \otimes V \otimes V^* \arrow[r, "e_V \otimes \mathrm{Id}_{V^*}"] & V^* \end{tikzcd}$$

(Note that in these pictures we have identified V with $V \otimes \mathbf{R}$ and $\mathbf{R} \otimes V$ by the canonical isomorphisms. We have also implicitly used the associative and commutative isomorphisms.)

Example. The symmetric monoidal category $(\mathbf{VEC}, \otimes, \mathbf{R})$ is rigid. Let $V^* := \hom(V, \mathbf{R})$. For any $V \in \mathbf{VEC}$ we may fix a basis $\alpha_V = \{v_1, \ldots, v_n\}$. Then we may obtain the dual basis $\beta_{V^*} = \{v_1^*, \ldots, v_n^*\} = \{v_j \mapsto \delta_{ij} \mid 1 \leq i \leq n\}$ for V^*. Now define $e_V \colon V^* \otimes V \to \mathbf{R}$ by $v_i^* \otimes v_j \mapsto \delta_{ij}$. And define $i_V \colon \mathbf{R} \to V \otimes V^*$ by $1 \mapsto \sum_{i=1}^{n} v_i \otimes v_i^*$. First, we need to show that $(\mathrm{Id}_V \otimes e_V) \circ (i_V \otimes \mathrm{Id}_V) = \mathrm{Id}_V$. Let

$1 \leq j \leq n$, and consider the point $1 \otimes v_j \in \mathbf{R} \otimes V$. Then

$$\begin{aligned}
[(\mathrm{Id}_V \otimes e_V) \circ (i_V \otimes \mathrm{Id}_V)](1 \otimes v_j) &= \mathrm{Id}_V \otimes e_V[(\sum_{i=1}^{n} v_i \otimes v_i^*) \otimes v_j] \\
&= \mathrm{Id}_V \otimes e_V[v_1 \otimes v_1^* \otimes v_j + \cdots + v_n \otimes v_n^* \otimes v_j] \\
&= v_1 \otimes e_V(v_1^* \otimes v_j) + \cdots + v_n \otimes e_V(v_n^* \otimes v_j) \\
&= v_j \otimes 1 \ (= 1 \otimes v_j)
\end{aligned}$$

Since $\{v_i \otimes 1 \mid 1 \leq i \leq n\}$ is a basis for $V \otimes \mathbf{R}$, we have shown that $(\mathrm{Id}_V \otimes e_V) \circ (i_V \otimes \mathrm{Id}_V) = \mathrm{Id}_V$, as desired. In a similar way we can show that $(e_V \otimes \mathrm{Id}_{V^*}) \circ (\mathrm{Id}_{V^*} \otimes i_V) = \mathrm{Id}_{V^*}$.

Note that the choice of basis is arbitrary, but this does not matter here. The basis α_{V^*} we choose for V^* in finding V^{**} may very well be different from the basis β_{V*} that comes from α_V.

Proposition 19. *In a rigid category $(\mathbf{C}, \otimes, I)$ there is an internal hom, and it is given by $[B, \bullet] := \bullet \otimes B^*$.*

PROOF. See Bakalov p. 31. □

5.2 The polar lacks certain categorical properties

The *polar* of a polytope $P \subseteq \mathbf{R}^n$ is given by

$$P^\Delta := \{v \in \mathbf{R}^n \mid v \cdot p \geq -1 \ \forall p \in P\}$$

P^Δ is not always a polytope. It is always a finite intersection of closed halfspaces, but it is not necessarily bounded. If 0 is in the interior of P, then P^Δ is a polytope.[1]

There is a closely related notion called the *dual* of a cone (though not, as we shall see, a dual in the categorical sense). The dual of a cone C is defined as $C^* := \hom(C, \mathbf{R}_{\geq 0})$. In fact, if C is a spanning subset of a finite-dimensional inner product space V, then

$$C^* \cong \{v \in V \mid \langle v, c \rangle \geq 0 \ \forall c \in C\}$$

This isomorphism holds primarily because every linear mapping $f \colon C \to \mathbf{R}_{\geq 0}$ is actually a mapping of the form $\langle v, \bullet \rangle$ for a unique $v \in V$.

Proposition 20. *For any polytope $P \subseteq \mathbf{R}^n$*

$$P^\Delta = \{v \in \mathbf{R}^n \mid (v, 1) \cdot c \geq 0 \ \forall c \in \mathrm{C}(P)\}.$$

[1] For further discussion of the polar, see Ziegler ch. 3.

PROOF. For $v \in \mathbf{R}^n$ we have

$$\begin{aligned} v \in P^\Delta &\iff v \cdot p \geq -1,\ \forall p \in P \\ &\iff v \cdot rp \geq -r,\ \forall p \in P \text{ and } \forall r > 0 \\ &\iff (v,1)\cdot(rp,r) \geq 0,\ \forall p \in P \text{ and } \forall r > 0 \\ &\iff (v,1)\cdot c \geq 0,\ \forall c \in \mathrm{C}(P)\ \square \end{aligned}$$

Remark. If P is full-dimensional, then $\mathrm{C}(P)$ is full-dimensional in $\mathbf{R}^{n+1}$, and so $\mathrm{C}(P)^* \cong \{v \in \mathbf{R}^{n+1} \mid v \cdot c \geq 0,\ \forall c \in \mathrm{C}(P)\}$. Keeping this identification in mind, we see that the proposition says that $P^\Delta = \{v \in \mathbf{R}^n \mid (v,1) \in \mathrm{C}(P)^*\}$. Thus, in a sense P^Δ is a "slice" of $\mathrm{C}(P)^*$.

Even though P^Δ is not always a polytope, it is still conceivable that we would be able to make $\bullet^\Delta$ into a functor, because we may use translation to our advantage. For instance, we could define $\bullet^\Delta$ on objects by stating that $\bullet^\Delta(P) := (P - b_P)^\Delta + b_P$, where b_P is the barycenter of P. (The set $(P - b_P)^\Delta$ is always a polytope since 0 is in the relative interior of $P - b_P$.) Nevertheless, a problem arises when we try to define what $\bullet^\Delta$ should do to arrows. We would like a polar functor to be "compatible" with the dual cone functor, $\hom(\bullet, \mathbf{R}_{\geq 0})\colon \mathbf{CONE}^{\mathrm{op}} \to \mathbf{CONE}$. By this we mean:

- The functor $\bullet^\Delta$ should be contravariant, i.e., $\bullet^\Delta\colon \mathbf{POLY}^{\mathrm{op}} \to \mathbf{POLY}$, and
- If $f\colon P \to Q$ is an affine map between two full-dimensional polytopes, then the following diagram commutes:

$$\begin{tikzcd} \mathrm{C}(Q)^* \arrow[r, "\mathrm{C}(f)^*"] & C(P)^* \\ Q^\Delta \arrow[u, hook] \arrow[r, "f^\Delta"'] & P^\Delta \arrow[u, hook] \end{tikzcd}$$

  where the inclusion $Q^\Delta \hookrightarrow \mathrm{C}(Q)^*$ is given by $v \mapsto ((v,1)\cdot \bullet)$ and similarly for $P^\Delta \hookrightarrow \mathrm{C}(P)^*$.

But there is no such extension of $\bullet^\Delta$ to arrows with these properties, as can be seen by the following argument by contradiction: Let $P, Q := [-1,1] \subseteq \mathbf{R}$ and define an affine map $f\colon P \to Q$ by $f(-1) = 0$ and $f(1) = 1$. We can easily calculate that $P^\Delta = Q^\Delta = [-1,1]$. By the second requirement for compatibility, for any $v \in Q^\Delta$ and for any $p \in P$, we have $(f^\Delta(v),1)\cdot(p,1) = (v,1)\cdot \mathrm{C}(f)(p,1)$. Hence, $f^\Delta(v)\cdot p = v \cdot f(p)$ for all $v \in Q^\Delta$ and $p \in P$. However, $f^\Delta(1)\cdot 0 = 0 \neq 1/2 = 1 \cdot f(0)$. This is a contradiction, so there can be no polar functor compatible with the dual cone functor.

On the other hand, if we restrict **POLY** so that all the objects are full-dimensional and centered at the origin, and all the arrows are linear, then we can obtain a polar functor compatible with the dual cone functor. In this case, for any polytope P, there is a natural identification of the points $v \in P^\Delta$ with the linear maps $(v \cdot \bullet) = L\colon P \to \mathbf{R}$ for which $L(p) \geq -1$ for all $p \in P$. In other words, to specify a point in P^Δ, we need only specify a linear map $L\colon P \to \mathbf{R}$ with $L(p) \geq -1$ for all $p \in P$. Thus, for a linear map $f\colon P \to Q$, we may define $f^\Delta\colon Q^\Delta \to P^\Delta$ by stating that an element $v \in Q^\Delta$ is sent to the linear map $(v \cdot f(\bullet))$. That this gives a functor compatible with the dual cone functor can be easily verified.

The polar does not give a dual in the categorical sense because $Q \otimes P^\Delta$ is not isomorphic to $\hom(P, Q)$. As mentioned in Chapter 2, adjoints are unique up to isomorphism, so any internal homs should be isomorphic. If $\bullet^\Delta$ provided a dual for every polytope, then by Proposition 19 we would know that $\bullet \otimes P^\Delta$ provided an internal hom. Thus, for all polytopes P and Q, we would have $Q \otimes P^\Delta \cong \hom(P, Q)$ where $\hom(P, Q)$ is the internal hom we already defined. However, a simple dimension argument shows that this cannot hold. Let $P = Q = [-1, 1] \subseteq \mathbf{R}$. Then both P and Q have dimension 1, so by Proposition 9, $\hom(P, Q)$ has dimension $(1 + 1)1 = 2$. On the other hand, using Propostion 17 and the fact that $P^\Delta = P \cong \Delta_1$, we get $Q \otimes P^\Delta \cong Q^{\oplus(1+1)} = Q \oplus Q$. Since $Q \oplus Q = [-1, 1] \oplus [-1, 1] = \mathrm{conv}(\{(-1, 0, 0), (1, 0, 0), (0, -1, 1), (0, 1, 1)\})$, it follows that $Q \oplus Q$ has dimension 3. Hence, $\hom(P, Q)$ and $Q \otimes P^\Delta$ do not share the same dimension, so they cannot be isomorphic.

The dual of a cone also does not yield a dual in the categorical sense. Let D be the cone over a square, i.e., $D = \mathrm{C}(P)$ for some square $P \subseteq \mathbf{R}^2$. Then $D \cong D^*$, so both are minimally generated by four elements. Thus, $D \otimes D^*$ is minimally generated by at most sixteen elements. One can easily obtain an inequality description of an embedding of $\hom(D, D)$ in $\mathbf{R}^9$. The program "Polymake" can then calculate a minimal collection of generators for $\hom(D, D)$. The size of this collection turns out to be twenty-four, so $D \otimes D^* \ncong \hom(D, D)$.

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