Sandpiles: a Bridge Between Graphs and Toric Ideals

A Thesis
Presented to
The Division of Mathematics and Natural Sciences
Reed College

In Partial Fulfillment
of the Requirements for the Degree
Bachelor of Arts

Jacob G. Perlman

May 2009

Approved for the Division
(Mathematics)

David Perkinson

Acknowledgements

I would like to thank my adviser Dave Perkinson and my mother Felice Perlman both of whom have provided me with more support and education than they can possibly imagine. I would also like to thank Luis David Garcia-Puente for inspiring me to look more deeply at the connection between sandpiles and toric ideals.

Table of Contents

Abstract

We extend known properties of sandpile groups on undirected graphs to directed graphs by extending the concept of a set firing to that of a script firing which allows us to generalize the toppling ideal introduced by Cory, Rossin, and Salvy in [2] to the directed case and calculate a Gröbner basis using properties of the graph. In so doing, we extend our attention to toric ideals generated by laplacians of graphs. We then characterize complete intersection lattice ideals defining finite point sets by the structure of the graphs which have these lattice ideals as toppling ideals.

Introduction

In our first chapter, we present a realization of the Abelian Sandpile Model on directed multigraphs with sink and restate several known results, gathered from the work of Holroyd, Levine, Mészáros, Peres, Propp, and Wilson, in our context [5]. We conclude by embedding the more usual case of the Abelian Sandpile Model on an undirected graph as a special case within our more general setting by demonstrating a well-defined method for introducing a sink to an undirected graph. In our second chapter, we parallel the work of Cory, Rossin, and Salvy in constructing toric ideals relating to sandpiles, followed by new results extending the work in [2] to the directed case, culminating in theorem 2.5.3 which provides a Gröbner basis for the constructed toric ideals using properties from the graph. We also note the highly intriguing theorem 2.6.2, which gives a dual relation between two natural sets of representatives for the toric ideal. In our final chapter, we use tools presented by Morales and Thoma in [6] to give new results about the relationship between graphs and the toric ideals their laplacians generate, resulting in theorems 3.2.1 and 3.2.2, which together characterize complete intersection lattice ideals defining finite point sets as arising from graphs of a particular form, and corollary 3.2.4 which states that an undirected graph gives such a lattice ideal if and only if it is a tree.

Chapter 1

Sandpiles

The sandpile model we will consider, the “Abelian Sandpile Model,” was first introduce by Dhar in [3], as a generalization of the original sandpile model of Bak, Tang, and Wiesenfeld. Though originally proposed as a means of studying self-organized criticality[1], the Abelian Sandpile Model has found various applications in diverse branches of mathematics. Here we present the model as a game played on a graph.

1.1 The Pieces

Definition 1.1.1. When we say a *graph*, we mean a *directed multigraph* (or *multi digraph*), that is a finite set, called the vertex set, generally denoted V, and a weight function, $w : V{\times}V \longrightarrow \mathbb{Z}_{\geq 0}$. We will also refer to the edges of a graph, $E_w = \{(u, v) \in V{\times}V : w(u, v) > 0\}$.

Definition 1.1.2. Associated to each vertex of a graph are two important values, the *out-degree* and the *in-degree*.

$$\begin{aligned} \mathrm{out}\!-\!\mathrm{degree}(v) &= \sum_{u \in V} w(v, u) \\ \mathrm{in}\!-\!\mathrm{degree}(v) &= \sum_{u \in V} w(u, v) \end{aligned}$$

As the out-degree comes up more often in practice, we will abbreviate it d_v.

Definition 1.1.3. A graph, $\Gamma = (V, w)$, is *undirected* if for all (u, v) in $V{\times}V$, $w(u, v) = w(v, u)$.

Definition 1.1.4. On a graph $\Gamma = (V, w)$, a *path* of length n from v to u is a sequence of vertices $v_0, v_1, \ldots, v_n$ in V, such that $v_0 = v$, $v_n = u$ and for i from 1 to n, $(v_{i-1}, v_i) \in E_w$.

Definition 1.1.5. On a graph $\Gamma = (V, w)$, a path from v to v is a *cycle*. A graph that contains no cycles of length greater than 0 is *acyclic*.

[1] A system exhibits self-organized criticality if it naturally moves towards critical states, which are those where a small perturbation may have a large result.

Definition 1.1.6. On a graph $\Gamma = (V, E)$ a vertex s is a *sink* if $d_s = 0$ and for every v in $V \backslash \{s\}$ there is a path from v to s. In general if $\Gamma = (V, w)$ is a graph with a sink s we will use $\tilde{V}$ to denote $V \backslash \{s\}$.[2]

Notation. For any two sets X and Y, let X^Y denote the set of functions from Y to X. That is,

$$X^Y = \{f : Y \longrightarrow X\}$$

In the case that Y is finite then we may order Y, say $Y = \{y_0, \ldots, y_{n-1}\}$. Thus elements, f, of X^Y can be thought of as vectors in X^n by $f_i = f(y_i)$. If, in addition, $X = \mathbb{Z}$, then X^Y is isomorphic to the free abelian group on Y,

$$\mathbb{Z}Y = \{\textstyle\sum_{y \in Y} a_y y : a_y \in \mathbb{Z}\},$$

by:

$$\begin{aligned} \varphi &: \mathbb{Z}^Y \longrightarrow \mathbb{Z}Y \\ \varphi &: f \mapsto \sum_{y \in Y} f(y) y, \end{aligned}$$

which allows us to use Y as a basis for X^Y as an abelian group by defining:

$$y_i(y_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Definition 1.1.7. Given a graph $\Gamma = (V, w)$ we define the *full laplacian*, $\Delta : \mathbb{Z}^V \longrightarrow \mathbb{Z}^V$, by:

$$\Delta(v)(u) = \begin{cases} d_v - w(v, v) & \text{if } u = v \\ -w(v, u) & \text{otherwise,} \end{cases}$$

and requiring Δ to be a homomorphism. Note that for a $v \in V$, $\sum_{u \in V} \Delta(v)(u) = d_v - \sum_{u \in V} w(v, u) = 0$, and that if s is a sink, then $\Delta(s) = 0$.

If we have an order on the vertices $V = \{v_0, v_1, v_2, \ldots, v_n\}$, then we may wish to view Δ as a matrix:

$$\Delta_{ij} = \begin{cases} d_{v_i} - w(v_i, v_i) & \text{if } i = j \\ -w(v_j, v_i) & \text{otherwise.} \end{cases}$$

If Γ has a sink (say v_j), then we define the *reduced laplacian*, $\tilde{\Delta} : \mathbb{Z}^{\tilde{V}} \longrightarrow \mathbb{Z}^{\tilde{V}}$ by $\tilde{\Delta}(v) = \Delta(v)|_{\tilde{V}}$, (that is, for the matrix, by deleting the j^{th} row and column from Δ) and the *projective laplacian*, $\overline{\Delta} : \mathbb{Z}^{\tilde{V}} \longrightarrow \mathbb{Z}^V$ by $\overline{\Delta}(v) = \Delta(v)$ for all v in $\tilde{V}$, (that is by deleting only the j^{th} column (which would be all zero's) of the matrix). We may make reference to "the laplacian" in which case context should make clear to which we are referring (as each wholly determines the other two). [3]

[2] Note that a sink is unique if it exists, since there are no paths from a sink to anywhere.

[3] To readers experienced with laplacians of graphs: this may be the transpose of the definition you are used to, we began using that definition but found ourselves to be using Δ^t more than Δ, so we switched.

1.2 The Game

We will take a graph with a sink. Imagine the sink as a bottomless pit and the other vertices, v, as small platforms, each able to hold $d_v - 1$ grains of sand. If more sand than this is placed there, it will topple and one grain will flow down each of v's out-edges.

Definition 1.2.1. On a graph $\Gamma = (V, w)$ with sink, a function $c : \tilde{V} \longrightarrow \mathbb{Z}_{\geq 0}$ can be viewed as a *configuration*. Picture c as trying to place $c(v)$ grains of sand on each vertex.

Definition 1.2.2. For a configuration c, we say that a non-sink vertex v is *stable* in c if $c(v) < d_v$. Otherwise v is *unstable*. A configuration is which every vertex is stable is a stable configuration.

Definition 1.2.3. If c_0 is a configuration on a graph $\Gamma = (V, w)$ with an unstable vertex v, then we can *fire* v *legally* to reach a new configuration c_1 defined by $c_1 = c_0 - \tilde{\Delta}(v)$, and we denote this relation between c_0 and c_1 by $c_0 \xrightarrow{v} c_1$. If v is not unstable, then we would say that firing v is *illegal*. More generally, if $c_0, c_1, \ldots$ is a sequence (usually finite) of configurations with a vertex v_i unstable at each c_i such that $c_i \xrightarrow{v_i} c_{i+1}$ then we may write $c_0 \longrightarrow c_n$ and refer to $v_0, v_1, \ldots$ as a legal firing sequence. One should think of firing a vertex, v, as moving d_v grains of sand off v and placing $w(v, u)$ grains of sand on each vertex u (the sand flows down the edges).

Lemma 1.2.4. *Firings are commutative. That is, given a configuration c with two unstable vertices v and v', firing v and then v' gives the same configuration as firing v' and then v (and more importantly both orders are legal firings).*

Proof. Provided both orders of firings are necessarily legal, they will commute by the commutativity of addition. By assumption, v is unstable in c, so it is legal to fire v, going to $c' = c - \tilde{\Delta}(v)$, and then $c'(v') = c(v') - \tilde{\Delta}(v)(v') \geq c(v')$ since $v' \neq v$ implies that $\tilde{\Delta}(v)(v') \leq 0$. Thus firing v' is legal from c'. Similarly firing v is legal from $c - \tilde{\Delta}(v')$. $\square$

Definition 1.2.5. For any configuration, c, let c^o be the *stabilization* of c, the result of firing vertices for as long as is legal.

The reader should be highly skeptical of this definition, however ...

Theorem 1.2.6. *For a graph $\Gamma = (V, w)$ with sink, s, every configuration c on Γ has a unique stabilization.*

Proof. First we will show existence. Assume that c has no stabilization, then there exists an infinite legal firing sequence $v_1, v_2, \ldots$ Let V^∞ be the set of vertices appearing in the sequence infinitely often and let u be an element of V^∞ with a minimal length path to s. Now consider the sequence:

$$M_i = \sum_{v \in V^\infty} c_i(v)$$

Where c_i is c after firing $v_1, v_2, \ldots, v_i$. Thus M_i is the amount of sand on V^∞ after the i^{th} firing in the sequence. Firing u strictly decreases M, since some sand must move closer to the sink, and therefore off of V^∞. Firing any other vertex in V^∞ either decreases M or leaves it unchanged, since sand can stay on or leave V^∞ but not enter. At some point, vertices in $V \backslash V^\infty$ no longer fire. Thus, the M_i's decrease without bound. But each M_i is the sum of non-negative integers, and thus a non-negative integer, a contradiction. Therefore c has a stabilization. Now for uniqueness, we will rely on the following lemma.

Lemma 1.2.7. *Let $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_m$ be legal firing sequences from a configuration c. Then if $c \overset{v'_1,\ldots,v'_m}{\longrightarrow} c'$ is stable, then $n \leq m$ and no vertex appears more times in $v_1, \ldots, v_n$ than in $v'_1, \ldots, v'_m$.*

Proof. We will proceed by induction on n. If $n = 0$, then the conclusion is obvious. Suppose $n > 0$, since v_1 is unstable and c' is stable, v_1 must occur in $v'_1, \ldots, v'_m$. Let v'_i be the first occurrence of v_1. The sequence $v_i, v'_1, \ldots, v'_{i-1}, v'_{i+1}, \ldots, v'_m$ is still legal, and by lemma 1.2.4, still results in c'. Now we may apply the inductive hypothesis to the sequences $v_2, \ldots, v_n$ and $v'_1, \ldots, v'_{i-1}, v'_{i+1}, \ldots, v'_m$ from the configuration $c - \tilde{\Delta}(v_1)$. □

Thus if $c \longrightarrow c'$ and $c \longrightarrow c''$ with c' and c'' both stable, then the sequences of firings to reach them must just be reorderings of each other, and by lemma 1.2.4, $c' = c''$.

□

Note. It is a trivial consequence of the definition of c^o that $(c + c')^o = (c^o + c')^o$ since in performing all possible firings we can perform those that were possible at c first and then proceed to perform the others.

1.3 The Group

We will now introduce what is arguably our main object of study.

Definition 1.3.1. For a graph $\Gamma = (V, w)$ with sink, the *sandpile group* for Γ is:

$$\mathcal{S}(\Gamma) = \mathbb{Z}^{\tilde{V}} / \tilde{\Delta}(\tilde{V})$$

Definition 1.3.2. A configuration c is *accessible*, if for all configurations c' there exists a configuration c'' such that $c' + c'' \longrightarrow c$. A configuration that is both accessible and stable is *recurrent*. For a graph Γ let $R(\Gamma)$ be the set of recurrent configurations on Γ.

Proposition 1.3.3. *Given a Γ with sink, if r is a recurrent configuration and c is any configuration, then $(c + r)^o$ is a recurrent configuration.*

Proof. That r is accessible ensures that for any configuration c' there exists a configuration c'' such that $c' + c'' \longrightarrow r$, thus $c' + (c + c'') \longrightarrow c + r \longrightarrow (c + r)^o$, thus $(c + r)^o$ is accessible and stable, thus recurrent. □

Definition 1.3.4. On any graph $\Gamma = (V, w)$, the *maximal stable configuration*, $c_{\max}$, is given by $c_{\max}(v) = d_v - 1$ for all v in $\tilde{V}$.

Proposition 1.3.5. *Given a graph with a sink Γ, a configuration c on Γ is recurrent if and only if there exists a c' such that $(c_{\max} + c')^o = c$.*

Proof. $\Longrightarrow$: That c is accessible implies that there exists a c' such that $c_{\max} + c' \longrightarrow c$ and c stable along with theorem 1.2.6 ensures that $(c_{\max} + c')^o = c$.
$\Longleftarrow$: $c = (c_{\max} + c')^o$ implies that c is stable, and for any c'', $c''^o \leq c_{\max}$ so $c_{\max} - c''^o$ is a configuration. Thus $c_{\max} - c''^o + c'$ is a configuration, and $(c'' + (c_{\max} - c''^o + c')) \longrightarrow c_{\max} + c' \longrightarrow c$, therefore c is accessible. Hence, c is recurrent. ☐

Theorem 1.3.6. *Given a graph $\Gamma = (V, w)$ with a sink, each element of $\mathcal{S}(\Gamma)$ is represented by a unique element of $R(\Gamma)$.*

Proof. Let $i = (2c_{\max} + 1) - (2c_{\max} + 1)^o$. Then i is in $\tilde{\Delta}(\tilde{V})$ and $i > c_{\max}$. Then for any configuration c, $c + i > c_{\max}$. Then $(c + ki)^o$ will be recurrent by proposition 1.3.5 and $c = (c + ki)^o \mod \tilde{\Delta}(\tilde{V})$.
Now for uniqueness, let $c_1 = c_2 \mod \tilde{\Delta}(\tilde{V})$ both be recurrent. Then we have

$$f := c_1 + \sum\nolimits_{v \in \tilde{V}} \alpha_v \tilde{\Delta}(v) = c_2 + \sum\nolimits_{v \in \tilde{V}} \gamma_v \tilde{\Delta}(v)$$

with all α_v and γ_v non-negative and for all v in $\tilde{V}$ either $\alpha_v = 0$ or $\gamma_v = 0$. Let k be large enough that for each v in $\tilde{V}$, $(f + ki)(v) \geq \max(\alpha_v, \gamma_v) d_v$, which is possible since $i > 0$. Then

$$f + ki \longrightarrow c_1 + ki$$
$$\text{and}$$
$$f + ki \longrightarrow c_2 + ki$$

since each vertex can be fired enough times from $f + ki$. Also, note that for any recurrent c, $(c + ki)^o = c$ since, by proposition 1.3.5 there is a c' such that $(c_{\max} + c')^o = c$, giving us:

$$c_{\max} + c' + i = c' + c_{\max} + (2c_{\max} - (2c_{\max})^o) \longrightarrow c' + c_{\max} + (2c_{\max})^o - (2c_{\max})^o \longrightarrow c$$

and

$$c_{\max} + c' + i \longrightarrow c + i \longrightarrow (c + i)^o$$

So by theorem 1.2.6 $(c + i)^o = c$, and thus $(c + ki)^o = c$ since $c + ki \longrightarrow c + (k-1)i \longrightarrow \ldots \longrightarrow c + i \longrightarrow c$.

Whence, $(f + ki)^o = c_1$ and $(f + ki)^o = c_2$. A final reference to theorem 1.2.6 completes the proof. ☐

Corollary 1.3.7. *Given a graph $\Gamma = (V, w)$ with $\tilde{V} = \{v_0, \ldots, v_{n-1}\}$,*

$$\operatorname{rank}(\tilde{\Delta}) = \operatorname{rank}(\overline{\Delta}) = \operatorname{rank}(\Delta) = n$$

Proof. There are only $\prod_{i=1}^{n} d_{v_{i-1}}$ stables, thus there are finitely many recurrents. Hence, by theorem 1.3.6, $\mathcal{S}(\Gamma)$ is finite. From this we get that $\operatorname{rank}(\tilde{\Delta}) \geq \dim(\mathbb{Z}^{\tilde{V}}) = n$.

Δ has column sums 0, so its rows are not linearly independent. Thus $\operatorname{rank}(\Delta) < n+1$, but Δ contains $\overline{\Delta}$ which contains $\tilde{\Delta}$, as submatrices, which implies that $n+1 > \operatorname{rank}(\Delta) \geq \operatorname{rank}(\overline{\Delta}) \geq \operatorname{rank}(\tilde{\Delta}) \geq n$. □

Definition 1.3.8. Exploiting theorem 1.3.6, given a graph $\Gamma = (V, E)$ with sink, let ϵ be the recurrent configuration such that $\epsilon = 0 \mod \tilde{\Delta}(\tilde{V})$, the *recurrent identity.*

Note. In light of theorems 1.3.3 and 1.3.6, some may prefer to view $\mathcal{S}(\Gamma)$ as the ideal $R(\Gamma)$ in the abelian monoid of all stable configurations on Γ with operation $\oplus$ defined by $c \oplus c' = (c + c')^o$. We encourage a reader who wishes to think through issues involving sandpiles to try to be comfortable switching freely between these equivalent formulations.

Definition 1.3.9. Given a graph $\Gamma = (V, E)$ with a sink, a configuration β is *burning* if,

1. β is in $\operatorname{span}\{\tilde{\Delta}(\tilde{V})\}$ and

2. for all v in V there exists a u such that there is a path from u to v and $\beta(u) > 0$

We refer to $\sigma_\beta = \tilde{\Delta}^{-1}(\beta)$ as a *burning script.*[4]

Theorem 1.3.10. *Given a graph Γ with sink, there exists a burning configuration. Furthermore, if β is a burning configuration with burning script σ_β, then*

1. *$(k\beta)^o = \epsilon$ for sufficiently large k,*

2. *A configuration c is recurrent if and only if $(c + \beta)^o = c$,*

3. *A configuration is recurrent if and only if while going from $c + \beta$ to $(c + \beta)^o$ each vertex v fires $\sigma_\beta(v)$ times, and*

4. *$\sigma_\beta \geq 1$.*

Proof. That there exists a burning configuration is shown by $c_{\max} + 1 - (c_{\max} + 1)^o$.

1. For each vertex v in $\tilde{V}$, there exists a u such that there is a path from u to v and $\beta(u) > 0$. Thus there exists a k such that $k\beta \longrightarrow c$ with $c(v) \geq d_v$ for all v in $\tilde{V}$, then $c = c_{\max} + c'$ for some configuration c'. Thus $(k\beta)^o$ is recurrent and β in $\tilde{\Delta}(\tilde{V})$ implies $k\beta$ in $\tilde{\Delta}(\tilde{V})$ which implies $(k\beta)^o$ in $\tilde{\Delta}(\tilde{V})$. Thus $(k\beta)^o = \epsilon$.

[4] Scripts will be defined generally later. Given that definition, to perform σ_β from a configuration c we will subtract β from c. This note is intended only to assure the reader that our usage of the word "script" is consistent.

2. If c is recurrent, then, by proposition 1.3.3, $(c+\beta)^o$ is recurrent. Now, $(c+\beta)^o = c \mod \tilde{\Delta}(\tilde{V})$, so by theorem 1.3.6, $c = (c+\beta)^o$. Conversely, if $(c+\beta)^o = c$, then $(c+k\beta)^o = c$ for all $k > 0$. But if we take a k as in (1), then $c + k\beta \longrightarrow c' + c_{\max} \longrightarrow (c' + c_{\max})^o$ so by theorem 1.2.6 we get $c = (c' + c_{\max})^o$ and by 1.3.5 we get that c is recurrent.

3. Let $\sigma : \tilde{V} \longrightarrow \mathbb{N}$ be defined by letting $\sigma(v)$ be the number of times v fires when performing $c + \beta \longrightarrow (c+\beta)^o$. Then,

$$\begin{aligned} c \text{ is recurrent} &\iff c = (c+\beta)^o && \iff c = c + \beta - \tilde{\Delta}(\sigma) \\ &\iff \tilde{\Delta}(\sigma) = \beta && \iff \sigma = \sigma_\beta \end{aligned}$$

4. Add β to $c_{\max}$, then for any v in $\tilde{V}$, let $v_1, v_2, \ldots, v_n$ be a cycle free path from a u with $\beta(u) > 0$ to v. Then each v_i fires whiles stabilizing $c_{\max} + \beta$, since at first v_1 is unstable and each time v_i fires it makes v_{i+1} unstable. Thus v fires, thus $\sigma_\beta(v) \geq 1$.

□

1.4 Undirected Graphs

Much of our work has been extending that of previous authors who considered only connected undirected multi-graphs to the more general definition of graph we have been using. However, at this point in the exposition it is unclear how our work relates to theirs, since our set-up requires Γ to have a sink, something which an undirected graph can only have if it has no edges. In this section we will show how the sandpile model functions on undirected graphs.

Theorem 1.4.1. *Given an undirected graph $\Gamma = (V, w)$, for any v in V, let $\tilde{\Delta}_v$ be the result of deleting the row and column corresponding to v from Δ. Then, for any v, v' in V we have,*

$$\mathbb{Z}^{V\setminus\{v\}}/\mathrm{im}(\Delta_v) \cong \mathbb{Z}^{V\setminus\{v'\}}/\mathrm{im}(\Delta_{v'})$$

as groups.

Thus we can arbitrarily declare any vertex to be the sink by deleting all of its out edges and we will get an isomorphic sandpile group $\mathcal{S}(\Gamma)$.

Proof. The function

$$\varphi : \mathbb{Z}^{V\setminus\{v\}} \longrightarrow \mathbb{Z}^{V\setminus\{v'\}}$$
$$\varphi(f)(u) = \begin{cases} f(u) & \text{if } u \neq v \\ -\sum_{u' \neq v} f(u') & \text{if } u = v \end{cases}$$

induces the desired isomorphism. This is the case since Δ having zero column sums implies that $\varphi(\Delta_v) = \Delta_{v'}$, and φ is clearly a one-to-one and onto homomorphism.

□

Theorem 1.4.2. *If a connected undirected graph* $\Gamma = (V, w)$ *is made into a directed graph with sink* $\Gamma' = (V, w')$ *by selecting an* s *in* V *and defining*

$$w'(u,v) = \begin{cases} w(u,v) & \text{if } u \neq s \\ 0 & \text{if } u = s, \end{cases}$$

then there exists a burning configuration, β*, with burning script* $\sigma_\beta = 1$*.*

Proof. Let $\tilde{\Delta}'$ be the reduced laplacian of Γ', then let

$$\begin{aligned} \beta &= \sum_{v \in \tilde{V}} \tilde{\Delta}'(v) \\ V_s &= \{v \in \tilde{V} : w'(v,s) > 0\} \\ V_{\neg s} &= \{v \in \tilde{V} : w'(v,s) = 0\}. \end{aligned}$$

Our program is to show that β is a burning configuration. Clearly it is in the span of $\tilde{\Delta}(\tilde{V})$. Now, note that for v in $V_{\neg s}$, and any u in V, $w'(u,v) = w(u,v) = w(v,u) = w'(v,u)$. Thus,

$$\begin{aligned} \beta(v) &= \sum_{u \in \tilde{V}} \tilde{\Delta}'(u)(v) \\ &= d_v - \sum_{u \in \tilde{V}} w'(u,v) \\ &= \sum_{u \in \tilde{V}} w'(v,u) - \sum_{u \in \tilde{V}} w'(v,u) \\ &= 0 \end{aligned}$$

and that for v in V_s, the same is true for u in $\tilde{V}$. Also, for v in V_s, $w(v,s) > 0$ and $w(s,v) = 0$. Thus,

$$\begin{aligned} \beta(v) &= \sum_{u \in \tilde{V}} \tilde{\Delta}'(u)(v) \\ &= d_v - \sum_{u \in \tilde{V}} w'(u,v) \\ &= w'(v,s) + \sum_{u \in \tilde{V}} w'(v,u) - \sum_{u \in \tilde{V}} w'(v,u) \\ &= w'(v,s) > 0. \end{aligned}$$

So in total we have that $\beta \geq 0$ and that $\beta(v) > 0$ if and only if v is in V_s. The final condition requires that for every u in $V_{\neg s}$ there be a v in V_s such that there is a path from v to u. This is fulfilled since Γ was originally a connected undirected graph, so there was a path from s to u (which we may assume without loss of generality had no loops), but s was only directly connected to vertices in V_s, thus this path must have gone through some v in V_s. The original loop free path minus that first connection will be present in Γ', thus there will be the desired path from v to u. ☐

Chapter 2

The Toppling Ideal

One might recall from the abstract that we purported to relate graphs to ideals in polynomial rings, so far we have introduced graphs and sandpiles. This has been done because thinking of sand and using techniques which reference it will make reasoning about the upcoming polynomials easier.

2.1 Moving to Polynomials

The concept of viewing sandpiles as polynomials and introducing the toppling ideal, as we will shortly, was first done in [2], however they only considered undirected graphs and we here extend their results to graphs in our sense of the term (that being multi digraphs with sink).

Definition 2.1.1. Given a graph $\Gamma = (V, w)$ with $\tilde{V} = \{v_0, \ldots, v_{n-1}\}$, we work in the *associated polynomial ring*, $k(\Gamma) = k[x_0, \ldots, x_{n-1}]$, where k is a field. If we have not specified an order on the vertices, then adjoin to k one variable, x_v, for each v in $\tilde{V}$.

To go from a configuration c on Γ to a monomial in $k(\Gamma)$, when we write x^c we intend $\prod_{v\in\tilde{V}} x_v^{c(v)}$. If we write x alone, we intend $x^1 = \prod_{v\in\tilde{V}} x_v$.

Definition 2.1.2. Given a graph $\Gamma = (V, w)$, every element ℓ of $\mathbb{Z}^{\tilde{V}}$ can be split uniquely into configurations ℓ^+ and ℓ^- with disjoint support such that $\ell = \ell^+ - \ell^-$, and we would like to define an operator to do this for us as we go from dealing with elements of $\mathcal{S}(\Gamma)$ to dealing with elements of $k(\Gamma)$, so let

$$\begin{aligned} t &: \mathbb{Z}^{\tilde{V}} \longrightarrow k(\Gamma) \\ t &: \ell \mapsto x^{\ell^+} - x^{\ell^-} \end{aligned}$$

with, $\ell^+(v) = \max(\ell(v), 0)$, and $\ell^-(v) = \max(-\ell(v), 0)$ as we have just explained. The immediate usefulness of this will be in allowing us to use the laplacian to go from vertices to binomials,

$$T : \tilde{V} \longrightarrow k(\Gamma)$$
$$T : v \mapsto t(\tilde{\Delta}(v))$$

We have perhaps been overly coy in refusing to nail down a ordering on vertex sets, after all they are just finite sets. This is because we have a specific way of ordering the vertices we would like to use, but did not want to introduce it before its utility would actually appear.

2.2 A Useful Term Ordering

Let us introduce a monomial ordering on $k(\Gamma)$. We would like this order to have the following property: $c \to c' \implies x^c > x^{c'}$. Conveniently enough, graded reverse lexicographic order (a standard monomial ordering with several efficiency properties relating to Gröbner bases) has this property provided we number the vertices correctly, with the highest index vertices having the shortest paths to the sink

Definition 2.2.1. *Graded reverse lexicographic* order, degrevlex, is defined by: $x^a > x^b$ if $|a| > |b|$ or both $|a| = |b|$ and the last non-zero entry in $a - b$ is negative. In other words, monomials are first sorted by degree and then those with fewest of the latest variables are greatest.

Why does this work? Well, $c \to c'$ means that c' is reached from c by a sequence of vertex firings, but each firing reduces the associated monomial in degrevlex ordering, since either the vertex that fired was adjacent to the sink, in which case the degree of the monomial reduces, or at least some sand gets closer to the sink which will introduce more of a later indexed variable into the monomial.

Definition 2.2.2. Given a monomial ordering and a polynomial, p, the leading term of p, denoted $\mathrm{LT}(p)$, is the monomial in p that is greatest according to the monomial ordering.

Definition 2.2.3. Given a monomial ordering and two polynomials p and q, the remainder after performing polynomial long division of p by q is p *reduced by* q, which we will notate $p\%q$.

Germane to our purposes, if the first and second term of p share no monomial divisors (fulfilled trivially if p is a monomial) and q is not a constant, then

$$p\%q = \begin{cases} p - \frac{\mathrm{LT}(p)}{\mathrm{LT}(q)}q & \text{if } \mathrm{LT}(q) \text{ divides } \mathrm{LT}(p) \\ p & \text{if } \mathrm{LT}(q) \text{ does not divide } \mathrm{LT}(p) \end{cases}$$

As we will use graded reverse lexicographic order exclusively, it is unproblematic that our notation makes no reference to the monomial ordering.

This gives the nice property that $x^c\%T(v) = x^{c'}$, where the configuration $x^{c'}$ is reached from the configuration x^c by firing vertex v until it is stable.

As we now have an order we would like to put on vertex sets we will freely switch between viewing the laplacian as a map between functions and as a matrix and viewing $\mathbb{Z}^{\tilde{V}}$ as $\mathbb{Z}^n$. By convention we will now reserve the variable n for the size of $\tilde{V}$, with $V = \{v_0, v_1, \ldots, v_n\}$ and v_n the sink.

2.3 The Toppling Ideal

Definition 2.3.1. Given a graph $\Gamma = (V, w)$, we define the *toppling ideal* of Γ to be $I(\Gamma) = (T(\tilde{V}), x^\beta - 1)$, where β is any burning configuration. We will of course show that which burning configuration is chosen does not matter.

Lemma 2.3.2. *Given a graph* $\Gamma = (V, w)$*, if*

$$I = \operatorname{span}\{t(\ell) : \ell \in \operatorname{span}\{\tilde{\Delta}(\tilde{V})\}\}$$

then $I(\Gamma) = I$*.*

Note that the right hand side of this equality makes no reference to any specific β, hence this shows as a corollary that $I(\Gamma)$ is well-defined.

Proof. We will make use of the fact that if $\alpha_1, \ldots, \alpha_k$ are generators for a $\mathbb{Z}$-submodule $A \subset \mathbb{Z}^n$, then

$$\begin{aligned} I :&= \operatorname{span}\{x^a - x^b : a, b \in \mathbb{N}^n, a - b \in A\} \\ &= \{f : x^m f \in J \text{ for some } m \in \mathbb{N}\} \end{aligned}$$

where

$$J = (\{x^{\alpha_i^+} - x^{\alpha_i^-} : i = 1, \ldots, k\}).$$

That is, I is the saturation of J with respect to the ideal (x). This is a standard property of lattice ideals [1].

Noting that $\operatorname{span}\{\tilde{\Delta}(\tilde{V})\}$ is a lattice with $\mathbb{Z}$-module generators $T(\tilde{V})$, and clearly $I(\Gamma) \subset I$, all that needs to be shown is that $I(\Gamma)$ is already saturated with respect to (x). Let $x^\ell f \in I(\Gamma)$, for some f in $k(\Gamma)$ and ℓ in $\mathbb{N}$. Then, for an m in $\mathbb{N}$, consider the monomial $x^{m\beta}$ corresponding to the configuration with $m\beta(v)$ grains of sand at each vertex v. If v is unstable then firing v results in replacing $x_v^{m\beta(v)}$ with $x_v^{m\beta(v)-d_v} \prod_{u \in \tilde{V}} x_v^{w(v,u)}$. Performing this replacement gives an equivalent monomial modulo $I(\Gamma)$. Since there is a path to every vertex in $\tilde{V}$ from some vertex, v, such that $\beta(v) > 0$, by taking a sufficiently large m and performing firings we can arrive at x^c equivalent to $x^{m\beta}$ with $c \geq \ell$. Thus working modulo $I(\Gamma)$,

$$\begin{aligned} 0 &= x^\ell f = x^c f \\ &= x^{m\beta} f = f \end{aligned}$$

Thus, f is in $I(\Gamma)$. Thus $I(\Gamma)$ is saturated, as desired.

∎

Definition 2.3.3. We define the *homogeneous toppling ideal* of a graph Γ, by

$$\overline{I(\Gamma)} = \mathrm{span}\{t(\ell) : \ell \in \mathrm{span}\{\overline{\Delta}(\tilde{V})\}\}$$

2.4 Script Firings and Super-Stability

The requirement that a vertex be unstable in order to fire ensures that no vertex has negative sand on it. But if there were two vertices, v_1 and v_2, and edges (v_1, v_2) and (v_2, v_1), then in a configuration, c, with $c(v_1) = d_{v_1} - 1$ and $c(v_2) = d_{v_2} - 1$ firing either v_1 or v_2 would result in negative sand on a vertex, but firing both of them would give a configuration. So while firing either vertex is illegal, we may want to say that firing both is legal. Situations like this lead us to consider script firings.

Definition 2.4.1. An element σ of $\mathbb{N}^{\tilde{V}}$ can be viewed as a *firing script*. In order to perform σ, fire each vertex v a total of $\sigma(v)$ times. This firing is *legal* for a configuration c provided a configuration is reached by performing σ from c.

We will now extend the function T to aid in the consideration of scripts:

$$T : \mathbb{N}^{\tilde{V}} \longrightarrow k(\Gamma)$$
$$T : \sigma \mapsto t(\textstyle\sum_{v \in \tilde{V}} \sigma(v) \tilde{\Delta}(v))$$

The $T(v_i)$ defined earlier corresponds to the special case where $\sigma = e_i$ where e_i is the i-th standard basis element for $\mathbb{N}^n$, the one corresponding to v_i. Now we note that our earlier property extends, if we have a script $\sigma \in \mathbb{N}^{\tilde{V}}$, then $x^c \% T(\sigma)$ is the result of performing the script firing σ as many times as is legal from c.

Definition 2.4.2. A configuration, c, is *super-stable* if there is no legal firing script, σ, for c such that $T(\sigma) \neq 0$. That is, for every σ in $\mathbb{N}^{\tilde{V}}$, either $T(\sigma) = 0$ (that is $\sigma = 0$ since $\tilde{\Delta}$ is of full rank) or $c + \tilde{\Delta}\sigma \not\geq 0$

2.5 A Gröbner Basis for $I(\Gamma)$

Gröbner bases are of great use in computational algebra. For example, they allow one to give a well-defined division algorithm for multivariate polynomials. So we present here a way to arrive at a Gröbner basis for $I(\Gamma)$ which uses the properties of $\mathcal{S}(\Gamma)$.

Definition 2.5.1. For any two polynomials p and q, the *S-polynomial* of p and q, is

$$S(p, q) = \frac{\mathrm{LCM}(\mathrm{LT}(p), \mathrm{LT}(q))}{LT(p)} p - \frac{\mathrm{LCM}(\mathrm{LT}(p), \mathrm{LT}(q))}{LT(q)} q.$$

Definition 2.5.2. Given an ideal I in a polynomial ring R, a subset G of I is a *Gröbner basis* if any of the following equivalent conditions hold (for a proof that they are equivalent the reader is directed to [4]):

1. $(\{\mathrm{LT}(g) : g \in G)\}) = \{\mathrm{LT}(p) : p \in I\}$,
2. for all p in I, there is a g in G such that $\mathrm{LT}(g)$ divides $\mathrm{LT}(p)$,
3. all p in I reduce to 0 by G, or
4. for all g_1, g_2 in G, $S(g_1, g_2)$ reduces to 0 by G and G is a generating set for I.

The reader may wish to note that a Gröbner basis depends on a choice of monomial order, but, as we only use one monomial order here, it is not a concern for us.

Theorem 2.5.3. *If σ_β is the script for some burning configuration β, then, $\mathcal{T} = \{T(\sigma) : \sigma \leq \sigma_\beta\}$ is a Gröbner basis for $I(\Gamma)$.*

Proof. Note that lemma 2.3.2 is equivalent to the statement $I(\Gamma) = \mathrm{Span}\{\mathrm{im}(T)\}$. Thus $\mathcal{T} \subset I(\Gamma)$, which implies $(\mathcal{T}) \subset I(\Gamma)$. Also, $\mathcal{T} \supset \{T(e_i), T(\sigma_\beta)\}$. Noting that $T(e_i) = T(v_i)$ and $T(\sigma_\beta) = x^\beta - 1$ we see that $(\mathcal{T}) \supset I(\Gamma)$. So $(\mathcal{T}) = I(\Gamma)$, that is $\mathcal{T}$ is a basis for $I(\Gamma)$. So it only remains to show that $\mathcal{T}$ is a Gröbner basis, which can be done by showing that all the S-polynomials of $\mathcal{T}$ reduce to 0 by $\mathcal{T}$.

Let $\sigma, \tau \leq \sigma_\beta$. The polynomials $T(\sigma)$ and $T(\tau)$ break into positive and negative parts:

$$T(\sigma) = P(\sigma) - N(\sigma)$$
$$T(\tau) = P(\tau) - N(\tau)$$

Note that since P is a configuration before firing a script and N is the same configuration after firing that script, P will be the leading term.

Now, let $m(\sigma)$ and $m(\tau)$ be the minimal monomials such that $m(\sigma)P(\sigma) = m(\tau)P(\tau)$, and let c be the element of $\mathbb{N}^n$ such that $x^c = m(\sigma)P(\sigma) = m(\tau)P(\tau)$. Since both scripts σ and τ are legal from the configuration x^c, so is σ_0, where σ_0 is defined by $\sigma_0(v) := \max(\sigma(v), \tau(v))$. Let $x^{c'}$ be the result of firing σ_0 from x^c. Then

$$\begin{aligned} S(T(\sigma), T(\tau)) &= m(\sigma)T(\sigma) - m(\tau)T(\tau) \\ &= m(\tau)N(\tau) - m(\sigma)N(\sigma) && \text{(WLOG assume } m(\tau)N(\tau) \text{ is the leading term)} \\ &\longrightarrow -m(\sigma)N(\sigma) + x^{c'} && \text{(reducing by } T(\sigma_0 - \tau)\text{)} \\ &\longrightarrow 0 && \text{(reducing by } T(\sigma_0 - \sigma)\text{).} \end{aligned}$$

□

2.6 Another Version of $\mathcal{S}(\Gamma)$

Theorem 2.6.1. *Each element of $\mathcal{S}(\Gamma)$ has a unique super-stable representative.*

Proof. Two configurations differ by an element of $\mathrm{span}\{\tilde{\Delta}(\tilde{V})\}$ if and only if one can be reached from the other by some series of firings and reverse firings (without regard to legality), which is the same as two monomials in $k(\Gamma)$ being equivalent modulo $I(\Gamma)$. So for any κ in $\mathcal{S}(\Gamma)$, let c be its recurrent representative, and let $x^{c'}$ be the remainder upon reducing x^c by $\mathcal{T}$. Then c' is super stable and a representative of κ. Since $\mathcal{T}$ is a Gröbner basis for $I(\Gamma)$, we know that $x^{c'}$ is the only representative of the equivalence class for x^c (that is, for κ) which is super-stable. $\square$

From this we see that the set of super-stables can also be thought of as $\mathcal{S}(\Gamma)$ just like $R(\Gamma)$ could (this time with the group action of $c \oplus c' = log((x^c x^{c'})\% I(\Gamma)))$. But there is also an interesting, and opposing, duality between the super stables and the recurrents.

Theorem 2.6.2. *A configuration c is recurrent if and only if $c_{\max} - c$ is super-stable.*

Proof. By theorem 2.6.1, we know that we have the same number of super-stables and recurrents, and $c \mapsto c_{\max} - c$ is clearly one-to-one, so if we show one direction of the implication we are done. To that end we will show that c recurrent implies that $c_{\max} - c$ is super-stable.

Assume $c_{\max} - c$ is not super-stable, then, by theorem 2.5.3 there is a non-zero $\sigma \leq \sigma_\beta$ such that we may perform σ from $c_{\max} - c$. Thus $c_{\max} - c - \tilde{\Delta}(\sigma) \geq 0$, which gives us that $c + \tilde{\Delta}(\sigma)$ is stable. But, if we add β to c, and then perform $\sigma_\beta - \sigma$ we arrive at the stable configuration $c + \tilde{\Delta}(\sigma)$. Unique stabilizations from theorem 1.2.6 and the properties of recurrents and burning configurations from theorem 1.3.10 means that c is not recurrent.

$\square$

Chapter 3

Of Lattice Ideals and Graphs

We now have lattice ideals coming out of graphs by way of the sandpile group. In this chapter we exploit relationships established between lattice ideals and their matrices in [6]. In particular if a lattice ideal is a complete intersection its defining matrix must have a very particular form. Then, in section 2, we show how these considerations about a matrix inform what sort of graph could have such a laplacian.[1]

3.1 The Set-Up

Definition 3.1.1. A homogeneous ideal, I, is a *complete intersection* if it has a basis consisting of a number of polynomials equal to the codimension of its associated variety (the set of points vanishing on all polynomials in I). As to why one would be interested in which ideals are complete intersections, the reader is directed to [4].

Definition 3.1.2. A matrix with entries in $\mathbb{Z}$, is *mixed* if each column contains both positive and negative entries. A matrix is *dominating* if it does not contain a square mixed submatrix. By convention, empty $d \times 0$ matrices are considered mixed dominating.

Notation. By $M(\alpha_1, \ldots, \alpha_m)$ we mean the matrix with columns $\alpha_1, \ldots, \alpha_m$.[2]

We now cite three theorems from [6].[3]

Theorem 3.1.3. *If $M(\alpha_1, \ldots, \alpha_m)$ is mixed dominating, then $\{\alpha_1, \ldots, \alpha_m\}$ is linearly independent.*

Theorem 3.1.4. *For an $(n+1) \times n$ matrix M of rank n, M is mixed dominating if and only if the lattice ideal*

$$I_M = \operatorname{span}\{x^{m^+} - x^{m^-} : m \in \operatorname{im}(M)\},$$

is a complete intersection.

[1] Once again the reader is warned when referring to our sources that most of our definitions are the transpose of theirs due to our preference for left-sided matrix multiplication.

[2] As a warning, this means that $M_{ij} = (\alpha_j)_i$.

[3] The second of these is actually the synthesis of a theorem, a remark during an example, and a definition.

Theorem 3.1.5. *If $M(\alpha_1, \ldots, \alpha_m)$ is mixed dominating with n rows and $n > m$, then there exist disjoint non-empty subsets of $\{1, \ldots, n\}$, E_1 and E_2, such that $E_1 \cup E_2 = \{1, \ldots, n\}$ and disjoint, possibly empty, subsets of $\{1, \ldots, m\}$, S_1 and S_2, such that $\{1, \ldots, m\} \backslash (S_1 \cup S_2)$ has one element, say q. These E_1, E_2, S_1, S_2 are such that:*

1. *the matrices $M(\alpha_i : i \in S_1)$ and $M(\alpha_i : i \in S_2)$ are mixed dominating,*
2. *if $i \notin S_k$ and $i \neq q$, then α_{ij} is zero for all $j \in E_k$,*
3. *if $(a_q^+)_j$ is non-zero then $j \in E_1$, and*
4. *if $(a_q^-)_j$ is non-zero, then $j \in E_2$.*

More comprehensibly, if M is a mixed dominating matrix, then by reordering of columns and rows we can get

$$M = \begin{bmatrix} M_1 & 0 & | & \alpha_m^+ \\ 0 & M_2 & | & \alpha_m^- \end{bmatrix}$$

where M_1 and M_2 are mixed dominating.

The applicability of theorem 3.1.4 to our purposes is almost immediate; as $\overline{\Delta}$ is of the correct size and rank, it will allow us to reason about $\overline{I(\Gamma)}$.

Theorem 3.1.5 is quite useful because it allows us to make inductive arguments about mixed dominating matrices, and thus about complete intersection lattice ideals.

Definition 3.1.6. Given a graph, $\Gamma = (V, w)$, with sink, s, and another graph, $\Gamma' = (V', w')$, with $V \cap V' = \emptyset$, we say that $\Gamma'' = (V'', w'')$ is made by *wiring* Γ *into* Γ' if $V'' = V \cup V'$ and

$$w''(u, v) = \begin{cases} w(u, v) & \text{if } (u, v) \in \tilde{V} \times V \\ w'(u, v) & \text{if } u, v \in V' \\ 0 & \text{if } u \in \tilde{V}, v \in V' \text{ or vice versa.} \end{cases}$$

And, $w''(s, v) > 0$ for at least one v in V', so that Γ'' inherits the sink of Γ'. Note that this does not put any other restrictions on the values for $w''(s, v)$ for v in V''.

Definition 3.1.7. We say that a graph is *completely wired* if it can be made by wiring a completely wired graph into another completely wired graph. By convention, we say that a graph with only one vertex is completely wired.

3.2 New Results

Theorem 3.2.1. *For every homogeneous complete intersection lattice ideal, I_M, with M having one more row than column and being of full rank, there exists a completely wired graph Γ with $\overline{I(\Gamma)} = I_M$.*

Proof. That I_M is a complete intersection ensures that, by theorem 3.1.4, M is mixed dominating, and that I_M is homogeneous ensures that M has zero column sums. We will now show that there exists a graph with $\overline{\Delta}$ such that $\operatorname{im}(M) = \operatorname{im}(\overline{\Delta})$, which will complete the proof.

We proceed to do this by using theorem 3.1.5. The base case is $M = \left[\begin{smallmatrix} n \\ -n \end{smallmatrix}\right]$, which is the $\overline{\Delta}$ for a graph made by wiring one vertex, v, into another, u, with $w(v,u) = n$. For the inductive step, by theorem 3.1.5, we may assume that

$$M = \left[\begin{array}{cc|c} M_1 & 0 & \alpha^+ \\ 0 & M_2 & \alpha^- \end{array}\right]$$

with M_1 and M_2 smaller zero column sum mixed dominating matrices. Thus we may replace M_1 and M_2 by $\overline{\Delta}_1$ and $\overline{\Delta}_2$, from some Γ_1, with sink s, and Γ_2, without changing the span

$$\operatorname{im}(M) = \operatorname{im} \left[\begin{array}{cc|c} \overline{\Delta}_1 & 0 & \alpha^+ \\ 0 & \overline{\Delta}_2 & \alpha^- \end{array}\right].$$

Now let β be a burning configuration on Γ_1, then by a similar argument to that used in the proof of theorem 2.3.2, we find an m such that $m\beta \longrightarrow c \geq \alpha^+$, viewing α^+ as a configuration on Γ_1 by ignoring its value at s. Then c is in the span of $\tilde{\Delta}_1$ so $\overline{c} = \left[\begin{smallmatrix} c \\ -|c| \end{smallmatrix}\right]$ is in the span of $\overline{\Delta}_1$, where $|c| = \sum_{v \in \tilde{V}_1} c(v)$. Thus $\left[\begin{smallmatrix} \overline{c} \\ 0 \end{smallmatrix}\right]$ is in the span of $\left[\begin{smallmatrix} \overline{\Delta}_1 \\ 0 \end{smallmatrix}\right]$, which gives us that

$$\operatorname{im}(M) = \operatorname{im} \left[\begin{array}{cc|c} \overline{\Delta}_1 & 0 & \alpha^+ - \overline{c} \\ 0 & \overline{\Delta}_2 & \alpha^- \end{array}\right],$$

which is the laplacian for a graph which is the wiring of Γ_1 into Γ_2 by defining $w(s,v) = (\alpha - \left[\begin{smallmatrix} \overline{c} \\ 0 \end{smallmatrix}\right])(v)$. □

Theorem 3.2.2. *For any completely wired graph $\Gamma = (V, w)$ with sink, there exists a mixed dominating matrix with zero column sums M such that* $\operatorname{im}(M) = \operatorname{im}(\overline{\Delta}(\Gamma))$*, and, therefore, $\overline{I(\Gamma)}$ is a complete intersection.*

Proof. We use induction on the size of V. If Γ has only one vertex then $\overline{\Delta}$ is empty, which is, by definition, mixed dominating.

Otherwise, $\Gamma = (V, w)$ is the wiring of a completely wired $\Gamma_1 = (V_1, w_1)$ (with sink s) into a completely wired $\Gamma_2 = (V_2, w_2)$ with projective laplacians $\overline{\Delta}_1$ and $\overline{\Delta}_2$ respectively. Defining

$$\alpha_1 : V_1 \longrightarrow \mathbb{Z} \qquad\qquad \alpha_2 : V_2 \longrightarrow \mathbb{Z}$$

by

$$\alpha_i : v \mapsto \begin{cases} -w(s,v) & \text{if } v \neq s \\ \sum_{u \in V} w(s,u) & \text{if } v = s \end{cases}$$

Then it is possible, by ordering the columns and the rows[4], to write

$$\overline{\Delta} = \left[\begin{array}{cc|c} \overline{\Delta}_1 & 0 & \alpha_1 \\ 0 & \overline{\Delta}_2 & \alpha_2 \end{array}\right].$$

[4]Right now we do not care about getting a Gröbner basis so this order that we wish for the rows will be our criterion for ordering the vertices. Reordering the columns does not affect the span and so may be done freely.

Then, as in theorem 3.2.1 but in reverse, we use a burning configuration on Γ_1 to find an element, $\bar{c}$, in the span of $\overline{\Delta}_1$ such that $\alpha_1 + \bar{c} \geq 0$. Now, by our inductive assumption, there exist M_1 and M_2 which are mixed dominating and zero column sum such that $\operatorname{im}(M_i) = \operatorname{im}(\overline{\Delta}_i)$. Thus, by theorem 3.1.5,

$$M = \left[\begin{array}{cc|c} M_1 & 0 & \alpha_1 + \bar{c} \\ 0 & M_2 & \alpha_2 \end{array}\right]$$

is mixed dominating, has zero column sums, and $\operatorname{im}(M) = \operatorname{im}(\overline{\Delta})$. □

Theorem 3.2.3. *An undirected graph, $\Gamma = (V, w)$, is completely wired if and only if it is a tree (modulo self-loops).*

Proof. That a tree is completely wired is obvious, so all that needs to be shown is that a completely wired undirected graph can only be a tree.

We proceed by induction on the size of V. As a base case, if Γ has only a single vertex, then it is completely wired and a tree.

Inductively, if $\Gamma_1 = (V_1, w_1)$ and $\Gamma_2 = (V_2, w_2)$ are trees, and we wish to wire Γ_1 into Γ_2—giving them sinks by altering w as usual—the only way to do it that gives an undirected graph (except for the sink) results in a tree. To see this, let s_1 be the sink of Γ_1 and s_2 be the sink of Γ_2, then in order for Γ_1 wired into Γ_2 to be undirected (except of course for having no out-edges from its sink s_2), we must define

$$w(s_1, v) = \begin{cases} w_1(v, s_1) & \text{if } v \in \tilde{V}_1 \\ 0 & \text{if } v \in \tilde{V}_2. \end{cases}$$

Thus the sink of G_1 can only be wired back into G_1 in such a way as to undirect all the edges into itself and can only go into s_2 without introducing directed edges that do not go into the sink. This means that if one wishes to wire one undirected completely wired graph into another, one can attach one vertex (any one since theorem 1.4.1 tells us we can select any vertex to be the sink) from one into any one vertex of the other, but that this is all one can do. Doing this between two trees results in a tree. □

Corollary 3.2.4. *An undirected graph, Γ, has a complete intersection $\overline{I(\Gamma)}$ (when given a sink) if and only if Γ is a tree (modulo self-loops).*

Proof. This is an immediate consequence of theorems 3.1.4, 3.2.1, 3.2.2, and 3.2.3. □

References

[1] Anna Bigatti and Lorenzo Robbiano. Toric ideals. *Mat. Contemp.*, 21:1–25, 2001. 16th School of Algebra, Part II (Portuguese) (Brasília, 2000).

[2] Robert Cori, Dominique Rossin, and Bruno Salvy. Polynomial ideals for sandpiles and their Gröbner bases. *Theoret. Comput. Sci.*, 276(1-2):1–15, 2002.

[3] Deepak Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64(14):1613–1616, 1990.

[4] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[5] Alexander E. Holroyd, Lionel Levine, Karola Mészáros, Yuval Peres, James Propp, and David B. Wilson. Chip-firing and rotor-routing on directed graphs. In *In and out of equilibrium. 2*, volume 60 of *Progr. Probab.*, pages 331–364. Birkhäuser, Basel, 2008.

[6] Marcel Morales and Apostolos Thoma. Complete intersection lattice ideals. *J. Algebra*, 284(2):755–770, 2005.