Kahzdan-Lusztig Polynomials on Matroids

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Abstract

This thesis investigates the log-concavity of Kazhdan-Lusztig (KL) polynomials in the context of matroids, exploring the connection to combinatorial Hodge theory. We first review the theory of KL-polynomials and their recursive definition in relation to characteristic polynomials. Our primary focus is on a specific class of graphic matroids. We compute the KL-polynomials for matroids in this class, specifically for this class of graphs, and conjecture a simplified form for the coefficients of these polynomials. The main result of our study is a formula that characterizes the KL-polynomial for a certain class of graphs based on deletion-contraction, with a correction term. Our work contributes to the ongoing investigation of the log-concavity conjecture for KL-polynomials of matroids and offers suggestions for future research directions.

Dedication

To my loving parents, Monroe and Lena, and grandparents, Monroe, Robert, Barbara, and Shirley.

Introduction

The theory of Kazhdan-Lusztig polynomials for matroids was introduced in 2014 by Elias, Proudfoot, and Wakefield Elias et al. (2016). These polynomials were directly inspired by the Kazhdan-Lusztig polynomials found in representation theory Kazhdan & Lusztig (1979).

Kazhdan-Lusztig polynomials are a central object of study in algebraic combinatorics and representation theory. They were introduced in the 1970s by David Kazhdan and George Lusztig as a tool to understand the representation theory of semisimple Lie algebras and their associated algebraic groups Kazhdan & Lusztig (1979).

The main idea behind Kazhdan-Lusztig polynomials is to study the cohomology of the flag variety associated to a semisimple Lie algebra. This cohomology has a rich structure, known as the Kazhdan-Lusztig basis, which is indexed by pairs of elements in the Weyl group. Kazhdan and Lusztig discovered that certain combinatorial data associated to this basis—namely, the structure constants of the multiplication in cohomology—could be expressed in terms of certain polynomials, now known as Kazhdan-Lusztig polynomials.

Kazhdan-Lusztig polynomials are defined recursively in terms of certain other polynomials, known as the Kazhdan-Lusztig R-polynomials. These polynomials are intimately related to the combinatorial structure of the Weyl group, and in particular to the Bruhat ordering on its elements. The recursive definition of Kazhdan-Lusztig polynomials involves a combinatorial algorithm.

The Kazhdan-Lusztig (KL) polynomials also have a natural interpretation in the context of matroids, which are combinatorial objects that generalize the notion of linear independence in linear algebra. In this setting, the KL polynomials encode important information about the topology and geometry of the matroid.

Given a matroid M, one can associate to it a cell complex known as the matroid flag variety. The cohomology of this cell complex has a natural basis indexed by the flags of flats in M. The KL polynomials of M are then defined as the structure constants of the multiplication in cohomology with respect to this basis.

The KL polynomials of matroids also satisfy a recursive definition in terms of the characteristic polynomial, which are defined in a manner similar to that in the Lie algebra setting. The recursive definition involves certain operations on matroids known as contraction and deletion, which are used to generate smaller matroids from a given one.

The theory of KL-polynomials lies in the larger context of combinatorial Hodge theory. Combinatorial Hodge theory is a theory that relates combinatorial objects, such as simplicial complexes, matroids, and hyperplane arrangements, to the classical theory in algebraic topology known as Hodge theory. The combinatorial theory was developed by June Huh in his work on the Rota conjecture and has since been used to prove a number of important results in algebraic combinatorics Huh (2012).

In classical algebraic topology, Hodge theory is concerned with the study of the cohomology of smooth complex projective varieties. It provides a powerful tool for understanding the geometry and topology of these varieties by decomposing their cohomology groups into a direct sum of subspaces with special geometric properties, known as the Hodge decomposition Huybrechts (2005).

Combinatorial Hodge theory transfers this classical theory to the setting of combinatorial objects. Instead of smooth complex projective varieties, one considers simplicial complexes, matroids, and hyperplane arrangements, which are combinatorial objects that capture certain aspects of the geometry of algebraic varieties. The goal is to define a Hodge decomposition for the cohomology of these objects, which can be used to study their geometry and topology Adiprasito et al. (2018).

The basic idea of combinatorial Hodge theory is to define a combinatorial analogue of the Laplacian operator, which is a key ingredient in the classical Hodge theory. This operator is defined in terms of certain combinatorial data associated to the combinatorial object, such as the face lattice of a simplicial complex or the incidence matrix of a hyperplane arrangement. The combinatorial Laplacian is then used to define a Hodge decomposition for the cohomology of the object, which decomposes the cohomology groups into a direct sum of subspaces with combinatorial properties analogous to the classical Hodge decomposition.

Combinatorial Hodge theory has important applications in algebraic combinatorics, such as the proof of log-concavity of coefficients of characteristic polynomials, and has also found connections to other fields such as algebraic geometry and representation theory.

Our goal in this thesis is to explore the log-concavity of the Kazhdan-Lusztig polynomials, as this is directly inspired from the log-concavity of characteristic polynomial Elias et al. (2016).

The proof that characteristic polynomials are log-concave by Huh, Adiprasito, and Katz is a groundbreaking result in algebraic combinatorics and has important applications in various fields such as algebraic geometry, representation theory, and convex geometry Adiprasito et al. (2018).

The main result of their paper is the log-concavity of the coefficients of characteristic polynomials of certain classes of combinatorial objects, including matroids, simplicial complexes, and hyperplane arrangements. More precisely, the authors show that the logarithm of the coefficients of the characteristic polynomial form a concave function, which is a fundamental property in convex geometry and optimization Adiprasito et al. (2018).

The main idea of their proof is to express the coefficients of the characteristic polynomial as the evaluation of a certain multivariate polynomial, called the *Ehrhart polynomial.* The combinatorial Hodge theory provides a way to express this polynomial as a sum of *Hodge integrals*, which are certain combinatorial integrals over a polytope. By analyzing the properties of these integrals, the authors are able to prove the log-concavity of the coefficients of the characteristic polynomial.

The proof has important applications in algebraic geometry, such as the study of positivity properties of line bundles and the existence of canonical metrics on Fano varieties (Adiprasito (2018), Adiprasito et al. (2019)). It also has connections to representation theory, where it has been used to prove log-concavity results for characters of certain representations of Lie groups Proudfoot et al. (2016). In this thesis we explore a particular case, a certain cycle with a double cord called $B_{n,2,2}$. We proceed first by investigating the theory of KL-polynomials in Chapter 2, then look towards a formula that was found in a previous paper that characterizes the formula for the KL-polynomial for graphic matroids based on deletion-contraction, Theorem 25. While with characteristic polynomials we have the deletion-contraction formula,

$$\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k),$$

for the KL-polynomials there exists a similar formula but with a correction term.

In this thesis we consider a certain class of of graphic matroids, $B_{n,m,\ell}$, Definition 40. In Theorem 42, we compute the KL-polynomials for matroids in this class, specifically $B_{n,2,2}$. We then conjecture a simpler form for the coefficients of these polynomials (Conjecture 43). We conclude with suggestions for further work.

Chapter 1

Matroid Theory

Matroid theory is a powerful and elegant framework for studying the combinatorial structure of abstract objects such as graphs, matrices, and sets. Matroids capture the essential properties of linear independence and dependence in a way that generalizes and unifies many fundamental concepts from linear algebra, graph theory, and combinatorics. In this chapter, we explore the theory of matroids with a focus on graphic matroids, which arise from graphs and are a fundamental example of matroids. We define the key concepts of rank, flats, lattice of flats, ground set, and independent sets in the context of graphic matroids, and provide examples and illustrations throughout. A more general reference for matroid theory would be Oxley (1992).

Definition 1. A *matroid* M is an ordered pair, (E, I) where E is a finite set, called the *groundset*, and a nonempty collection of subsets of E, called the *independent sets* I, where we have the following conditions:

1. $\varnothing \in I$.

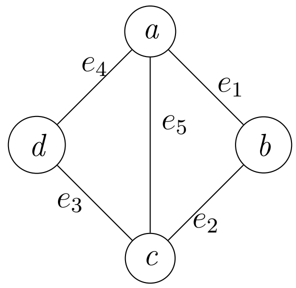
2. If $X \in I$, then every subset of X is in I.

3. If two subsets, X, Y, are in I and $|X| > |Y|$, then there exists $x \in X \setminus Y$ such that $Y \cup \{x\}$ is in I.

In this thesis, matroids are our primary object, and more specifically graphic matroids.

Definition 2. Given an undirected graph, $G = (V, E)$ with vertex set V and edge set E, we define the *graphic matroid*, denoted as $M(G) = (E, \mathcal{I})$, where $\mathcal{I} = \{F \subseteq E \mid F \text{ is acyclic}\}$, or in other words the forests of the graph are the independent sets.

Example 3. *The* diamond graph G *can be represented as the following graph:*



As a graphic matroid, the edges, e_1, e_2, e_3, e_4, e_5*, are the ground set. The matroid is defined by the following maximal independent sets:*

$$
\begin{aligned}
I_1 &= \{e_1, e_2, e_3\}, & I_2 &= \{e_1, e_2, e_4\}, \\
I_3 &= \{e_1, e_3, e_4\}, & I_4 &= \{e_2, e_3, e_4\}, \\
I_5 &= \{e_1, e_4, e_5\}, & I_6 &= \{e_2, e_3, e_5\}, \\
I_7 &= \{e_1, e_3, e_5\}, & I_8 &= \{e_2, e_4, e_5\}.
\end{aligned}
$$

The rest of the independent sets are the subsets of the above maximal sets.

We move our attention to some essential definitions for our work.

Definition 4. The *rank* function of a subset of a matroid $M = (E, I)$ is the function, $\mathrm{rk}\colon 2^E \to \mathbb{Z}_{\geq 0}$, defined as

$$\mathrm{rk}(A) := \max\{|J| \mid J \subseteq A, J \in I\} \text{ for all } A \subseteq E.$$

The rank of the matroid M is then $\mathrm{rk}(M)$. We define the *corank* to be $\mathrm{crk}(A) = \mathrm{rk}(M) - \mathrm{rk}(A)$.

Example 5. *Consider a set of vectors in* $\mathbb{R}^3$, $\{v_1, v_2, v_3, v_4\}$*. We can define a matroid over this set of vectors as follows:*

1. *The independent sets of the matroid are the sets of vectors that are linearly independent.*

2. *The rank function of the matroid is the rank of the matrix whose columns are the vectors in a given set.*

For example, if we take $\{v_1, v_2\}$ *as a subset of our vector set, the matrix formed by these vectors is:*

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}.$$

If this matrix has rank 2 (i.e., its columns are linearly independent), then $\{v_1, v_2\}$ *is an independent set of the matroid, and its rank is 2. Similarly, the rank of the matrix whose columns are formed by* $\{v_1, v_2, v_3, v_4\}$ *determine the rank of the matroid.*

Definition 6. Let $M = (E, I)$ be a matroid, the *closure* for any $A \subseteq E$ is defined as,

$$\mathrm{cl}(A) = \{x \in E \mid \mathrm{rk}(A) = \mathrm{rk}(A \cup \{x\})\}.$$

A subset of $A \subseteq E$ is a *flat* if the set is closed, i.e., $\mathrm{cl}(A) = A$. The set of flats, $\mathcal{L}(M)$, will form a lattice.

Example 7. *Suppose we have a matroid defined over a set of four elements: $\{a, b, c, d\}$, whose independent sets are*

$$\{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}.$$

We will look at $A = \{a, b\}$ and take its closure. Consider that $\mathrm{rk}(A \cup \{c\}) = 2$ since the largest independent subset is A. We can do this once more to find that $\mathrm{cl}(A) = \{a, b, c, d\}$. In other words, the set $\{a, b, c, d\}$ is a flat, and the only other flats are the following: $\{\}, \{a\}, \{b\}, \{c\}, \{d\}$.

Definition 8. A *basis* of a matroid is a maximal independent set. Similarly, a *circuit* is a minimally dependent set.

Definition 9. Let M be a matroid, for any flat $F \in \mathcal{L}(M)$ define $I^F = I \setminus F$ and $I_F = F$. We define the *restriction to F of M*, denoted as M^F, to be the matroid on I^F consisting of subsets of I^F whose union with a basis for F are independent. Now consider the *localization to F*, denoted as M_F, which is the matroid on I_F consisting of subsets of I_F which are independent in M.

Example 10. *Consider the flat $F = \{e_1, e_2, e_5\}$ in the matroid of the diamond graph $M(G)$ from Example 3. From here we will simply refer to $M(G)$ as G.*

Then $I^F = I \setminus F = \{e_3, e_4\}$ and $I_F = F = \{e_1, e_2, e_5\}$. The restriction of G with respect to F, denoted by G^F, is the matroid on $I^F = \{e_3, e_4\}$ consisting of subsets X of I^F such that $X \cup F$ is a independent set in G. Note, that while in this case G^F is graphic, this is not necessarily the case. We can verify that the independent sets of G^F are:

$$\begin{aligned} I_1^F &= \{\}, & I_2^F &= \{e_3\}, \\ I_3^F &= \{e_4\}, & I_4^F &= \{e_3, e_4\}. \end{aligned}$$

The localization of G with respect to F, denoted by G_F, is the graphic matroid on $I_F = \{e_1, e_2, e_5\}$ consisting of subsets X of I_F that form a forest in G. We can verify that the independent sets of G_F are:

$$\begin{aligned} I'_1 &= \{\}, & I'_2 &= \{e_1\}, \\ I'_3 &= \{e_2\}, & I'_4 &= \{e_5\}, \\ I'_5 &= \{e_1, e_2\}, & I'_6 &= \{e_1, e_5\}, \\ I'_7 &= \{e_2, e_5\}. \end{aligned}$$

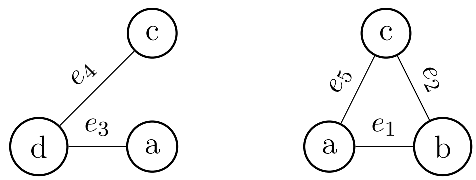


Figure 1.1: Right: Restriction G^F, Left: Localization G_F.

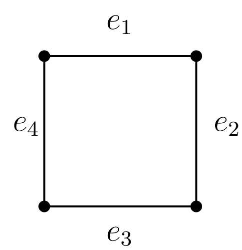


Figure 1.2: Graph on four edges forming a cycle, C_4.

Example 11. *Now we look at the lattice of flats of the graph on four edges forming a cycle, denoted C_4, as seen in Figure 1.2.*

For the matroid $M(C_4)$, there is one flat of rank 3 *(the matroid itself), six of rank* 2 *(all subsets of two edges), four of rank* 1 *(all singleton sets), and one of rank* 0 *(the empty set).*

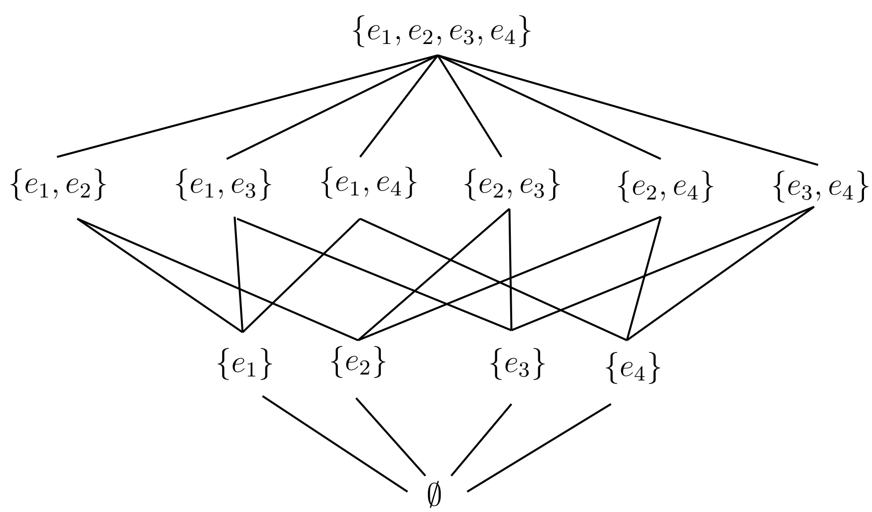


Figure 1.3: Lattice of flats for the cycle graph on four vertices.

Definition 12. Let M be a matroid. The *dual matroid*, denoted as $M^* = (E^*, I^*)$, has the same groundset as M, and has its independent sets defined as $S \in I^*$ if and only if M has a basis set that is disjoint from S.

Definition 13. Let M be a matroid. The *deletion* of an element e from M, denoted as $M \setminus e$ is the localization of M at $E \setminus e$, or $M_{E \setminus e}$. The *contraction* of an element e from M, denoted as M/e is the dual operation, $M/e = (M^* \setminus e)^*$.

Definition 14. Given two matroids, $M_1 = (E_1, I_1)$ and $M_2 = (E_2, I_2)$, we have the direct sum of the two matroids, denoted as $M_1 \oplus M_2$, as the matroid whose groundset is $E_1 \sqcup E_2$ and independent sets are $I_1 \sqcup I_2$.

Definition 15. Let M be a matroid. An element $i \in E$ is a *loop* if the singleton set $\{i\}$ is not independent. An element $i \in E$ is a *coloop* if $E \setminus i$ is a flat.

Chapter 2

Kazhdan-Lusztig Polynomials of Matroids

In this chapter, we delve into the fascinating world of Kazhdan-Lusztig polynomials in the context of matroid theory. Kazhdan-Lusztig polynomials, originally introduced by David Kazhdan and George Lusztig in 1979, have played a significant role in the realm of algebraic combinatorics, representation theory, and geometric group theory Kazhdan & Lusztig (1979). Over the past few decades, these polynomials have been studied extensively, unveiling a rich tapestry of connections among various mathematical disciplines. In this chapter, we embark on an investigation of Kazhdan-Lusztig polynomials of matroids, starting with a brief overview of the fundamental concepts and definitions. We then proceed to examine the crucial results and breakthroughs that have contributed to our current understanding of these objects. Among these milestones, we will highlight key properties, invariants, and conjectures related to Kazhdan-Lusztig polynomials of matroids. Furthermore, we will discuss prominent examples that showcase the elegance and complexity of these polynomials and their interplay with matroid structures.

Definition 16. The *characteristic polynomial of a matroid*, M, is as follows:

$$\chi_M(\lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{\text{corank}(A)}.$$

Example 17. *Consider the graph on four edges forming a cycle as in Example 11. Let $M = M(C_4)$ be the associated graphic matroid. Since the rank of $A \subseteq E$ is the rank of the smallest flat containing A,*

$$\chi_M(\lambda) = \sum_{|A|=0} \lambda^3 - \sum_{|A|=1} \lambda^2 + \sum_{|A|=2} \lambda - \sum_{|A|=3} \lambda^0 + \sum_{|A|=4} \lambda^0 = \lambda^3 - 4\lambda^2 + 6\lambda - 3.$$

Proposition 18 (Oxley (1992))**.** *Let $M = (E, I)$ be a matroid, and $e \in E$. Then the following holds:*

$$\chi_M(\lambda) = \chi_{M \setminus e}(\lambda) - \chi_{M/e}(\lambda),$$

where $M \setminus e$ is deletion and M/e is contraction. The formula is called the deletion-contraction formula.

Example 19. *We will compute the characteristic polynomial of the matroid of the diamond graph, $M(G)$ from Example 3. First note that the characteristic polynomial of the path with k edges is $(\lambda-1)^k$. From there (choosing the middle edge to delete/contract),*

$$\begin{aligned}\chi_{M(G)}(\lambda) &= \chi_{M(G)\backslash e}(\lambda) - \chi_{M(G)/e}(\lambda)\\ &= \chi_{M(C_4)}(\lambda) - \chi_{M(P_2)}(\lambda)\\ &= \lambda^3 - 4\lambda^2 + 6\lambda - 3 - (\lambda-1)^2\\ &= \lambda^3 - 5\lambda^2 + 8\lambda - 4.\end{aligned}$$

Now, with the essential definitions, we can define Kazhdan-Lusztig Polynomials.

Definition 20 (Kazhdan-Lusztig Polynomial of M, Elias et al. (2016))**.** For all matroids M, there exists a polynomial $P_M(t) \in \mathbb{Z}[t]$ such that,

1. If $\operatorname{rk}(M) = 0$, then $P_M(t) = 1$.
2. If $\operatorname{rk}(M) > 0$, then $\deg P_M(t) < \frac{1}{2}\operatorname{rk} M$.
3. For all M, $t^{\operatorname{rk} M} P_M(t^{-1}) = \sum_F \chi_{M_F}(t) P_{M^F}(t)$.

Proposition 21 (Elias et al. (2016))**.** *Let M_1, M_2 be matroids, then $P_{M_1 \oplus M_2}(t) = P_{M_1}(t) P_{M_2}(t)$.*

Proof. We do this by induction on the ranks of M_1 and M_2. The formula holds automatically when $\operatorname{rk} M_1 = 0$ or $\operatorname{rk} M_2 = 0$. Now assume that the statement holds for matroids M_1', M_2' if $\operatorname{rk} M_1' \leq \operatorname{rk} M_1$ and $\operatorname{rk} M_2' \leq \operatorname{rk} M_2$ where at least one of the inequalities is strict.

We know that $\mathcal{L}(M_1 \oplus M_2) = \mathcal{L}(M_1) \times \mathcal{L}(M_2)$. We also have the following identities,

$$(M_1 \oplus M_2)_{(F_1,F_2)} \cong (M_1)_{F_1} \oplus (M_2)_{F_1} \qquad (M_1 \oplus M_2)^{(F_1,F_2)} \cong (M_1)^{F_1} \oplus (M_2)^{F_1}.$$

We also have the following,

$$\chi_{(M_1)_{F_1} \oplus (M_2)_{F_2}}(t) = \chi_{(M_1)_{F_1}}(t) \cdot \chi_{(M_2)_{F_2}}(t) \qquad P_{(M_1)^{F_1} \oplus (M_2)^{F_2}}(t) = P_{(M_1)^{F_1}}(t) \cdot P_{(M_2)^{F_2}}(t)$$

by the induction hypothesis if $F_1 \neq \varnothing$ or $F_2 \neq \varnothing$. These observations with the recursive definition of the KL polynomials tells us,

$$t^{\operatorname{rk} M_1 + \operatorname{rk} M_2} P_{M_1 \oplus M_2}(t^{-1}) - t^{\operatorname{rk} M_1} P_{M_1}(t^{-1}) \cdot t^{\operatorname{rk} M_2} P_{M_2}(t^{-1}) = -P_{M_1}(t) P_{M_2}(t) + P_{M_1 \oplus M_2}(t).$$

The left-hand side has degree strictly greater than $\frac{1}{2}\operatorname{rk} M_1 + \frac{1}{2} M_2$ while the right-hand side is concentrated in degree strictly less than $\frac{1}{2}\operatorname{rk} M_1 + \frac{1}{2}\operatorname{rk} M_2$, so both sides must vanish, and we have the desired result. □

Proposition 22 (Elias et al. (2016))**.** *The constant term of $P_M(t)$ is 1.*

Proof. We do this by induction on the rank of M. If rank $M = 0$, then $P_M(t) = 1$ by the definition. Now assume $\mathrm{rk}(M) > 0$, use the recursion

$$t^{\mathrm{rk}(M)} P_M(t^{-1}) = \sum_F \chi_{M_F}(t) P_{M^F}(t).$$

Since we know that $\deg P_M(t) < \mathrm{rk}(M)$, the left-hand side has no constant term, so we have

$$\sum_F \chi_{M_F}(0) P_{M^F}(0) = 0$$

By the inductive hypothesis, we know $P_{M^F}(0) = 1$ for all nonempty flats F, so then we need to show that

$$\sum_F \chi_{M_F}(0) = 0.$$

This follows from the fact that $\chi_{M_F} = \mu(\varnothing, F)$ and $\mathrm{rk}\, M > 0$ where μ is the Möbius function by definition Oxley (1992). □

Proposition 23 (Proudfoot et al. (2016)). *For a cycle graph on n vertices, the KL-polynomial is the following:*

$$P_{C_n}(t) = \sum_i c_{n,i} t^i$$

where

$$c_{n,i} = \frac{1}{i+1} \binom{n-i-2}{i} \binom{n}{i}.$$

Example 24. *Consider the cycle graph on 4 vertices, then,*

$$c_{4,0} = \frac{1}{1}\binom{4-2}{0}\binom{4}{0} = 1 \quad \textit{and} \quad c_{4,1} = \frac{1}{2}\binom{4-1-2}{1}\binom{4}{1} = 2.$$

So we have that

$$P_{C_4}(t) = 2t + 1.$$

Now, consider the cycle graph on 5 vertices. Then,

$$c_{5,0} = \frac{1}{1}\binom{5-2}{0}\binom{5}{0} = 1 \quad \textit{and} \quad c_{5,1} = \frac{1}{2}\binom{5-1-2}{1}\binom{5}{1} = 5.$$

So we have

$$P_{C_5}(t) = 5t + 1.$$

Now, we may move to the deletion-contraction formula for Kazhdan-Lusztig polynomials.

Theorem 25 (Braden & Vysogorets (2019)). *If $e \in E$ is not a coloop in M, then*

$$P_M(t) = P_{M\setminus e}(t) - tP_{M_e}(t) + \sum_{F \in S} \tau(M_{F\cup e}) t^{\mathrm{crk(F)}/2} P_{M^F}(t)$$

where

$$S := \{F \in \mathcal{L}(M) \mid e \notin F \text{ and } F \cup e \in \mathcal{L}(M)\}$$

and

$$\tau(M) = \begin{cases} \text{coefficient of } t^k \text{ in } P_M(t) & \text{if } \operatorname{rk}(M) = 2k+1 \\ 0 & \text{if } \operatorname{rk}(M) = 2k \end{cases}.$$

Corollary 26 (Parallel Edge Connection, Braden & Vysogorets (2019))**.** *For graphs such that $G = H_1 \cup H_2$, and $H_1 \cap H_2 = e$, a single edge,*

$$P_G(t) = P_{G\setminus e} - tP_{H_1}(t)P_{H_2}(t).$$

Example 27. *We first do an example of the parallel edge connection corollary. Take the matroid of the diamond graph, $M(G)$ from Example 3. Let e be the middle edge, then we can see the top graph is C_3 and the bottom is C_3. We can then compute the KL-polynomial using Corollary 26,*

$$P_G(t) = P_{C_4} - t \cdot P_{C_3}(t)^2 = (2t+1) - t \cdot 1^2 = t + 1.$$

Example 28. *Let M be the graphic matroid in the graph picture in Figure 2.1.*

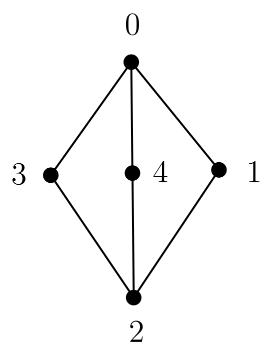


Figure 2.1: Graph with two vertices such that there are three paths of length 2 between them.

Now we may construct S from Theorem 25.

$$S = \{\varnothing, \{\overline{03}\}, \{\overline{01}\}, \{\overline{24}\}, \{\overline{23}\}, \{\overline{12}\}, \{\overline{01}, \overline{03}\}, \{\overline{01}, \overline{23}\}, \{\overline{03}, \overline{12}\}, \{\overline{12}, \overline{23}\}\}.$$

Then, we may note that $P_{M\setminus e} = P_{C_4 \oplus e} = P_{C_4}$ and $P_{M_e} = t+1$ by previous calculations.

Now to find $P_M(t)$ we simply need to calculate $\tau(M_{F\cup e})$ and apply Theorem 25. For all the flats consisting of an individual edge, i.e. $F = \overline{03}$ for example, $\tau(M_{F\cup e}) = 0$ since $M_{F\cup e}$ is of rank 2. If F is one of the two edge sets, then $M_{F\cup e}$ is not 0 since it is rank 3. Performing the contractions, we see $M_{F\cup e} \cong P_2$. In this case, we have a rank 1 matroid so $\tau(M_{F\cup e}) = 1$. Note also $P_{M^F} = 1$ since $M^F \cong P_2 \oplus P_2$. Finally, consider the case $F = \varnothing$. In this case, we see $\tau(M_{F\cup e}) = 1$ since $M_{F\cup e} \cong K_4 \setminus e$. Then we may see the following calculation,

$$P_M(t) = (2t+1) - t(t+1) + 4t + t^2 = 5t + 1,$$

as confirmed by Sage calculations.

Definition 29. A sequence of real numbers, $e_0, \ldots, e_n$, is *unimodal* if

$$e_0 \leq e_1 \leq \ldots \leq e_k \geq e_{k+1} \geq \ldots \geq e_n$$

for some $0 \leq k \leq n$.

Definition 30. A sequence of real numbers, $e_0, \ldots, e_n$, is *log-concave* if for all $0 < i < n$,

$$e_{i-1}e_{i+1} \leq e_i^2$$

and is said to have no *internal zeros* if there do not exist $i < j < k$ such that

$$e_i \neq 0, \quad e_j = 0, \quad e_k \neq 0.$$

Proposition 31. *Let* $\mathbf{e} = (e_0, \ldots, e_n)$ *be a sequence of nonnegative real numbers with no internal zeros. If* $\mathbf{e}$ *is log-concave, then* $\mathbf{e}$ *is unimodal.*

Proof. If at most two values of $\mathbf{e}$ are nonzero, then $e_{i-1}e_{i+1} = 0$ for all $1 \leq i \leq n-1$. Now assume that $\mathbf{e}$ is not unimodal, then there exists some i such that $e_{i-1} > e_i < e_{i+1}$. Then since the elements are non-negative we have that $e_i^2 < e_{i-1}e_{i+1}$. □

In late 2015, Adiprasito, Huh, Katz jointly proved that the coefficients of the characteristic polynomial of a matroid is log-concave with no zeros Adiprasito et al. (2018).

Later, Elias, Proudfoot, Wakefield stated the following conjecture.

Conjecture 32 (Elias et al. (2016))**.** For any matroid M, the coefficients of $P_M(t)$ form a log-concave sequence with no internal zeros.

This is the conjecture we are most interested in. However, they also conjectured the following.

Conjecture 33 (Elias et al. (2016))**.** For any matroid M, the coefficients of $P_M(t)$ are non-negative.

Conjecture 33 was proved in 2020 by Braden, Huh, Matherne, Proudfoot, and Wang (Braden et al. (2020)). They did so by showing a bijjection between the coefficients of the KL-polynomials and dimensions of the graded pieces of a $\mathbb{K}$-algebra. Specifically, they defined $\mathrm{IH}(M) \otimes_{H(M)} \mathbb{Q}$, where the dimension of the graded pieces of this algebra will coincide with the coefficients of the Kazhdan-Lusztig polynomial. For the remainder of the chapter, our goal is to construct $\mathrm{IH}(M)$ to give an outline of the construction for Conjecture 33. In pursuing this, we will refer to tools that are not fully explained in this thesis. Some important references for the material include *Toric Varieties* by Cox et al. (2011), and *Matroid Polytopes, Nested Sets and Bergman Fans* by Feichtner & Sturmfels (2005).

We will begin by looking at the Chow ring of a matroid and the associated theory behind it.

For the remainder of this chapter, let M be a matroid of rank d, with cardinality n for the groundset E. Moreover, let M be a loopless matroid. Now we may define two polynomial rings before defining our Chow ring of M,

$$\begin{aligned}\underline{S}_M &:= \mathbb{Q}[x_F \mid F \text{ is a nonempty proper flat of } M], \text{ and}\\ S_M &:= \mathbb{Q}[x_F \mid F \text{ is a proper flat of } M] \otimes \mathbb{Q}[y_i \mid i \in E].\end{aligned}$$

Now, with these definitions in mind, we may define the Chow ring, and the augmented Chow ring of M.

Definition 34. The *Chow ring* of M is the quotient,

$$\underline{\mathrm{CH}}(M) := \underline{S}_M/(\underline{I}_M + \underline{J}_M),$$

where

$$\underline{I}_M := \left\langle \sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F \mid \text{ for all } i_1, i_2 \in E, i_1 \neq i_2 \right\rangle,$$

and

$$\underline{J}_M := \langle x_{F_1} x_{F_2} \mid \text{ for all incomparable nonempty flats } F_1, F_2 \text{ of } M \rangle .$$

Note that when $d > 0$, then the Chow ring of M coincides with the usual Chow ring of an $(n-1)$-dimensional smooth toric variety defined by the $(d-1)$-dimensional fan $\underline{\Pi}_M$ called the Bergman fan of M (Feichtner & Sturmfels (2005)).

Definition 35. The *augmented Chow ring* of M is the quotient,

$$\mathrm{CH}(M) := S_M/(I_M + J_M),$$

where

$$I_M := \left\langle y_i - \sum_{i \notin F} x_F \mid \text{ for all } i \in E \right\rangle,$$

and

$$J_M := \langle x_{F_1} x_{F_2}, y_i x_F \mid \text{ for all incomparable nonempty flats } F_1, F_2 \text{ of } M, \forall i \in E, i \notin F \subseteq M \rangle .$$

Note that when $d > 0$, the Chow ring of M coincides with the usual Chow ring of an n-dimensional smooth toric variety defined by the d-dimensional fan Π_M called the augmented Bergman fan of M (Feichtner & Sturmfels (2005)).

We can note that,

$$\underline{\mathrm{CH}}(M) \cong \mathrm{CH}(M)/\langle y_i, y_i - y_j \mid i, j \in F, \mathrm{rk}(F) = 1 \rangle.$$

An important fact is that $\mathrm{IH}(M)$ is an $\mathrm{H}(M)$-module, where $\mathrm{H}(M)$ is the graded Möbius algebra defined as

$$\mathrm{H}(M) := \bigoplus_{F \in \mathcal{L}(M)} \mathbb{Q} y_F.$$

Moreover, $\mathrm{CH}(M)$ is in fact a $\mathrm{H}(M)$-module. Specifically, there is a unique graded algebra homomorphism,

$$\mathrm{H}(M) \to \mathrm{CH}(M), \quad y_{\bar{i}} \mapsto y_i$$

where $\bar{i}$ is the unique rank 1 flat of M containing i of E. Hence, $\mathrm{H}(M)$ is the same as the subalgebra of the augmented Chow ring generated by the $y_i s$.

It turns out that using the Bergman fan, we can see that $\underline{\mathrm{CH}}(M)$ vanishes for $\geq d$ and similarly $\mathrm{CH}(M)$ vanishes for $> d$. There is also a Poincaré algebra structure to the Chow ring as there exists isomorphisms from $\underline{\mathrm{CH}}^{d-1}(M)$ and $\mathrm{CH}^d(M)$ to $\mathbb{Q}$. We can do this by constructing a degree map (Adiprasito & Yashfe (2021), Adiprasito et al. (2018)).

Definition 36. For the Chow ring, when $d > 0$, we can define the degree map to be the unique linear map

$$\underline{\deg}_M \colon \underline{\mathrm{CH}}^{d-1}(M) \to \mathbb{Q}, \quad \prod_{F \in \underline{\mathcal{F}}} x_F \mapsto 1$$

where $\underline{\mathcal{F}}$ is any complete flag of nonempty proper flats of M.

Similarly, for the augmented Chow ring, we have the degree map as the unique linear map,

$$\deg_M \colon \mathrm{CH}^d(M) \to \mathbb{Q}, \quad \prod_{F \in \mathcal{F}} x_F \mapsto 1$$

where $\mathcal{F}$ is any complete flag of proper flats of M.

Now we will focus on defining the pullback and pushforward maps for the Chow ring. For this definition, we need to define a few Chow classes, α, $\underline{\alpha}$, $\underline{\beta}$ defined as,

$$\begin{aligned}
\alpha = \alpha_M &:= \sum_{G \in \mathcal{L}(M), G \neq M} x_G \in \mathrm{CH}^1(M) \\
\underline{\alpha} = \underline{\alpha}_M &:= \sum_{i \in G \in \mathcal{L}(M)} x_G \in \underline{\mathrm{CH}}^1(M) \\
\underline{\beta} = \underline{\beta}_M &:= \sum_{i \notin G \in \mathcal{L}(M)} x_G \in \underline{\mathrm{CH}}^1(M).
\end{aligned}$$

The last two are summing over the nonempty proper flats of G. Due to the linear relations of $\underline{\mathrm{CH}}(M)$, $\underline{\alpha}$ and $\underline{\beta}$ do not depend on the choice of i. Note, the map defined above will map α to $\underline{\alpha}$ and $-x_\varnothing$ to $\underline{\beta}$.

Now we may define the pullback and pushforward.

Definition 37. The *pullback* of a non-empty flat F, $\phi^F = \phi^F_M$, is the unique surjective graded algebra homomorphism

$$\mathrm{CH}(M) \to \underline{\mathrm{CH}}(M_F) \otimes \mathrm{CH}(M^F)$$

where M_F is contraction and M^F is restriction. The pullback ϕ^F needs to satisfy the following:

1. If G is a flat properly contained in F, then $\phi^F(x_G) = 1 \otimes x_G$.

2. If G is a flat properly containing in F, then $\phi^F(x_G) = x_{G\setminus F} \otimes 1$.

3. If G is a flat incomparable to F, then $\phi^F(x_G) = 0$.

4. If G is the flat F, then $\phi^F(x_G) = -1 \otimes \alpha_{M^F} - \underline{\beta}_{M_F} \otimes 1$.

The *pushforward* of a proper flat F, $\psi^F = \psi^F_M$, is the unique degree 1 linear map

$$\underline{\mathrm{CH}}(M_F) \otimes \mathrm{CH}(M^F) \to \mathrm{CH}(M)$$

determined by

$$\prod_{F'} x_{F'\setminus F} \otimes \prod_{F''} x_{F''} \mapsto x_F \prod_{F'} x_{F'} \prod_{F''} x_{F''}.$$

Now we may define the intersection homology of M. First, we need to define $\underline{\mathrm{IH}}(M)$ which lies inside $\underline{\mathrm{CH}}(M)$. Let $\underline{\mathrm{H}}(M)$ be the unital subalgebra of $\underline{\mathrm{CH}}(M)$ generated by $\underline{\beta}$.

For all subspaces of $\underline{\mathrm{CH}}(M)$, we let the orthogonal space be defined as

$$V^\perp := \left\{\eta \in \underline{\mathrm{CH}}(M) \mid \underline{\deg}_M(v\eta) = 0 \text{ for all } v \in V\right\}.$$

Now we may construct the subspace $\underline{\mathrm{IH}}(M)$.

Definition 38. Let M be a loopless matroid of positive rank d.

1. For a nonempty proper flat F of M, we define

$$\underline{\mathrm{K}}_F(M) := \underline{\psi}^F(\underline{\mathrm{J}}(M_F) \otimes \underline{\mathrm{CH}}(M^F)).$$

   We know this is a $\underline{\mathrm{H}}(M)$-submodule of $\underline{\mathrm{CH}}(M)$ by Proposition 2.9 in Braden et al. (2020).

2. Define the $\underline{\mathrm{H}}(M)$-submodule $\underline{\mathrm{IH}}(M)$ of $\underline{\mathrm{CH}}(M)$ by

$$\underline{\mathrm{IH}}(M) := \left(\sum_{\varnothing < F < E} \underline{\mathrm{K}}_F(M)\right)^\perp,$$

   noting that we are summing over all nonempty proper flats F of M.

3. We define the graded subspace $\underline{\mathrm{J}}(M)$ of $\underline{\mathrm{CH}}(M)$ by setting

$$\underline{\mathrm{J}}^k(M) := \begin{cases} \underline{IH}^k(M) & \text{if } k \leq (d-2)/2, \\ \underline{\beta}^{2k-d+2}\underline{\mathrm{IH}}^{d-k-2}(M) & \text{if } k \geq (d-2)/2 \end{cases}.$$

We can look at two simple examples. If $\mathrm{rk}(M) = 1$, then

$$\underline{\mathrm{IH}}(M) = \underline{\mathrm{CH}}(M) = \mathbb{Q} \text{ and } \underline{\mathrm{J}}(M) = 0.$$

If $\mathrm{rk}(M) = 2$, then

$$\underline{\mathrm{IH}}(M) = \underline{\mathrm{CH}}(M) = \mathbb{Q} \oplus \mathbb{Q}\underline{\beta} \text{ and } \underline{\mathrm{J}}(M) = \mathbb{Q}.$$

Later in Braden et al. (2020), the authors prove that $\underline{\mathrm{IH}}(M)$ satisfies the Kähler package, and therefore the Hard Lefscetz Theorem, specifically with respect to $\underline{\beta}$. Precisely this means, for all $0 \leq k < d/2$, we have that the following map is an isomorphism,

$$\underline{\mathrm{IH}}^k(M) \to \underline{\mathrm{IH}}^{d-k-1}, \quad \eta \mapsto \underline{\beta}^{d-2k-1}\eta.$$

We need to give the definition of two graded algebras,

$$\mathrm{H}(M) := \langle y_i \mid y_i \in \mathrm{CH}(M), \text{ and } i \in E \rangle \text{, and}$$

$$\mathrm{H}_\circ(M) := \langle y_i, x_\varnothing \mid y_i \in \mathrm{CH}(M), \text{ and } i \in E \rangle .$$

Now we may look at the definition of $\mathrm{IH}(M)$, as this will give us the coefficients of the Kazhdan-Lusztig polynomials (after slight modification). If $E = \varnothing$, then $\mathrm{H}_\circ(M)$ cannot be defined.

Similar to before, for a subspace V of $\mathrm{CH}(M)$ we set

$$V^\perp := \{\eta \in \mathrm{CH}(M) \mid \deg_M(v\eta) = 0 \text{ for all } v \in V\} .$$

Again, if V is an $\mathrm{H}(M)$-submodule or an $\mathrm{H}_\circ(M)$-module, then so is $V^\perp$.

Definition 39. Let M be a loopless matroid.

1. For all proper flats F of M, we let

$$\mathrm{K}_F(M) := \psi^F(\underline{\mathrm{J}}(M_F) \otimes \mathrm{CH}(M^F)).$$

   Braden et al. (2020) shows that this is an $\mathrm{H}(M)$-submodule of $\mathrm{CH}(M)$, and when $F \neq \varnothing$, then it is also $\mathrm{H}_\circ(M)$-submodule of $\mathrm{CH}(M)$.

2. We define the $\mathrm{H}(M)$-submodule of $\mathrm{IH}(M)$ of $\mathrm{CH}(M)$ as

$$\mathrm{IH}(M) := \left(\sum_{F \in \mathcal{L}(M)} \mathrm{K}_F(M) \right)^\perp ,$$

   where the sum is over the proper flats of M.

3. If E is nonempty, we may define the $\mathrm{H}_\circ(M)$-submodule $\mathrm{IH}_\circ(M)$ of $\mathrm{CH}(M)$ by

$$\mathrm{IH}_\circ(M) := \left(\sum_{\varnothing < F \in \mathcal{L}(M)} \right)$$

   where the sum is over all the nonempty proper flats of M.

Our key interest in the intersection homology is the following:

$$P_M(t) = \sum_{k\geq 0} \dim\left(\mathrm{IH}^k(M)_\varnothing\right) t^k \text{ and } Z_M(t) = \sum_{k\geq 0} \dim\left(\mathrm{IH}^k(M)\right) t^k$$

where

$$\mathrm{IH}(M)_\varnothing := \mathrm{IH}(M) \otimes_{\mathrm{H}(M)} \mathbb{Q} \cong \mathrm{IH}(M)/\mathfrak{m}\,\mathrm{IH}(M)$$

where $\mathfrak{m}$ is the graded maximal ideal of $\mathrm{H}(M)$ and $\mathbb{Q}$ is short-hand for the one-dimensional graded $\mathrm{H}(M)$-module in degree 0. The proof for this is essentially the entirety of Braden et al. (2020). In doing so, since the dimension of a vector space is non-negative, they confirmed Theorem 33.

Chapter 3

Kazhdan-Lusztig Polynomials of $B_{n,m,\ell}$

In this chapter, we focus on the explicit computation of the Kazhdan-Lusztig polynomial for a certain class of graphs, characterized by two special vertices connected through paths of length n, 2, and 2. To achieve this, we employ a two-pronged approach. Firstly, we utilize a deletion-contraction-like formula, which offers a systematic methodology for exploring the polynomial's properties and relationships with the graph structure. This formula, inspired by classical graph theory, allows us to break down the problem into smaller, more manageable components. Secondly, we leverage the power of computational tools, specifically the SageMath software, to support our analytical findings and provide a robust framework for performing complex calculations. By harnessing SageMath's capabilities, we can efficiently compute Kazhdan-Lusztig polynomials for our chosen class of graphs, validate our theoretical results, and uncover novel insights.

Definition 40. The graph $B_{n,m,\ell}$ is formed by starting with two vertices, u and v, and then joining them with three paths: one of length n, one of length m, and one of length ℓ (here, *length* denotes the number of edges in a path).

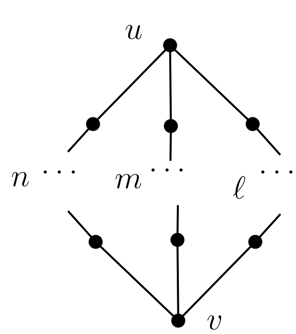


Figure 3.1: Graph $B_{n,m,\ell}$.

Example 41. *Consider the graph $B_{7,2,2}$. The associated KL-polynomial for this graph is*

$$P_{B_{7,2,2}} = 28t^4 + 293t^3 + 255t^2 + 40t + 1.$$

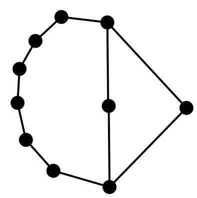


Figure 3.2: Graph $B_{7,2,2}$.

We consider the following graph for the remainder of the thesis, $B_{n-2,2,2}$.

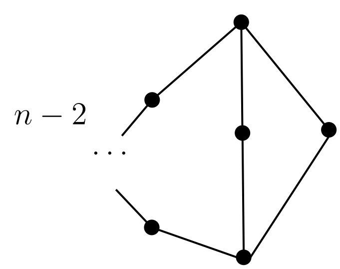


Figure 3.3: Graph with two vertices with three paths between them of length $n-2$, 2, and 2.

Our motivation for investigating this case is to look to address the most general case of computing the KL-polynomial for $B_{n,m,\ell}$. In this case, there are two cycles of length $m+\ell$ and $n+\ell$ connected by a cord of length ℓ. If this case is resolved, then by the cord cases and direct sums, we would be able to compute the KL-polynomial of many planar graphs. Our work in this thesis can be regarded as the first step in computing the KL-polynomials for the class $B_{n,m,\ell}$.

Relying on Theorem 25, we derive the following theorem,

Theorem 42. *Consider the graph $G = B_{n-2,2,2}$ and its associated matroid $M = M(G)$. Then the KL-polynomial is as follows,*

$$\begin{aligned} P_M(t) &= P_{C_n} - t(P_{C_n} - tP_{C_{n-2}}) \\ &\quad + 2 \cdot \sum_{k=1, n-k \ \mathit{even}}^{n-2} \binom{n-2}{k-1} c_{n-k,\ell} \cdot t^{(n-k)/2} + \sum_{k=0, n-k \ \mathit{even}}^{n-4} \binom{n-2}{k} (c_{n-k,\ell} - c_{n-k-2,\ell}) \cdot t^{(n-k)/2}. \end{aligned}$$

Proof. First, we invoke the deletion-contraction for KL-polynomials, Theorem 25. If we choose one of the edges on a path of length 2, then $M \setminus e \cong C_n$. For M/e, we can use the parallel connection theorem since $M/e \cong B_{n-2,1,2}$. Specifically, the two graphs that union to M/e will be C_{n-2} and C_3, so we get that $(M/e) \setminus e' \cong C_n$. Hence, in total the first term is $P_{C_n} - t(P_{C_n} - tP_{C_{n-2}})$. Now we must compute S as defined in Theorem 25.

Suppose a flat F has k edges. For $0 \leq k \leq n-2$ there are two cases. For the first case we can pick the k edges from the $n-2$ path. For the second case we can pick 1 edge from either of the two length 2 paths, and $k-1$ edges from the $n-2$ path (we cannot pick more than 1 from the two 2 paths because that would not be a flat once e is included). Hence, for a general k we have $2 \cdot \binom{n-2}{k-1} + \binom{n-2}{k}$ flats F with cardinality k.

Now we have figured out what S looks like, we need to figure out what the terms in the correction term must be. For any of the flats in S, we know that $P_{M^F} = 1$ since all of the flats are trees.

Now we take a look at $\tau(M_{F\cup e})$. For the case that we choose k edges, then we are simply getting a parallel edge connection. In this case, the polynomial is $P_{n-k} - tP_{n-2-k}$. Both terms here have the same degree, so to find the coefficient of the top term, we simply take the difference of the top two terms, i.e. $\tau(M_{F\cup e}) = c_{n-k,\ell} - c_{n-k-2,\ell}$ where $\ell(n) = \lfloor \frac{n}{2} \rfloor - 1$. For the other case, i.e., once we contract the middle edge and one of the other edges, and then simplify the matroid, we are left with a cycle of length $n-k$. Hence, $\tau(M_{F\cup e} = c_{n-k,\ell})$. Hence, the result follows. $\square$

We can take a look at the coefficients of $P_M(t) = \sum_{k\geq 0} f_n(k)t^k$. Let $k = (n-j)/2$ for some j where k is an integer, then the coefficient for $f_n(k)$ is

$$f_n(k) = c_{n,k} - c_{n,k-1} + c_{n-2,k-2} + 2\cdot \binom{n-2}{n-2k-1} c_{2k,k-1} + \binom{n-2}{n-2k} c_{2k,k-1} - c_{2(k-1),k-2}.$$

After computing these coefficients, we asked if there is a more explicit way to compute them. In other words, is there a simple polynomial that can compute a polynomial for each of the coefficients. Consider the sequence $\{f_n(2)\}_{n\geq 0}$:

$$3, 19, 58, 132, 255, 443, \ldots.$$

The successive differences of this sequence appears in Figure 3.4. Since the differences are all 0 after the fifth line, we suspect that there is a fourth degree polynomial defining this sequence (we are not certain since we are working with data generated by Sage using the formula in Theorem 42). For example, we have that for $n = 3$, we expect our polynomial, $an^4 + bn^3 + cn^2 + dn + e$, to output 3 (since this is the third element of our sequence). Specifically,

$$a(3)^4 + b(3)^3 + c(3)^2 + d(3) + e = 3.$$

By setting $n = 3, 4, 5, 6, 7$, we then have a system of linear equations that we can solve for $a, b, \ldots, e$. In doing so, we get a formula for the coefficient of t^2 in $P_M(t)$,

$$\frac{1}{12}(n^3 + 17n^2 + 60n + 36)(n+1).$$

$$
\begin{array}{ccccccccccccc}
3 & & 19 & & 58 & & 132 & & 255 & & 443 & & \cdots \\
& 16 & & 39 & & 74 & & 123 & & 188 & & \cdots & \\
& & 23 & & 35 & & 49 & & 65 & & \cdots & & \\
& & & 12 & & 14 & & 16 & & \cdots & & & \\
& & & & 2 & & 2 & & \cdots & & & & \\
& & & & & 0 & & \cdots & & & & &
\end{array}
$$

Figure 3.4: Finite Differences of the Sequence $\{f_n(2)\}$.

Repeating the calculation, we find the coefficients of t^n in $P_M(t)$ (where $M = M(B_{n-2,2,2})$) for $n = 2, 3, 4$, and 5:

$$\frac{1}{12}(n^3 + 17n^2 + 60n + 36)(n+1) \tag{3.1}$$

$$\frac{1}{144}(n^4 + 30n^3 + 257n^2 + 792n + 648)(n+2)(n+1) \tag{3.2}$$

$$\frac{1}{2880}(n^4 + 39n^3 + 430n^2 + 1720n + 1920)(n+7)(n+3)(n+2)(n+1) \tag{3.3}$$

$$\frac{1}{86400}(n^4 + 48n^3 + 647n^2 + 3180n + 4500)(n+9)(n+8)(n+4)(n+3)(n+2)(n+1). \tag{3.4}$$

Note, for $n = 0$ and $n = 1$, this pattern does not continue. Specifically, since t^0 will always be 1, and t^1 does not follow a pattern that will form this quartic structure.

Except in the case of the coefficient of t^2, the quartics that appears (though, we can expand the polynomial in the case of $n = 2$ to be a quartic). Further computations indicate that this pattern continues. Could it be possible that the coefficients of these quartics can be written as polynomials? For instance, the coefficients of the n^3 in these quartics are

$$30, 39, 48, \ldots.$$

Here are the coefficients of n^k of the quartic factors appearing in (3.1, 3.2, 3.3, 3.4),

$$
\begin{aligned}
k = 3, &\quad 30, 39, 48, \ldots \\
k = 2, &\quad 257, 430, 647, \ldots \\
k = 1, &\quad 792, 1720, 3180, \ldots \\
k = 0, &\quad 648, 1920, 4500, \ldots.
\end{aligned}
$$

In this case, we get polynomials for the sequences as,

$$\begin{aligned} k = 3, &\quad 3(3k+1) \\ k = 2, &\quad 22k^2 + 19k + 2 \\ k = 1, &\quad 2k(k+1)(10k+3) \\ k = 0, &\quad 6k^3(k+1). \end{aligned}$$

Note, these polynomials for the coefficents only begins to work for $n = 3$.

Looking at the other pieces of these equations (3.1, 3.2, 3.3, 3.4), we have a coefficient out front, and the linear factors. The coefficient out front is, $\frac{1}{n!\cdot(n+1)!}$. The first few linear pieces follow the pattern (using i for n as to not cause confusion),

$$(i+1)(i+2)\dots(i+(n-1)).$$

The last few linear terms seem to follow this pattern,

$$(i+(n+3))(i+(n+4))\dots(i+(2n-1)).$$

From here, we may make the following conjecture:

Conjecture 43. Consider the graph $G = B_{n-2,2,2}$ and its associated matroid $M = M(G)$. Then the KL-polynomial is

$$P_M(t) = \sum_{k\geq 0}^{\lfloor \frac{n}{2} \rfloor} f_n(k) t^k,$$

where

$$\begin{aligned} f_n(i) = &\frac{1}{n!(n+1)!}\left(\prod_{k=1}^{n-1}(i+k)\right)\left(\prod_{j=n+3}^{2n-1}(i+j)\right) \\ &\cdot \big(i^4 + (3(3n+1))i^3 + (22n^2+19n+2)i^2 + 2n(n+1)(10n+3)i + 6n^3(n+1)\big) \end{aligned}$$

for $n > 2$, $i \geq 0$.

It is expected that this formula coincides with the formula found in Theorem 42.

First thoughts on a generalization. From the case of $B_{n-2,2,2}$, it seems natural to want to address $B_{n-2,m-2,2}$. We call this case the double-cord case as seen in Figure 3.5. While this case is difficult, it appears that the $B_{n,m,2}$ generalization should follow

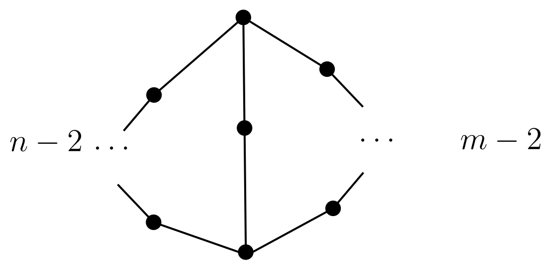


Figure 3.5: Double-Cord Graph.

closely to Theorem 42. One point that needs to be addressed is that the most general case, i.e., $B_{n,m,\ell}$, has a hurdle that is not the case for the double-cord. The first term,

$$P_{M\backslash e} - t(P_{M/e})$$

will require computation of $P_{M/e}$, which is the the same as computing $P_{B_{n,m,\ell}}$ (assuming $\ell \leq m, n$). Hence, we see, this requires a recursion of Theorem 25 ℓ many times to finally reduce to the Parallel Edge Connection case. This recursion requires computing an S set, from Theorem 25, ℓ many times. Therefore, computing $P_{B_{n,m,\ell}}$ does not seem easily feasible using Theorem 25.

Conclusion

In this thesis, we have studied the Kazhdan-Lusztig (KL) polynomials of matroids, with a particular focus on planar graphs with two vertices connected by three paths of length n, 2, and 2. Through the analysis and computation of the KL-polynomial for this specific case, we have laid the foundation for generalizing our findings to $B_{n,m,\ell}$ so that the KL-polynomial of many planar graphs may be computed

The eventual goal is to show log-concavity of matroids in general, but a stepping stone would be that of proving log-concavity of the KL-polynomial of $B_{n,m,\ell}$. Key steps to do this include proving Conjecture 43. To do this, one must show that the formula in Theorem 42 coincides with the formula found in Conjecture 43. After doing this, using Theorem 42 and Conjecture 43, one should prove log-concavity of $B_{n-2,2,2}$.

If the previous points have been resolved, the next step would be to resolve the double-cord case entirely, i.e., compute the KL-polynomial for $B_{n,m,2}$. While challenging, this computation should be doable in a similar manner to $B_{n-2,2,2}$. Fully solving the KL-polynomial for $B_{n,m,\ell}$ would be the end goal, however using Theorem 25 to do so seems extremely difficult. Specifically because one would have to recursively use Theorem 25 ℓ many times since $B_{n,m,\ell}/e \cong B_{n,m,\ell-1}$. Performing this recursion seems like an arduous task, so the author encourages exploring other methods to resolve $B_{n,m,\ell}$.

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