Matroids and their cycle systems: the h-vector conjecture for a new class of matroids

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Finally, thank you to mi querida Fiona. Thank you for being patient; I should have more free time now.

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Abstract

In a 1977 paper by Richard Stanley, it was conjectured that the h-vector—the coefficients of the h-polynomial—of any matroid M is a pure $\mathcal{O}$-sequence. Stanley's "h-vector conjecture" became a reoccurring subject of research in matroid theory. Over the past 40 years, proofs of the conjecture have been developed for some classes of matroids, including cographic matroids. In this paper, we introduce cycle systems and coparking functions for arbitrary matroids and prove the h-vector conjecture for matroids with cycle systems, using recurrence of the Tutte polynomial. We also describe an algorithm for a bijection between matroid bases and coparking functions using a deletion-contraction diagram with respect to a cycle system, results previously known only for linear matroids.

Dedication

To all the lesbians of the world.

Preface

In 1977, combinatorialist Richard Stanley conjectured that the h-vector of a matroid is a pure $\mathcal{O}$-sequence (Stanley (1977)). Since then, much research in the field of matroid theory has been in pursuit of proving Stanley's conjecture. Thus far, it has been proved only for some classes of matroids, including cographic matroids (López (1997)), paving matroids, rank 3 matroids, corank 2 matroids, and positroids (cf. He et al. (2021)). In this paper, we provide the necessary background to understand the conjecture and its claim before introducing the notion of a *cycle system* for a matroid. Our main result, Theorem 3.14, shows that the h-vector conjecture holds for all matroids with cycle systems: including all cographic matroids, generalizing the chip-firing proof by Merino López. We also present an algorithmic process for constructing a bijection between matroid bases and coparking functions using a deletion-contraction diagram with respect to a cycle system, as well as several examples of how cycle systems lead to deletion-contraction trees.

The theory behind this thesis has been in development for several years. We continue from the work in Albers (2023) on cycle systems for linear matroids, which was itself derived from a research project conducted in the summer of 2023 by Lily Factora, Sanay Sehgal, and Lixing Yi under the guidance of David Perkinson. Their efforts were built upon previous contributions by Perkinson, Scott Corry, and Anton Dochtermann who introduced cycle systems and coparking functions for graphic matroids. Similar ideas are contained in Dong (2014).

In Chapter 1, we introduce matroids and provide some foundational theory, including the principle of duality and recurrence relations via deletion-contraction. Chapter 2 defines the Tutte polynomial and the h-vector of a matroid M and, after introducing multicomplexes and $\mathcal{O}$-sequences, describes Merino's proof of the h-vector conjecture for cographic matroids. Chapter 3 introduces the notion of a cycle system for general matroids, presents some examples of matroids with cycle systems, and proves several surprising properties of the structure, culminating in a recursive proof of the h-vector conjecture for matroids with cycle systems. Chapter 4 discusses the construction of bijections between leaves of the deletion-contraction tree, bases of M, and elements of the associated multicomplex. Finally, Chapter 5 considers remaining questions and ideas for further work to generalize the construction further.

Thank you for reading, and I hope you enjoy my thesis.

Chapter 1

Introduction and Definitions

1.1 Matroids

You might wonder—what is a matroid? The answer is... a lot of different things! A matroid is an abstract structure within combinatorics, the field of mathematics concerned with counting. While first conceptualized in the 1930s, the amount of research on matroids has grown tremendously within the past decade, as this abstract construction is applied everywhere from geometry, topology, and algebraic varieties to more practical problems in network and coding optimization theory. An exceptional characteristic of matroid theory is that researchers can independently arrive at identical results, because of the wide variety of ways one might conceptualize and encounter matroids.

Most significantly, a matroid is a structure that generalizes the notion of linear independence. Whether the matroid is a vector space or a graph, or exists as an extension of field theory, is up to whatever definition is most useful. There is a lot to be said about the wide world of matroids; this author recommends Oxley (2011) for a thorough background. This paper will only really include the parts of matroid theory relevant to our main results, such as the following:

Definition 1.1.1. A (finite) *matroid* is a pair $M = (E, \mathcal{I})$ consisting of a finite set E, the *ground set*, and a collection $\mathcal{I}$ of subsets of E, the *independent sets*, which satisfies the following properties:

(a) $\emptyset \in \mathcal{I}$;

(b) $\mathcal{I}$ is closed under taking subsets: if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$;

(c) the exchange axiom: if $I, J \in \mathcal{I}$ and $|I| > |J|$, there exists $e \in I \setminus J$ such that $J \cup \{e\} \in \mathcal{I}$.

We will often write $E(M)$ or $\mathcal{I}(M)$ when more than one matroid is being considered; for now, let $M = (E, \mathcal{I})$ be our matroid.

If $B \subseteq E$ is a maximal element of I under inclusion of subsets of E, then B is a *basis* for M. The set of all bases is denoted $\mathcal{B}(M)$. By axiom (c), all bases of M have the same cardinality, thus the bases of M are exactly the independent sets of maximal cardinality.

The *rank* of a subset $A \subseteq E$, denoted $\mathrm{rank}(A)$, is the cardinality of a maximal independent subset of A, where maximal can again be taken with regard to inclusion or cardinality. The rank of M is defined by $\mathrm{rank}(M) = \mathrm{rank}(E)$, which is to say the rank of M is the cardinality of any basis of M. It follows that an independent subset of size $\mathrm{rank}(B)$ is a basis for M, and that any subset with cardinality greater than $\mathrm{rank}(B)$ is dependent.

Example 1.1.2. Consider the following classes of matroids:

(a) *Uniform matroids.* Let $E = \{1, ..., n\}$, and let $k \leq n$ be a natural number. We say $U_{k,n}$ is the *uniform matroid* on E with every subset of size at most k taken as independent.

(b) *Vector matroids.* Let A be a $k \times n$ matrix with coefficients in the field F. Let $E = \{e_1, ..., e_n\}$ be the column labels for A, and let $\mathcal{I}$ be the subsets of E whose corresponding columns are linearly independent. Then $M = (E, \mathcal{I})$ is a vector matroid. For example, the following matrix defines a matroid:

$$\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

where $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $\mathcal{I}$ is all subsets of the following sets: $\{e_1, e_2, e_4\}, \{e_1, e_2, e_6\}, \{e_1, e_3, e_4\}, \{e_1, e_3, e_6\}, \{e_1, e_4, e_6\}, \{e_2, e_3, e_4\}, \{e_2, e_3, e_6\}$.

(c) *Cycle matroids.* Let E be the edges of an undirected multigraph (a graph which is permitted to have parallel edges and loops) and $\mathcal{I}$ the subsets of edges comprising the forests (acyclic subgraphs). Then $M = (E, \mathcal{I})$ is the *cycle matroid* of graph G, denoted $M(G)$. An example can be seen in Fig. 1.1, the diamond graph with an added loop. It is straightforward to show that $M(G)$ is exactly the same matroid as described in example 1.1.2(b) above.

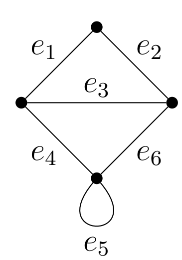


Figure 1.1: A graph G whose cycle matroid is comprised of the same ground set and collection of independent sets as in Example 1.1.2(b).

Two matroids $(E, \mathcal{I})$ and $(E', \mathcal{I}')$ are *isomorphic* if there is a bijection $f\colon E \to E'$ such that

$$A \in \mathcal{I} \Leftrightarrow f(A) \in \mathcal{I}'.$$

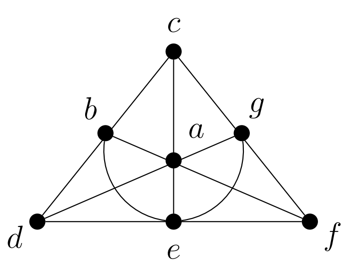
Any matroid isomorphic to $U_{k,n}$ is called k-uniform. A matroid isomorphic to a vector matroid is called a *linear* matroid, and similarly we call any matroid isomorphic to the cycle matroid of a graph a *graphic matroid.* Not all linear matroids can be represented graphically like in Example 1.1.2; consider for counterexample the Fano matroid, which we will examine in Example 1.1.5. However, the converse holds.

Definition 1.1.3. For a field F, a matroid is said to be *F-representable* or *F-linear*, or *representable over F* if it is isomorphic to the vector matroid of a matrix over field F.

For example, a *binary matroid* is a matroid which is representable over $GF(2) = \mathbb{Z}/2\mathbb{Z}$.

Proposition 1.1.4 (Oxley, Proposition 5.1.2)**.** *All graphic matroids are F-representable for every field F.*

Proof. The complete proof, found in Oxley (2011), is summarized here. Let $M(G)$ be a graphic matroid. Then it is isomorphic to a cycle matroid for some graph G; the signed adjacency matrix $A(G)$ is a linear representation of the graphic matroid $M(G)$. $\square$



$$\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Figure 1.2: (Left) The Fano matroid, formed by taking the Fano plane's seven points as the ground set and the three-element noncollinear subsets as bases. (Right) The representation of the Fano matroid as a binary matrix. Note that we can add a 1 before each of the vectors to turn it from a question of affine independence to linear independence.

Example 1.1.5. The Fano matroid, also called F_7, is an exceptional matroid for several reasons. Though it is often depicted as in Fig. 1.2, do not be fooled into thinking it is a graphic matroid! Rather, we call this diagram a *geometric representation*: the seven vertices represent elements while the seven "lines" represent affine dependencies or minimally dependent sets. Notice that there are three points on every line and three lines through every point; these points and lines cannot exist with this pattern of incidences in Euclidean geometry, and instead live in the projective two-space over the field with two elements. Moreover, the Fano matroid is actually F-representable if and only if the characteristic of F is two (Oxley (2011), Proposition 6.4.8).

If a subset of the ground set is not independent, it is *dependent*. A minimal dependent set of matroid M is called a *circuit*, the collection of which we denote $\mathcal{C}(M)$. A *cycle* refers to a union of circuits[1]. As mentioned previously, there are several equivalent ways of defining the same matroid. We will show that a matroid can defined by its independent subsets $\mathcal{I}$ or its circuits $\mathcal{C}$. To do this, we demonstrate the properties of the set of circuits are derivable from the axioms that define $\mathcal{I}$.

Theorem 1.1.6. *Let $M = (E, \mathcal{I})$ be a matroid and $\mathcal{C}(M)$ its circuits. Then the following properties of $\mathcal{C}$ hold:*

(a) $\emptyset \notin \mathcal{C}$;

(b) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$;

(c) circuit elimination axiom: If $C_1, C_2 \in \mathcal{C}$ such that $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists some $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Proof. Property 1.1.6(a) follows from the first axiom of $\mathcal{I}$, $\emptyset \in \mathcal{I}$. Since $C \in \mathcal{C}$ is minimal, 1.1.6(b) follows by definition. Then it remains to be shown that the properties of $\mathcal{I}$ imply the circuit elimination axiom.

Suppose M is a matroid such that $C_1, C_2 \in \mathcal{C}(M)$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$. A *restriction* of M to $S \subseteq E$ is defined $M|S = \{S, \mathcal{I}'\}$ where $\mathcal{I}' = \{x \in \mathcal{I} : x \subseteq S\}$. Let $S = C_1 \cup C_2$ and let $M' = M|S$. Since $C_1 \neq C_2$, we can pick some $f \in C_1 \setminus C_2$ and since $C_1 \in \mathcal{C}$ is minimally dependent $C_1 \setminus f$ must be independent. Furthermore, $M' \setminus f$ is dependent, since $C_2 \subseteq M' \setminus f$. Let B be a basis for M' containing $C_1 \setminus f$; then $f \notin B$, and it follows $B \subseteq M' \setminus f$. However since $M' \setminus f$ is dependent, $B \neq M' \setminus f$. Thus, $|B| < |M'| - 1$. Since $|(C_1 \cup C_2) \setminus \{e\}| = |M'| - 1$, we see that $(C_1 \cup C_2) \setminus \{e\}$ has too many elements to be a basis, and thus cannot be independent. $\square$

Corollary 1.1.7. *Let E be an arbitrary finite set and $\mathcal{C} \subseteq 2^E$ the power set of E satisfying (a),(b),(c) above. Then $M = (E, \mathcal{C})$ characterizes a matroid.*

The circuit elimination axiom extends to imply that for any basis B and $e \notin B$, there exists a unique circuit $C \subseteq B \cup \{e\}$. The following proposition falls out.

Proposition 1.1.8. *Let B be a basis, and let e be such that $e \notin B$. Then there exists a unique circuit $C \subseteq B \cup \{e\}$.*

Proof. Note first that since B is a basis, $B \cup \{e\}$ must be dependent and hence there exists some circuit $C \subseteq B \cup \{e\}$. Moreover, $e \in C$. Suppose there exist distinct $C_1, C_2 \in \mathcal{C}$ such that $C_1, C_2 \subseteq B \cup \{e\}$. Since any circuit contained in $B \cup \{e\}$ must contain $\{e\}$, then by Theorem 1.1.6 there would exist some $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$, contradicting the independence of $(C_1 \cup C_2) \setminus \{e\} \subseteq B$. $\square$

[1]Note this is distinct from its usage in graph theory where these terms are used in reverse, so that a *circuit* is defined as a union of *cycles*. We will proceed using the matroid terminology.

The circuit C as defined in 1.1.8 is also referred to as the *fundamental circuit of e with respect to B*, and written $C := C(B, e)$.

Proposition 1.1.9 (Strong circuit elimination axiom)**.** *Let $\mathcal{C}$ be the set of circuits of a matroid M. Then if C_1, C_2 are elements of $\mathcal{C}$ such that $e \in C_1 \cap C_2$ and $f \in C_1 \setminus C_2$, there is a member C_3 of $\mathcal{C}$ such that $f \in C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.*

Proof. First note that by definition since C_2 is a circuit, $C_2 \setminus \{e\}$ must be independent. Since any independent set can be extended to a basis, we can construct basis $B \supseteq C_2 \setminus \{e\}$ for $(C_1 \cup C_2) \setminus \{e, f\}$. It follows that B is also a basis for $(C_1 \cup C_2) \setminus \{f\}$, since $C_2 \subseteq B \cup \{e\}$ is dependent.

Similarly, $C_1 \setminus \{f\}$ must be independent, as $f \in C_1$, and so we can complete $C_1 \setminus \{f\}$ to a basis B' for $(C_1 \cup C_2) \setminus \{f\}$. This is to say, B' is maximally independent subset of $(C_1 \cup C_2) \setminus \{f\}$; moreover, since $C_1 \subseteq B' \cup \{f\}$ is dependent, it follows B' must also be a basis for $C_1 \cup C_2$. Since B and B' are both bases for $(C_1 \cup C_2) \setminus \{f\}$, it follows from the exchange axiom that they have the same size. Thus, B is also a basis for $C_1 \cup C_2$ not containing e, and $B \cup \{f\}$ is dependent, containing some circuit $C_3 \ni f$ by Proposition 1.1.8. □

Now that we've begun to get used to the vocabulary of matroids and what can be done with them, the question may arise if we can define new matroids from those that are given. As it turns out, there are several ways we can form new matroids from those that already exist.

Suppose we wish to combine two matroids by adding them. We define the *direct sum* of matroids $M = (E, \mathcal{I})$ and $N = (F, \mathcal{J})$, denoted $M \oplus N$, as the matroid whose ground set is the disjoint union of E and F and whose independent sets are disjoint unions of an independent set of M with an independent set of N. Equivalently, the independent sets of $M \oplus N$ are elements of $\mathcal{I} \times \mathcal{J}$. For example, if $G \sqcup H$ is the disjoint union of graphs G and H, then

$$M(G \sqcup H) = M(G) \oplus M(H).$$

1.2 Deletion and contraction.

Another recipe for constructing matroids from subsets of others is by deleting and contracting on elements of the ground set. A *minor* of a matroid M is another matroid N that is obtained from M by a sequence of deletion and contraction operations (described below). This method is inspired again by graph theory, in which one can delete and contract on edges of a multigraph to form new graphs. We will show that deletion and contraction generalizes to apply to all matroids.

Let $M = (E, I)$ define a matroid. A *bridge* (also called an *isthmus* in some literature[2]) is an element $e \in E$ contained in every basis. A *loop* of M is an element $e \in E$ contained in no basis (rank$(\{e\}) = 0$). If $e \in E$ is not a bridge we can get a new matroid by *deleting* e:

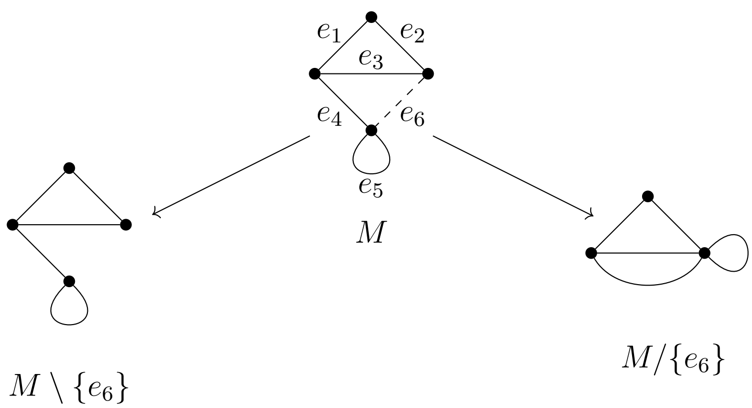
$$M \setminus e := (E \setminus \{e\}, \{I \in \mathcal{I} \colon e \notin I\})$$

[2]It is also called a *coloop*, for reasons we will learn in Section 1.3.

is the matroid formed from M by removing e from the ground set and removing all independent sets containing e. Similarly if $e \in E$ is not a loop, we can form a new matroid by *contracting* e,

$$M/e := (E \setminus \{e\}, \{I \subseteq E \setminus \{e\} : I \cup \{e\} \in \mathcal{I}\}),$$

the matroid formed from M by removing e from the ground set and all the independent sets in which it appears. In the case of a linear matroid such as in Fig. 1.3, we might observe that deletion removes a column from the matroid, while contraction adds together the rows that were separated by the contracted element.



$$\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ M \end{array}$$

$$\swarrow \qquad \searrow$$

$$\underset{M \setminus \{e_6\}}{\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}} \qquad \underset{M/\{e_6\}}{\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}}$$

Figure 1.3: (Above) Deletion-contraction decomposition on element e_6 (dashed) of the cycle matroid from Fig. 1.1. (Below) Deletion-contraction on the vector matroid isomorphic to the cycle matroid.

The majority of properties necessary to prove that $M \setminus e$ and M/e are indeed matroids follow immediately from the definition of the collection of independent sets. However, for the sake of completeness we provide an overview of the proof here.

Proposition 1.2.1. *Fix $e \in E$, and let $M' := M \setminus e$ and $M'' := M/e$.*

(a) Suppose e is a non-loop. Then $M' = (E', \mathcal{I}')$ defines a matroid.

(b) Suppose e is a non-bridge. Then $M'' = (E'', \mathcal{I}'')$ defines a matroid.

Proof. (a) $\emptyset \in \mathcal{I}'$ follows immediately, as does the fact that $e \notin J \subseteq I \in \mathcal{I}$ implies $J \in \mathcal{I}'$. Then suppose $I, J \in \mathcal{I}'$ and $|I| > |J|$; since $e \notin I, J$, there exists some $f \neq e$ such that $f \in I \setminus J$ and $J \cup \{f\} \in \mathcal{I}$. Moreover, this implies $e \notin J \cup \{f\} \in \mathcal{I}'$. (b) Since e is a non-loop, we know $e \in \mathcal{I}$ and so $\emptyset \cup \{e\} \in \mathcal{I}$, implying $\emptyset \in \mathcal{I}''$. Next, if $I \in \mathcal{I}''$ then $J \subset I$ implies $J \cup \{e\} \subset I \cup \{e\} \in \mathcal{I}$ as well, and thus $J \in \mathcal{I}''$. Finally, suppose $I, J \in \mathcal{I}''$ and $|I| > |J|$. Then there exists some $f \in (I \cup \{e\}) \setminus (J \cup \{e\})$ such that $J \cup \{e, f\} \in \mathcal{I}$, and as such $J \cup \{f\} \in \mathcal{I}''$. $\square$

One may recursively apply the process of deletion and contraction to create a *deletion-contraction tree*. This tree is easy to visualize in the case of a graph, as seen in Fig. 1.4. It is straightforward to show that if e is not a loop, G/e is the graph obtained by removing e from G and uniting its two vertices, such that $M(G/e) = M(G)/e$. Likewise, if e is not a bridge then $M(G \setminus e) = M(G) \setminus e$.

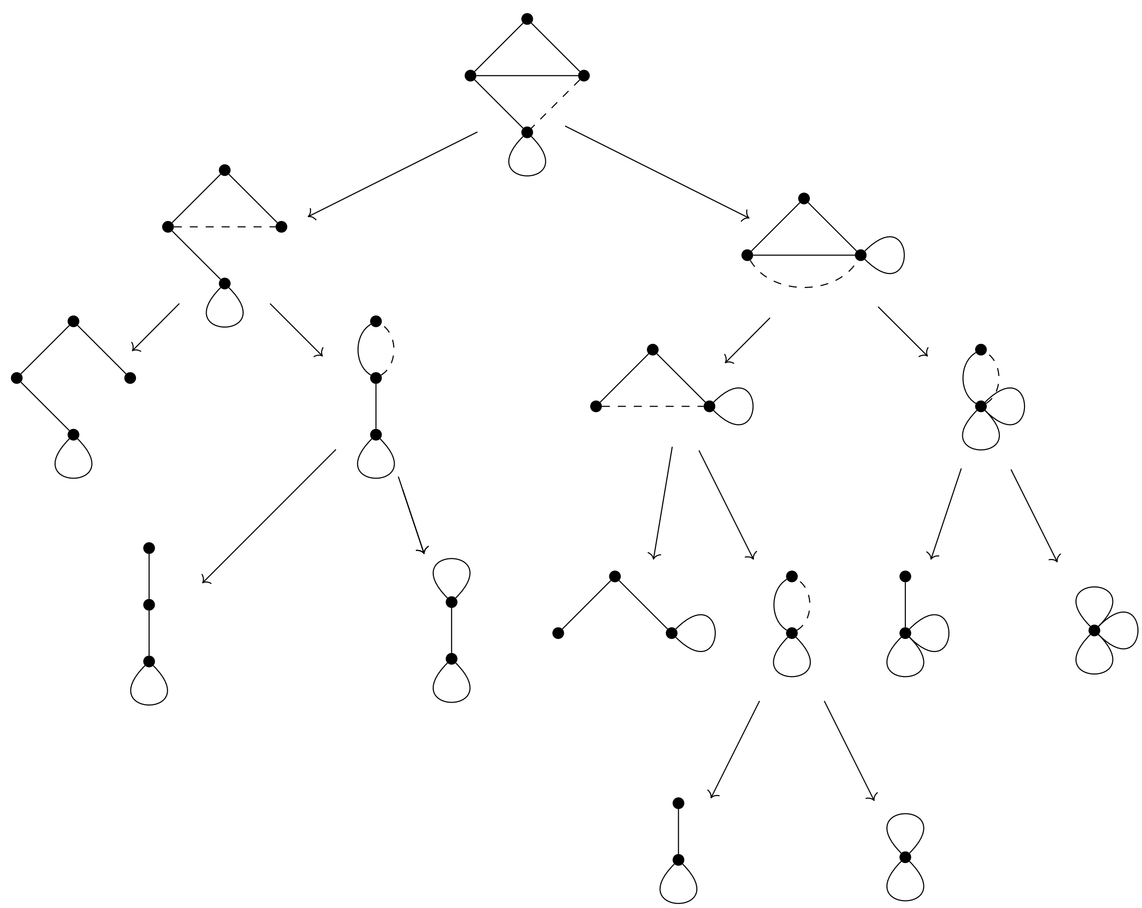


Figure 1.4: A complete deletion-contraction tree on the cycle matroid in Example 1.1.2. The "leaves" of the tree are graphs with no non-bridge or non-loop edges.

Corollary 1.2.2.

(a) For a non-bridge edge e of M, the bases of M not containing e are exactly the bases of $M \setminus e$.

(b) For a non-loop edge e of M, the bases of M containing e are exactly the bases of M/e with e added.

Corollary 1.2.2(a) follows immediately from Proposition 1.1.9, and Corollary 1.2.2 (b) follows from our definition of contraction.

Our motivation for introducing deletion and contraction for matroids comes from one recursive definition of the *Tutte polynomial*. While originally constructed for graphs, the Tutte polynomial also contains important information about matroids. We will see in Chapter 2 how to compute the Tutte polynomial $T_M(x, y)$ using deletion-contraction operations on M.

1.3 Duality

The *dual* of finite matroid M, denoted M^*, is a matroid on the same set E whose bases are the complements of the bases of M and whose independent sets are the subsets of those bases. There are many useful properties of the dual matroid that are straightforward to verify: for instance,

Proposition 1.3.1 (Oxley (2011), Chapter 2)**.** *Let $\mathcal{B}(M)$ and $\mathcal{C}(M)$ denote the collection of bases and circuits for M, respectively.*

(a) $(M^*)^* = M$.

(b) $\mathcal{B}^*(M) = \mathcal{B}(M^*)$.

(c) $\mathcal{C}^*(M) = \mathcal{C}(M^*)$.

(d) $\mathcal{B}(M) = (\mathcal{C}(M))^*$.

(e) C is a circuit of M^* if and only if it is a minimal set that intersects every basis in M.

(f) If M is representable over the field F, then M^* is also representable over F.

(g) $(U_{k,n})^* = U_{n-k,n}$.

(h) If e is not a loop, then $(M/e)^* = M^* \setminus e$. If e is not a bridge, then $(M \setminus e)^* = M^*/e$.

We call a matroid M *self-dual* if M is isomorphic to its dual M^*; for example a uniform matroid is self-dual if and only if $k = n/2$.[3] If $M(G)$ is a cycle matroid for some graph G, then $M^*(G)$ is the *cocycle matroid.* Similarly, the dual of a graphic matroid is called a *cographic matroid*. The bases, circuits, and loops of M^* are the respective *cobases*, *cocircuits*, and *coloops* of M. Recall that a loop appears in no

[3] Alternatively, some authors reserve the term self-dual to refer to matroids for which $M = M^*$.

subset of E that forms a basis for M. Then analogously a coloop appears in every subset of E forming a basis for M.

By the same naming convention, we refer to $g := \text{rank}(M^*)$ as the *corank* of matroid M. Evidently,

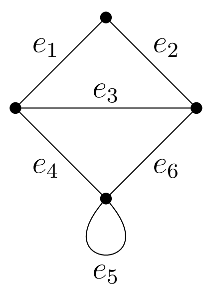
$$\text{rank}(M) + \text{rank}(M^*) = |E(M)|.$$

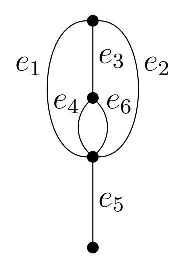
The corank will contain relevant information for us in Chapter 3 when we look at matroid cycle spaces. We can also calculate the rank of a subset of the dual matroid using the following formula.

Proposition 1.3.2 (Oxley (2011), Proposition 2.1.9)**.** *Let* rank *denote the rank function of a matroid $M = (E, \mathcal{I})$. Then the function* rank^* *for $U \subseteq E$ is*

$$\text{rank}^*(U) = |U| - \text{rank}(E) + \text{rank}(E \setminus U).$$

Example 1.3.3. Pictured left is our example $M(G)$ from Fig. 1.1, while on the right is $M(G^*)$. Note that they both have the same number of edges and vertices. However, while e_5 is a loop in $M(G)$, it is a bridge in $M(G^*)$ (equivalently a "cobridge" of M). It is important to note that the dual of a graphic matroid is not necessarily graphic.





In fact, the cographic matroid $M^*(G)$ is itself a graphic matroid if and only if G is planar. (Oxley (2011), Theorem 2.3.4).

Sometimes it may be advantageous to consider a matroid dual, and then draw conclusions about the original matroid based on properties of duality. We will see this more explicitly in Chapter 2 as we prove the h-vector conjecture for cographic matroids.

Chapter 2

Stanley's h-vector Conjecture

Now we've introduced the definition of a matroid as well as some of the important concepts from matroid theory. In this chapter, we introduce Stanley's h-vector conjecture and provide an exposition of Criel Merino's proof of the conjecture for cographic matroids. In order to state the conjecture and understand the cographic proof, we discuss the Tutte polynomial of a matroid in Section 2.1 and multicomplexes in Section 2.2.

2.1 The Tutte polynomial

The Tutte polynomial $T_M(x, y)$, defined for every matroid M, is a powerful tool for analyzing and understanding certain properties of matroids. While originally conceptualized for algebraic graph theory problems such as graph coloring and nowhere-zero flow, this special polynomial has been applied to other subjects such as statistical physics and theoretical computer science (Welsh (1999)).

We call the Tutte polynomial a *fundamental invariant* because it depends only on the isomorphism class of a matroid. For example, the Tutte polynomial of the cycle matroid for any given graph is the same regardless of edge or vertex labelings: thus, the Tutte polynomial can convey important information about a matroid without causing us to worry about the specific form or representation of the matroid from which the Tutte polynomial was derived.

The Tutte polynomial has several equivalent definitions; we present two here, in order to eventually demonstrate the connection to the h-vector via Merino's proof for cographic matroids. Proof of their equivalence can be found in Bollobás & Riordan (1999).

2.1.1 Definition by deletion-contraction

A special property of the Tutte polynomial is that it is the most general graph invariant that can be defined by a deletion–contraction recurrence.

Definition 2.1.1. The *Tutte polynomial* is computed recursively using deletion and contraction:

(a) $T_\emptyset(x, y) = 1$.

(b) If e is a loop, then $T_M(x, y) = yT_{M\backslash e}(x, y)$.

(c) If e is a bridge, then $T_M(x, y) = xT_{M/e}(x, y)$.

(d) If e is neither a loop nor a bridge, then

$$T_M(x, y) = T_{M\backslash e}(x, y) + T_{M/e}(x, y).$$

Note that if there are B edges and L loops and no other edges, $T_M(x, y) = x^B y^L$.

Proposition 2.1.2. $T_M(x, y) = T_{M^*}(y, x)$.

Proof. The proof follows by induction; we note first that $T_\emptyset(x, y) = 1 = T_{\emptyset^*}(y, x)$. Since bridges and loops are duals, it follows that if e is a loop of M (thus a bridge in M^*) then using our inductive hypothesis as well as Proposition 1.3.1(h) we may calculate directly:

$$\begin{aligned} T_M(x, y) &= yT_{M\backslash e}(x, y) \\ &= yT_{(M\backslash e)^*}(y, x) \\ &= yT_{M^*/e}(y, x) \\ &= T_M^*(y, x). \end{aligned}$$

A similar argument shows that if e is a bridge of M^* (thus a cobridge of M) then $T_{M^*}(y, x) = T_M(x, y)$. Finally, if e is neither a loop nor a bridge it follows

$$\begin{aligned} T_M(x, y) &= T_{M\backslash e}(x, y) + T_{M/e}(x, y) \\ &= T_{(M\backslash e)^*}(y, x) + T_{(M/e)^*}(y, x) \\ &= T_{M^*/e}(y, x) + T_{M^*\backslash e}(y, x) \\ &= T_{M^*}(y, x) \end{aligned}$$

as desired. $\square$

Example 2.1.3. Fig. 2.1 shows an example computation of the Tutte polynomial by deletion and contraction, for the cycle matroid of the diamond graph with loop as seen in Example 1.1.2. The resulting Tutte polynomial is

$$T_M(x, y) = x^3y + 2x^2y + 2xy^2 + xy + y^2 + y^3.$$

2.1.2 Definition by rank

Recall that the rank of a subset $S \subseteq E$ is the cardinality of a maximal independent subset of S.

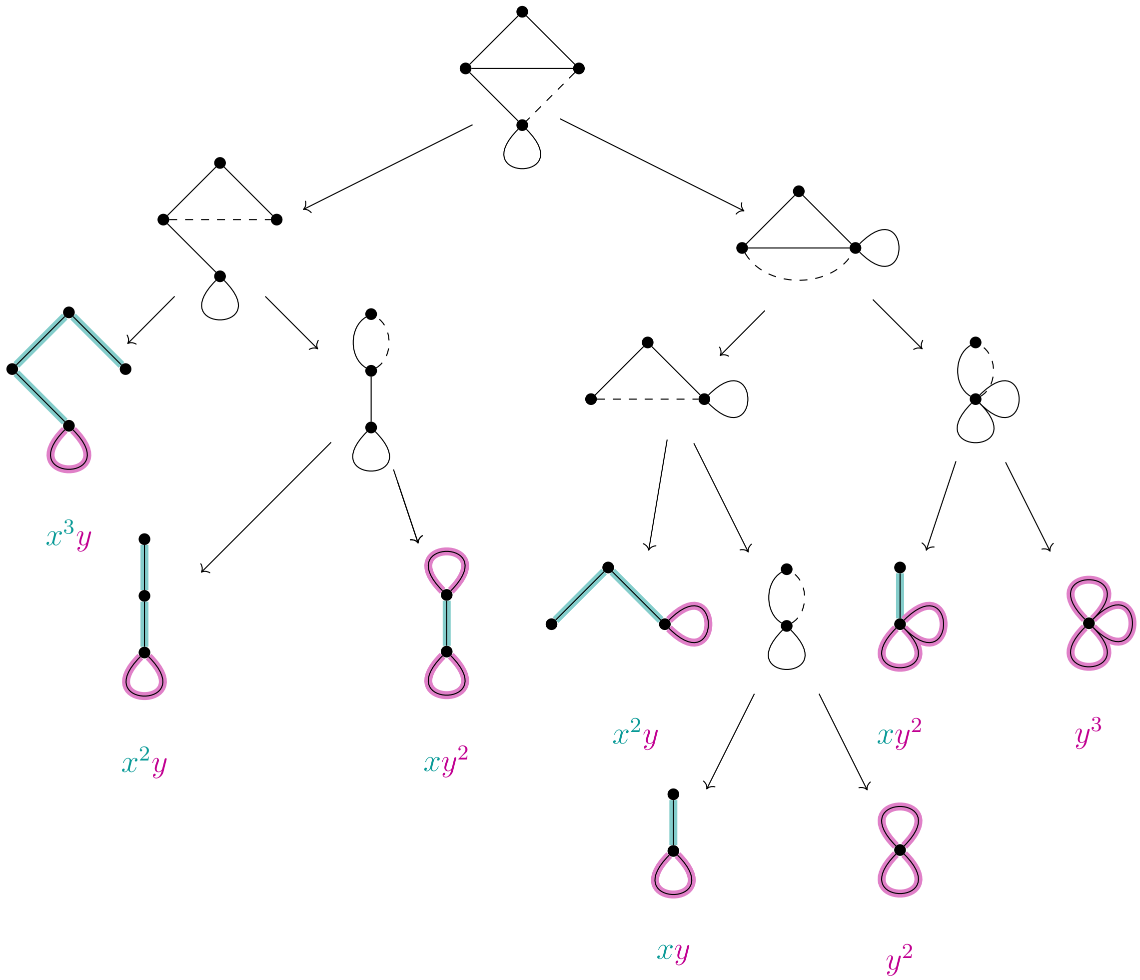


Figure 2.1: A deletion-contraction tree on the vector matroid in Example 1.1.2. The leaves of the tree correspond to monomials within the Tutte polynomial, labeled below each leaf. Note that a bridge corresponds to a power of x, while a loop corresponds to a power of y in the monomial; this follows from Definition 2.1.1.

Definition 2.1.4. The Tutte polynomial of M is

$$T_M(x,y) = \sum_{\text{subsets } S \subseteq E} (x-1)^{\text{rank}(E)-\text{rank}(S)}(y-1)^{|S|-\text{rank}(S)}.$$

Example 2.1.5. We compute the Tutte polynomial for the same matroid as in Example 2.1.3 using the rank definition. Since our summation is over all subsets $S \subseteq E$, we can partition the subsets by whether or not they contain the loop. First, consider the subsets with no loop; equivalently, this is the set of subsets of the cycle matroid of the diamond graph. These results appear in the table below.

$\lvert S\rvert$	rank(S)	polynomial term	number of subsets
0	0	$(x-1)^3$	1
1	1	$(x-1)^2$	5
2	2	$(x-1)$	10
3	2	$(x-1)(y-1)$	2
3	3	1	8
4	3	$(y-1)$	5
5	3	$(y-1)^2$	1

For the subsets that contain a loop, recall that loops have rank 0 and thus the addition of a loop to a set does not change the rank of that set. Then for each row in the previous table, we can add 1 to $|S|$ without changing the rank or number of subsets.

$\lvert S\rvert$	rank(S)	polynomial term	number of subsets
1	0	$(x-1)^3(y-1)$	1
2	1	$(x-1)^2(y-1)$	5
3	2	$(x-1)(y-1)$	10
4	2	$(x-1)(y-1)^2$	2
4	3	$(y-1)$	8
5	3	$(y-1)^2$	5
6	3	$(y-1)^3$	1

The sum of the polynomial terms in the first table is

$$\begin{aligned}1(x-1)^3+5(x-1)^2+10(x-1)+2(x-1)(y-1)+8(1)+5(y-1)+1(y-1)^2\\ = x^3+2x^2+x+2xy+y+y^2.\end{aligned}$$

Similarly, the sum of the polynomial terms in the second table is

$$x^3y-x^3+2x^2y-2x^2+2xy^2-xy-x+y^3-y.$$

Thus, their sum together is

$$x^3y+2x^2y+2xy^2+xy+y^2+y^3,$$

the same as in Example 2.1.3.

2.1.3 Specializations of the Tutte polynomial

At various points and lines of the (x, y)-plane, the Tutte polynomial evaluates to quantities that have been studied in diverse fields of mathematics and physics. Part of the appeal of the Tutte polynomial comes from the unifying framework it provides for analyzing these quantities. Examples of these include the chromatic polynomial, which specifies the number of graph vertex colorings using a set of n colors, as well as the Ising and Potts model from statistical mechanics (Welsh & Merino (2000)).

Proposition 2.1.6 (Welsh (1999))**.** *Let $M = (E, \mathcal{I})$ be a matroid.*

(a) $T_M(1, 1)$ *is the number of bases of* M*.*

(b) $T_M(2, 1) = |\mathcal{I}|$*.*

(c) $T_M(2, 2) = 2^{|E|}$ *the number of subsets of* M*.*

One explanation for the Tutte polynomial's wide range of applications is that it interacts favorably with matroid operations such as deletion and contraction. The following result demonstrates that in fact every function that 'behaves nicely' with respect to deletion and contraction arises from the Tutte polynomial.

Theorem 2.1.7. *Let f be a function from the isomorphism classes of matroids to a commutative ring R satisfying the following:*

(a) $f(M \oplus N) = f(M)f(N)$ *for all matroids* M, N*;*

(b) There exist $a, b, c, d \in R$ *such that for every matroid* M*,*

$$f(M) = \begin{cases} af(M \setminus e) + bf(M/e) & \text{if } e \text{ is not a loop or bridge,} \\ cf(M/e) & \text{if } e \text{ is not a loop,} \\ df(M \setminus e) & \text{if } e \text{ is not a bridge.} \end{cases}$$

Then for every matroid M, we can write the function f as

$$f(M) = a^{\operatorname{rank}(M^*)} b^{\operatorname{rank}(M)} T_M(c/b, d/a).$$

We may worry about the case where a or b does not have an inverse in R, which would cause their position in the denominator to become a problem. In that case, we can expand the Tutte polynomial according to its definition and cancel with the powers of a and b on the left side of the term, thus eliminating the issue.

It follows then that any function that satisfies the criteria (a) and (b) of Theorem 2.1.7 is a special evaluation of the Tutte polynomial. Some examples of such functions include Whitney's rank polynomial, the flow polynomial, and the random cluster model of Fortuin–Kasteleyn.

Corollary 2.1.8. *If $G \sqcup H$ is the disjoint union of graphs G and H, then*

$$T_{M(G \sqcup H)} = T_{M(G)} T_{M(H)}.$$

Of particular concern to this thesis is the evaluation of the Tutte polynomial of M at $y = 1$:

$$T_M(x, 1) = h_0 x^r + h_1 x^{r-1} + \cdots + h_r$$

where $r = \operatorname{rank}(M)$. We call this the *h-vector* of M, and write it

$$h = h(M) := (h_0, ..., h_r, 0, 0, \dots).$$

Remark. For convenience, we will establish the convention that two h-vectors are the same if their leading non-zero coordinates are the same. This is to say, when writing the h-vector $h = (h_0, h_1, \dots, h_r, 0, 0, \dots)$ we may omit trailing zeros and write $h = (h_0, \dots, h_r)$.

The h-vector of a matroid contains important information about its simplicial homology; we will return to it when we state Stanley's conjecture.

2.2 Multicomplexes

A multicomplex is a type of set with a partial order, arising naturally in commutative algebra in the context of monomial ideals. While this thesis has little to do with its algebraic usage, we employ a more generalized definition for a matroid multicomplex compatible with the algebraic definition. For a more in-depth background on all things multicomplexes, the author recommends Boij et al. (2012) or Stanley (1977).

The following ideas will help us define what a multicomplex is, several classifications of multicomplexes, and finally their connection to $\mathcal{O}$-sequences.

Definition 2.2.1. A *partially ordered set* (also called a *poset*) is a set P and a relation $\leq$ such that for all $x, y, z \in P$,

(a) $x \leq x$ (reflexivity);

(b) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry);

(c) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

Example 2.2.2. Let $r \in \mathbb{N}$ and $D \subseteq \mathbb{N}^r$, with the relation that for $a, b \in \mathbb{N}^r$ we have $a \leq b$ if $a_i \leq b_i$ for all i. Then D is a poset.

Fix $\mathbb{N}^r$ with the order from the example. For x, y in any poset P, we write $x < y$ if $x \leq y$ and $x \neq y$. We also say y *covers* x if $x < y$ and there is no $z \in P$ such that $x < z < y$. We often find it convenient to think of a poset P in terms of its Hasse diagram: the graph with vertex set P in which two elements x, y are connected by an edge when y covers x (Fig. 2.2). When drawn in the plane, we draw the edges of the Hasse diagram so that the larger covering element is above the smaller. We call an element *maximal* if no other element is greater in the poset.

The subset $D \subseteq P$ is called an *order ideal* if whenever $a \in P$ and $b \in D$, then $a \leq b$ implies $a \in D$. An order ideal D is *finitely generated* by $G \subseteq D$ if $|G| < \infty$ and

$$D = \langle G \rangle := \{x \in D : \text{ there exists } y \in G \text{ such that } x \leq y\}.$$

Definition 2.2.3. A *multicomplex* $\mathcal{M}$ is a finitely generated order ideal in $\mathbb{N}^r$. The *degree* of an element $a \in \mathcal{M}$ is the sum of its values, that is,

$$\deg(a) = \sum_{i=0}^{r} a_i.$$

We associate with $\mathcal{M}$ a vector $d = (d_0(\mathcal{M}), \ldots, d_r(\mathcal{M}))$ where d_i is the number of elements of $\mathcal{M}$ with degree i. We also call this the *degree sequence* of $\mathcal{M}$. If $r = 0$, then the only non-empty multicomplex is $\{()\}$, whose element has degree 0. A multicomplex is considered *pure* if all of its maximal elements have the same degree.

Example 2.2.4. Fig. 2.2 shows the Hasse diagram of a pure multicomplex, as well as a multicomplex that is not pure.

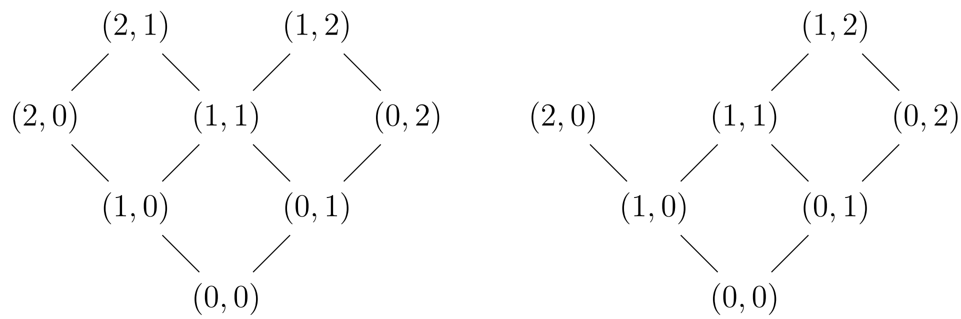


Figure 2.2: (Left) A pure multicomplex, generated by $\langle(2,1),(1,2)\rangle$, displayed as a Hasse diagram with degree sequence $(1,2,3,2)$. (Right) A multicomplex that is not pure: the maximal elements are $(1,2)$ and $(2,0)$ but $\deg(1,2) = 3 \neq \deg(2,0) = 2$.

2.2.1 Macaulay's theorem and $\mathcal{O}$-sequences

Let $\mathcal{M} \subset \mathbb{N}^n$ be a multicomplex, and let r be the highest degree of an element in $\mathcal{M}$ so that $\mathcal{M}$ has non-zero degree sequence $\deg(\mathcal{M}) = (d_0, \dots, d_r)$. A *pure $\mathcal{O}$-sequence* is a vector $h = (h_0, \dots, h_r)$ that is the degree sequence of a pure multicomplex.

Let n and d be positive integers. Every positive integer n can be written uniquely in the form

$$n = n_d := \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta}$$

such that $k_d > k_{d-1} > ... > k_\delta \geq \delta \geq 1$ are uniquely determined integers. We call this the the *d-binomial expansion of* n, or the *Macaulay expansion*, and call $k_d, k_{d-1}, ..., k_\delta$ as the *d'th Macaulay coefficients* of n.

Example 2.2.5. Let $n = 86$ and $d = 5$. The successive values of $\binom{n}{5}$ for $n = 4$, 5, 6, etc. are 0, 1, 6, 21, 56, 126, and so on, of which the largest one not exceeding 86 is 56, for $n = 8$. Therefore $k_5 = 8$, and we're left with $86 - 56 = 30$. Next, we choose $k_4 = 6$ as the greatest value for which $\binom{k_4}{4}$ is no greater than 30, take the difference, and repeat for k_3. Proceeding in this manner until the difference is 0 we get that the 5-binomial expansion of 86 is

$$86 = \binom{8}{5} + \binom{6}{4} + \binom{5}{3} + \binom{3}{2} + \binom{2}{1}.$$

The Macaulay coefficients of n are $(8, 6, 5, 3, 2)$.

Define the function ψ on $\mathbb{N}^2$ by

$$\psi(n,d) := \binom{k_d + 1}{d+1} + \binom{k_{d-1}+1}{d} + \cdots + \binom{k_\delta + 1}{\delta + 1}$$

where $k_d, \dots k_\delta$ are the d-th Macaulay coefficients of n.

Theorem 2.2.6 (Macaulay (1927)). *The vector $h = (h_0, \ldots, h_r)$ is an $\mathcal{O}$-sequence if and only if*

$$h_{d+1} \leq \psi(h_d, d)$$

for all d.

In a 1977 paper on Cohen-MaCaulay complexes, combinatorialist Richard P. Stanley conjectured the following relationship between h-vectors of a matroid and pure multicomplexes:

Conjecture 2.2.7 (Stanley (1977)). *The h-vector of a matroid is a pure $\mathcal{O}$-sequence.*

The claim is that, for any given matroid, there exists a pure multicomplex with the same degree sequence as the h-vector of that matroid.

Ever since, the so-called h-vector conjecture has been the subject of much scrutiny by mathematicians, who have proved the conjecture for various classes of matroids such as paving matroids, rank 3 matroids, corank 2 matroids, and positroids[1] (He et al. (2021)). In particular, Criel Merino's proof of the h-vector conjecture for cographic matroids utilizes the ideas of deletion and contraction, subsequently inspiring the method of proof for the main result of this thesis, Theorem 3.14.

2.2.2 Algebraic background for multicomplexes

Within the context of this thesis, it is sufficient to conceptualize multicomplexes as subsets of vectors in $\mathbb{N}^r$. However, readers with a strong background in algebra may not recognize this definition as it differs from those traditionally employed in commutative algebra. The following section attempts to reconcile the algebraic background of the multicomplex with its usage in this thesis.

Recall that Definition 2.2.3 defines a multicomplex to be a finitely generated order ideal. More precisely, we associate a monomial with each vector in the multicomplex $\mathcal{M}$ as follows:

$$\mathcal{M} \subset \mathbb{N}^d \to \Bbbk[x_1, \ldots, x_d]$$
$$a = (a_1, \ldots, a_d) \mapsto x^a := \prod_{i=1}^{d} x_i^{a_i}.$$

where $\Bbbk$ is an infinite field. We then associate a monomial ideal to $\mathcal{M}$:

$$I_{\mathcal{M}} = \{x^a : a \notin \mathcal{M}\}.$$

Let $S = \Bbbk[x_1, \ldots, x_d]$. We say S/I is the *graded* $\Bbbk$*-algebra* with dth graded piece $(S/I)_d$, defined as the span in S/I of the monomials of degree d.

The *Hilbert function* H_d is defined $H(d) = \dim_k(S/I)_d$ for $d \geq 0$. There are a limited number of values for which $H(d)$ is non-zero, which we record in the vector $h = (h_1, \ldots, h_r)$. It follows that $\mathcal{M}_d$ is a basis for $(S/I)_d$, which gives us that $\deg(\mathcal{M})$ is a $\mathcal{O}$-sequence.

[1] A *paving matroid* is a matroid with no circuit smaller than the matroid's rank; a *positroid* is a matroid realizable by a real matrix with all nonnegative submatrices.

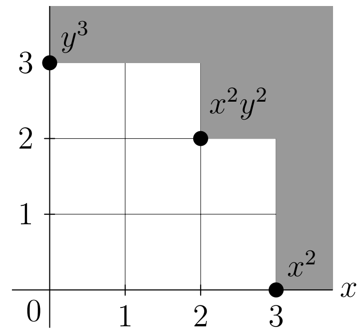


Figure 2.3: Another interpretation of the pure multicomplex in Fig. 2.2. The associated monomial ideal is $\{x^3, x^2y^2, y^3\}$, the degree vectors of which are $(3,0),(2,2),(0,3)$. If we quotient out $\mathbb{Z}^2$ by these vectors, then each point on the integer lattice equals an element in our multicomplex.

The h-vector also exists outside of its definition by the Tutte polynomial. An (abstract) *simplicial complex* Δ on $[n] := \{1,\dots,n\}$ is a subset of $2^{[n]}$ closed under inclusion. The simplicial complex is pure if all its maximal elements have the same cardinality. The *f-vector* of Δ is the vector $f_\Delta = (f_0, \dots f_d)$ where f_i where f_i is the number of faces of dimension i or the number of elements of Δ with cardinality $i+1$.

For a given matroid M, we can equivalently conceptualize M as a simplicial complex, so that a face of dimension i is one of the independent subsets having cardinality $i+1$. For a non-empty matroid $M = (E, \mathcal{I})$ of rank r, let $f_{-1} = 1$ and define

$$f(x) = \sum_{i=0}^{r} f_{i-1} x^{r-i} = \sum_{i=0}^{r} f_{r-i-1} x^i.$$

We can then define $h(x) = f(x-1)$, giving us the result

$$h(x) = f(x-1) = \sum_{i=0}^{r} h_{r-1} x^i.$$

The h-vector $(h_0, \dots, h_r)$ is the coefficients of the resulting polynomial, which we can show algebraically is equal to the definition via Tutte.

Thus, a pure $\mathcal{O}$-sequence is the h-vector of a pure order ideal or, equivalently, the h-vector of a simplicial complex. Furthermore, we see that a pure $\mathcal{O}$-sequence is also the degree vector of a pure multicomplex.

2.3 Merino's theorem

Having explained the statement of Stanley's conjecture, we may now move on to Merino's proof of the hypothesis for the class of cographic matroids. To outline Merino's inductive proof, we will use the construction of a chip-firing game, which works well with the deletion and contraction recurrence of the Tutte polynomial, as well as properties of matroid duality.

However, before we get into the details of Merino's method, we must define some vocabulary from graph theory. A *chip-firing game* is a mathematical model used to study various properties of graphs, especially in the context of algebraic combinatorics. In this game, chips or tokens are placed on the vertices of a graph G. Each turn, a vertex is selected that has at least as many chips as the number of edges incident to it. We then *fire* the vertex, and the chips are redistributed to the adjacent vertices such that each adjacent vertex receives one chip.[2]

Let $G = (E, V)$ be an undirected multigraph with finite edge set E and finite vertex set V. Of course, we are interested in the cycle matroid that arises from G more than the graph itself, but since the Tutte polynomial of an undirected multigraph is the Tutte polynomial of its cycle matroid $T_G(x, y) = T_{M(G)}(x, y)$ we can speak on the two interchangeably. Define the cycle rank (or as we know it, corank) $g = |E|-|V|+1$ and fix a *sink vertex* $s \in V$ that can never hold chips. Then $\tilde{V} := V \setminus \{s\}$ is the subset of non-sink vertices of G.

Definition 2.3.1. A *configuration* on (G, s) is a vector $c \in \mathbb{N}_{\geq 0}^{\tilde{V}}$ denoting the number of chips placed on vertex v. The set of configurations on a graph G is denoted $\mathrm{Config}(G)$. We say vertex $v \in \tilde{V}$ is *stable* if $c(v) < \mathrm{outdeg}_G(v)$, where $\mathrm{outdeg}_G(v)$ denotes the number of edges incident on v. Moreover, a configuration is called stable if every $v \in \tilde{V}$ is stable.

If c, c' are both configurations on G ($c, c' \in \mathrm{Config}(G)$), we write $c \leq c'$ if $c(v) \leq c'(v)$ for all $v \in \tilde{V}$.

Definition 2.3.2. Let $c \in \mathrm{Config}(G)$ and let $S \subseteq \tilde{V}$. Suppose c' is the configuration obtained from c by firing all the vertices in S. We call this a *legal set-firing* if $c'(v) \geq 0$ for all $v \in S$.

Definition 2.3.3. We call the configuration $c \in \mathrm{Config}(G)$ *superstable* if:

(a) $c(v) \geq 0$ for all $v \in \tilde{V}$,

(b) c has no legal non-empty set-firings, i.e. for all non-empty $S \in \tilde{V}$, there exists $v \in S$ such that $c(v) < \mathrm{outdeg}_G(v)$.

Proposition 2.3.4.

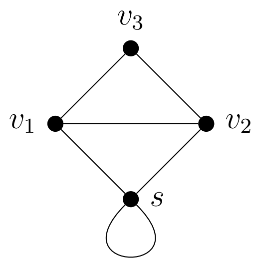
(a) Let ℓ denote the number of loops in graph G; then the maximal superstables of graph G have degree $g - \ell \leq g$.

(b) For any graph G, the collection of superstables on G forms a pure multicomplex.

Proof. This follows from our definition of superstable, in conjunction with Corry & Perkinson (2018) Corollary 4.9. □

For $i = 0, ..., g$, let d_i denote the number of superstables of degree i; we will call this the degree vector. We will show that in fact, the degree vector gives us the coefficients of the Tutte polynomial in reverse order.

[2]There are several alternative rule sets for different kinds of chip-firing games: for an extensive look at methods, see Corry & Perkinson (2018).



$$000$$
$$100, 010, 001$$
$$200, 020, 101, 011$$

Figure 2.4: (Left) The diamond graph with loop, as in Example 1.1.2, with sink s and non-sink vertices v_1, v_2, v_3. (Right) The superstable configurations on G, such that the ith digit read left-to-right is the number of chips on v_i. Note the maximal elements of this set have equal degree and that the addition of the loop does not change the superstables of the diamond graph.

Theorem 2.3.5 (Merino)**.** *Let $T(x, y) = T_G(x, y)$ be the Tutte polynomial of the undirected multigraph G. Then*

$$T(1, y) = \sum_{i=0}^{g} d_{g-i} y^i.$$

Proof. We provide a sketch of the complete proof, which can be found in Corry & Perkinson (2018). Fix s as the sink vertex of G. The proof goes by induction on the number of edges. Towards this end, we define $d_0 = 1$ when G has only a single vertex s (and possibly some loop edges) so that $T(1, y) = 1 = d_0$.

Suppose e is an edge incident to s and non-sink vertex v. There are two trivial cases: first, where e is a loop on s, we can use part (b) of Definition 2.1.1 to prove the theorem straightforwardly. Alternatively if e is a bridge, then by part (c) of Definition 2.1.1 we have $T_G(1, y) = T_{G/e}(1, y)$ and the result follows by induction.

The more interesting case, where e is neither a loop nor a bridge, has us divide the superstables into two sets:

$$A := \{c : c(v) = 0\}$$
$$B := \{c : c(v) > 0\}$$

We show that there exists a bijection between the elements of A and the superstables of G/e, as well as a bijection between the elements of B and the superstables of $G \setminus e$. The bijection between the elements of A and the superstables of G/e follows quite straightforwardly: since $c \in A$ implies $c(v) = 0$, any configuration $c \in A$ is also a configuration on G/e, and since the outdegree of $w \in W \subset \tilde{V} \setminus v$ is the same in G and G/e, then c is superstable on G/e if and only if c is superstable on G. The bijection between the elements of B and the superstables of $G \setminus e$ is also not difficult: for $c \in B$ we can subtract 1 from $c(v)$ to get a configuration on $G \setminus e$, which is superstable if and only if c is superstable on G.

Let d' and d'' be the degree vectors for G/e and $G \setminus e$ respectively. Then it follows

$$\begin{aligned}
d_i = d'_i + d''_i \Longrightarrow T(G;1,y) &= T(G \setminus e;1,y) + T(G/e;1,y) \\
&= \sum_{i=0}^{g-1} d'_{g-i-1} y^i + \sum_{i=0}^{g} d''_{g-i} y^i \\
&= \left[\sum_{i=0}^{g-1} \left(d'_{g-i-1} + d''_{g-i}\right) y^i\right] + d''_0 y^g \\
&= \left[\sum_{i=0}^{g-1} d'_{g-i} + d''_{g-i} y^i\right] + y^g \\
&= \sum_{i=0}^{g} d_{g-i} y^i. \qquad \square
\end{aligned}$$

Furthermore, by Proposition 2.1.2, we have

$$T_G(x,y) = T_{M(G)}(x,y) = T_{M(G)^*}(y,x)$$

implying

$$T_G(1,y) = T_{M(G)^*}(y,1) = \sum_{i=0}^{g} h_{g-i} y^i$$

where h is the *h-vector* for $M(G)^*$.

Merino's theorem works because chip-firing, like the Tutte polynomial, behaves nicely under the deletion-contraction recurrence. Our goal in the next chapter is to generalize Merino's theorem to a larger class of matroids. Our proof of our main result, Theorem 3.14, uses a deletion-contraction technique inspired by Merino's inductive proof by deletion and contraction on elements of the ground set of an arbitrary matroid. In an attempt to 'dualize' the proof for graphic matroids, the proof works for a general class of matroids.

Chapter 3

h-vector conjecture for matroids with cycle systems

The main result of the thesis, Theorem 3.14, demonstrates the validity of Stanley's h-vector conjecture for matroids with a cycle system. To outline our approach, we first define cycle systems and discuss the properties we will utilize. Next, we show how to associate a multicomplex of "coparking functions" to a cycle system. Through Lemma 3.13, we analyze the behavior of these coparking functions under deletion and contraction, culminating in a proof of our main theorem via induction.

Suppose $M = (E, I)$ is a vector matroid isomorphic to the $r \times n$ matrix A. Then, $E = \{1, ..., n\} = [n]$. The *cycle space* of M, denoted $\mathrm{Cyc}(M)$, is defined as the kernel of A. In Albers (2023), a cycle system is defined to be a basis of $\mathrm{Cyc}(M)$ meeting certain conditions for uniqueness. This thesis generalizes the construction of a cycle system for arbitrary matroids, including non-linear matroids for which a cycle space is not defined. Thus, we require a new definition of the cycle system.

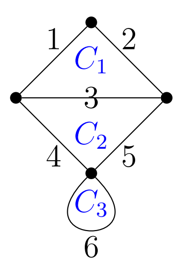
With that in mind, let $M = (E, \mathcal{I})$ be an arbitrary matroid.

Definition 3.1. Let $\mathcal{C}(M) = \{C_1, \ldots, C_g\}$ be a collection of cycles of M, where g is the corank of M. We call $\mathcal{C}$ a *cycle system* if it satisfies the **unique union property**: for all non-empty $\sigma \subseteq [g] = \{1, \ldots, g\}$, the set

$$\mathcal{C}_\sigma := \left\{ e \in \bigcup_{i \in \sigma} C_i : e \text{ appears in exactly one } C_i \text{ for } i \in \sigma \right\}$$

is dependent.

Example 3.2. Consider M the cycle matroid of the diamond graph with a loop. The following is a cycle system for M:



$\mathcal{C}(M)$	1	2	3	4	5	6
C_1	1	2	3	-	-	-
C_2	-	-	3	4	5	-
C_3	-	-	-	-	-	6

We can check that indeed for every $\sigma \subseteq [3]$ the set $\mathcal{C}_\sigma$ is dependent.

We will show more examples of matroids with cycle systems at the end of this chapter. For now, fix a cycle system $\mathcal{C} = \mathcal{C}(M) = \{C_1, \ldots, C_g\}$ for M.

Proposition 3.3. *An element e of M is a non-bridge if and only if $e \in C_i$ for some $i \in [g] := \{1, \ldots, g\}$.*

Proof. Suppose e is a non-bridge. Pick $f_g \in \mathcal{C}_{[g]}$. Up to a permutation of indices we may assume $f_g \in C_g$. If $e \in C_g$, we are done. Otherwise, consider the matroid $M \setminus \{f_g\}$ which we will show in Theorem 3.10 has cycle system $\mathcal{C}' = \{C_1, \ldots, C_{g-1}\}$.

We claim e is a nonbridge in $M \setminus \{f_g\}$. Since e is a non-bridge in M it must be contained in some circuit C in M. If C is contained in $M \setminus \{f_g\}$, the result follows immediately. Otherwise if $f_g \in C$, then $f_g \in C \cap C_g$, and we know there exists some cycle $C' \subset (C \cup C_g) \setminus f_g$ with $e \in C'$ by Proposition 1.1.9. Then C' is a cycle containing e in $M \setminus \{f_g\}$, and in this case, too, e is a non-bridge.

We now repeat this process, picking $f_{g-1} \in \mathcal{C}'_{[g-1]}$, assuming as before $f_{g-1} \in C_{g-1}$. If $e \notin C_{g-1}$, we consider $M \setminus \{f_{g-1}, f_g\}$ with cycle system $\{C_1, \ldots, C_{g-2}\}$. Continuing in this way, we will eventually find C_i such that $e \in C_i$. Otherwise, we end up with $\tilde{M} := M \setminus \{f_1, \ldots, f_g\}$ with empty cycle system and e a non-bridge element. However this is not possible since

$$g(\tilde{M}) = |E(\tilde{M})| - \operatorname{rank} \tilde{M} = 0.$$

If $|E(\tilde{M})| = \operatorname{rank}(\tilde{M})$, then there are no non-bridge elements of $\tilde{M}$. Thus, e a non-bridge must belong to some C_i.

Conversely, suppose $e \in C_i$. Then e is contained in some circuit $C \subseteq C_i$. Complete the independent set $C \setminus \{e\}$ to a basis B. Since C is dependent, we know $e \notin B$; therefore, e is a non-bridge. □

Proposition 3.4. *Choose i such that $S := \mathcal{C}_{[g]} \cap C_i$ is non-empty. Then there is a circuit in C_i containing S.*

Proof. Suppose the statement is false. Then there exist at least two distinct circuits $C, C' \subseteq C_i$ such that for $e, e' \in S$ we have $e \in C \setminus C'$ and $e' \in C' \setminus C$. Since e is a non-bridge contained in $C \setminus C' \subseteq C_i$, we may consider the matroid $N = M \setminus e$ with circuit system $\mathcal{C} \setminus \{C_i\}$. Note that e' is a non-bridge in N, since it is contained in the circuit C' of N. Then by Proposition 3.3, we have that e' belongs to a cycle in $\mathcal{C} \setminus \{C_i\}$. Hence, $e' \notin S$ since $e' \in C_j$. □

Corollary 3.5. *For each $i \in [g]$ there is at most one loop in $\mathcal{C}_{[g]} \cap C_i$.*

Definition 3.6. The set of *coparking functions* of matroid M with respect to cycle system $\mathcal{C}$, denoted $P^* = P^*(M, \mathcal{C})$, is the set of functions

$$\begin{aligned} a : \mathcal{C} &\to \mathbb{N} \\ C_i &\mapsto a_i \end{aligned}$$

where $a_i := a(C_i)$ with the property that for all non-empty $\sigma \subseteq [g]$, there exists some $i \in \sigma$ such that
$$a_i < |\,\mathcal{C}_\sigma \cap C_i|.$$
The *degree* of a coparking function a is $\deg(a) = \sum_{i=1}^{g} a_i$. The *coparking function degree sequence for* $\mathcal{C}$ is

$$\deg(a) := d(\mathcal{C}) = (d_0, d_1, \dots)$$

where d_i is the number of coparking functions having degree i:

$$d_i := |\{a \in P^*(\mathcal{C}) : \deg(a) = i\}|.$$

Define a partial order $\leq$ on $P^*(\mathcal{C})$ by $a \leq b$ if $a_i \leq b_i$ for $i = 1, \dots, g$. We write $a < b$ if $a \leq b$ and $a \neq b$, and we call b *maximal* if $a \leq b$ for all $a \in P^*$.

Remark. When it is clear from context, we may fix an ordering of $\mathcal{C} = \{C_1, \dots, C_g\}$ and write the coparking functions as vectors $a = (a_1, \dots, a_g) \in \mathbb{N}^g$.

To determine whether a non-negative integer vector $a = (a_1, \dots, a_g)$ is a coparking function with respect to $\mathcal{C}$, first set $\sigma = [g]$. Next, attempt to find $i \in \sigma$ such that $a_i < |\,\mathcal{C}_\sigma \cap C_i|$. If no such i exists, then σ demonstrates that a is not a coparking function. Otherwise, replace σ by $\sigma \setminus \{i\}$. (More generally, one may choose any non-empty subset γ of σ with the property that $a_i < |\,\mathcal{C}_\sigma \cap C_i|$, for all $i \in \gamma$ and replace σ by $\sigma \setminus \gamma$.) If $\sigma = \emptyset$, then a is a parking function. Otherwise, repeat the previous steps with the new σ.

The algorithm is finite since at each round either we have demonstrated that a is not a coparking function, or σ grows smaller. To prove the correctness of the algorithm, it remains to be shown that if the algorithm ends with $\sigma = \emptyset$, then a is a coparking function. So suppose $\sigma = \emptyset$ when the algorithm ends, and let τ be a non-empty subset of $[g]$. As the algorithm runs, elements are removed from σ until a first element $i \in \tau$ is removed. Consider the set σ just before i is removed. We have $a_i < |\,\mathcal{C}_\sigma \cap C_i|$. Then, since $i \in \tau \subseteq \sigma$, we have

$$a_i < |\,\mathcal{C}_\sigma \cap C_i| \leq |\,\mathcal{C}_\tau \cap C_i|.$$

Thus, a is a coparking function. The algorithm written as pseudocode is included as Algorithm 1.

Example 3.7. We will check that $(1, 2, 0)$ is a coparking function for Example 3.2. Set $\sigma = \{1, 2, 3\}$. Then $\mathcal{C}_\sigma = \{1, 2, 4, 5, 6\}$, which is dependent. Furthermore, $1 < |\mathcal{C}_\sigma \cap C_1| = 2$ so we can set $\sigma = \{2, 3\}$. Then $\mathcal{C}_\sigma = \{3, 4, 5, 6\}$, and $2 < |\mathcal{C}_\sigma \cap C_2| = 3$. We repeat for the final entry of a and verify that indeed $(1, 2, 0)$ is a coparking function for the fixed cycle system.

Proposition 3.8. *Let M be a matroid with cycle system $\mathcal{C} = \{C_1, ..., C_g\}$, and let $P^*(M, \mathcal{C})$ be the corresponding coparking functions. Let B be the set of bridges of M. Then P^* is a pure multicomplex whose maximal elements have degree $|E| - |B| - g$.*

Algorithm 1: Verify coparking function

```
input: C(M) = {C_1, ..., C_g}, a = (a_1, ..., a_g);
output: TRUE if a is a coparking function, FALSE if not;
σ ← [g];
while σ ≠ ∅ do
    for i ∈ σ do
        if a_i < |C_σ ∩ C_i| then
            σ = σ \ {i};
            break;                    /* break out of the for loop */
        end
        output: FALSE
    end
end
output: TRUE
```

Proof. We will first show P^* is a multicomplex; Take $a = (a_1, ..., a_g) \in P^*$ and let $b \in \mathbb{N}^g$ such that $b_i \leq a_i$ for all $i = 1, ..., g$. Because $a \in P^*$, given non-empty σ, there exists an i such that

$$a_i < |\,\mathcal{C}_\sigma \cap C_i|.$$

It then follows that

$$b_i \leq a_i < |\,\mathcal{C}_\sigma \cap C_i|,$$

and so $b \in P^*$.

Next, we will show that P^* is a pure multicomplex: that is, a coparking function is maximal if and only if it has maximal degree. Set $\sigma = [g]$. We will begin to build an ordered list $I = (i_1, ..., i_g)$ of indices of a. Starting with $I = \emptyset$, add to the set $i_1 \in \sigma$ such that

$$a_{i_1} < |\,\mathcal{C}_\sigma \cup C_{i_1}|,$$

which exists because a is a coparking function. Next, redefine $\sigma = \sigma \setminus \{i_1\}$, and identify i_2 such that $a_{i_2} < |\,\mathcal{C}_\sigma \cup C_{i_2}|$, adding i_2 to I and repeating this process until $\sigma = \emptyset$. Since by definition there exists an $i \in \sigma$ such that $a_{i_k} < |\,\mathcal{C}_\sigma \cap C_{i_k}|$ for any non-empty $\sigma \in [g]$, we know that such an i_k will always exist, guaranteeing $|I| = g$. For each $m \in [g]$, define the set

$$U_m = \mathcal{C}_{\{i_m,\ldots,i_g\}} \cap C_{i_m}$$

such that $U_1 = \mathcal{C}_{[g]} \cap C_{i_1}$, $U_2 = \mathcal{C}_{[g]\setminus\{i_1\}} \cap C_{i_2}$, and so on. Note that by our construction of the set I,

$$a_{i_m} < |\,\mathcal{C}_{\{i_m,\ldots,i_g\}} \cap C_{i_m}| = |U_m|$$

for all $m \in [g]$. We will show that we have a disjoint union

$$\bigsqcup_{m=1}^{g} U_m = E \setminus B. \tag{3.1}$$

Firstly, we note that if $e \in B$ then $e \notin C_{i_m}$ for all i_m, and thus $e \notin U_m$ for all m. So, suppose $e \notin B$. By Proposition 3.3, we know e is in some C_i. Let $k = \max\{j : e \in C_{i_j}\}$. It follows that

$$e \in \mathcal{C}_{\{i_k, i_{k+1}, \ldots, i_g\}} \cap C_{i_k} = U_k.$$

By the maximality of k, we know that $e \notin U_j$ for $j > k$. For $j < k$ such that $e \in C_{i_j}$, we see that $e \notin \mathcal{C}_{[g] \setminus \{i_1, \ldots, i_{j-1}\}}$ because $e \in C_{i_j} \cap C_{i_k}$ and thus is not unique to C_{i_j}. This establishes Eq. (3.1). We use the derived identity to calculate the maximal degree of any coparking function

$$\begin{aligned}\deg(a) = \sum_{m=1}^{g} a_{i_j} &\leq \sum_{m=1}^{g} (|U_m| - 1) \\ &= |E \setminus B| - g \\ &= |E| - |B| - g.\end{aligned}$$

Let $c = (c_1, \ldots, c_g)$ where $c_{i_m} = |U_m| - 1$ for each $i_m \in I$. The sequence of steps taken in Algorithm 1 used to verify that a is a coparking function is recorded in I; that same sequence of steps shows that c is a coparking function. By our definition of c_{i_m} we see c is maximal with degree $|E| - |B| - g$. Thus, we have proven every maximal coparking function has the same degree. Therefore, P^* is a pure multicomplex. $\square$

The following result is necessary for our proof of Theorem 3.10:

Proposition 3.9 (Oxley 9.3.6)**.** *Let M be a matroid on ground set E. Suppose C is a circuit of M and let $e \in E \setminus C$. Then either C is a circuit of M/e, or, for some integer $k > 1$, there is a partition of C into non-empty subsets $X_1, X_2, \ldots, X_k$ such that each of the sets $C \setminus X_1, C \setminus X_k, \ldots, C \setminus X_k$ is a circuit of M/e, and M/e has no other circuits contained in C.*

This proposition asserts that if $e \in E \setminus C$ then C is a union of circuits in M/e, i.e. C is a cycle in M/e. Now we can produce new cycle systems via deletion-contraction as follows.

Theorem 3.10. *Let M be a matroid with cycle system $\mathcal{C} = \{C_1, C_2, \ldots, C_g\}$.*

1. *If $e \in \mathcal{C}_{[g]} \cap C_i$, then*

$$\mathcal{C}' = \{C_1, \ldots, C_{i-1}, C_{i+1}, \ldots C_g\}$$

is a cycle system for $M \setminus e$.

2. *If e is any non-loop of M, then*

$$\mathcal{C}'' := \{C_1, \ldots, C_{i-1}, C_i \setminus \{e\}, C_{i+1}, \ldots, C_g\}$$

is a cycle system for M/e.

Proof. (1) For this part we may assume $i = g$ and let $e \in \mathcal{C}_{[g]} \cap C_g$ without loss of generality. Note that $\mathcal{C}' = \mathcal{C} \setminus C_g$. Let $\emptyset \neq \sigma \subseteq [g-1]$. Then since $\mathcal{C}$ is a cycle system and $\mathcal{C}' \subseteq \mathcal{C}$, the set $\mathcal{C}'_\sigma$ is dependent. Since $e \in C_g$, we have by Proposition 3.3 that e is not a bridge and thus

$$\begin{aligned} g(M \setminus e) &= |E \setminus \{e\}| - \operatorname{rank}(M \setminus e) \\ &= (|E| - 1) - \operatorname{rank}(M) \\ &= g(M) - 1. \end{aligned}$$

Thus, $\mathcal{C}'$ has the correct cardinality, $g - 1$.

(2) First we note that $C_i \setminus e$ is a cycle in M/e; if $e \in C_i$ the result is straightforward, and for $e \notin C_i$ it follows from Proposition 3.9. Then suppose e is a non-loop and let $\emptyset \neq \sigma \subseteq [g]$. Since $\mathcal{C}$ is a cycle system on M, we know $\mathcal{C}_\sigma$ contains a cycle C. It follows $\mathcal{C}''_\sigma \supseteq C \setminus e$, and by definition of M'' we know $C \setminus e$ is dependent in M'' since $(C \setminus e) \cup \{e\}$ is dependent in M. Furthermore,

$$g(M/e) = |E/\{e\}| - \operatorname{rank}(M/e) = (|E|-1) - (\operatorname{rank}(M)-1) = |E| - \operatorname{rank}(M) = g(M),$$

so we have the correct number of cycles. $\square$

Example 3.11. Recall the deletion and contraction on the diamond graph with loop in Fig. 1.3, where $e = \{5\}$. Then using the cycle system from Example 3.2, the cycle systems for $M \setminus e$ and M/e are as follows:

$\mathcal{C}'$	1	2	3	4	6
C_1	1	2	3	-	-
C_3	-	-	-	-	6

$\mathcal{C}''$	1	2	3	4	6
C_1	1	2	3	-	-
C_2	-	-	3	4	-
C_3	-	-	-	-	6

which we can verify are both valid cycle systems for their respective matroids.

Lemma 3.12. *Suppose $\ell \in \mathcal{C}_{[g]} \cap C_i$ is a loop, and let $a = (a_1, ..., a_g) \in P^*(M, \mathcal{C})$. Then $a_i = 0$.*

Proof. Let i be the unique element of $[g]$ such that $\ell \in C_i$; without loss of generality, $i = 1$. It suffices to show there exists $\sigma \subseteq [g]$ such that $\mathcal{C}_\sigma = \{\ell\}$, since

$$|C_j \cap \mathcal{C}_\sigma| = \begin{cases} 0 & \text{if } j \neq 1 \\ 1 & \text{if } j = 1, \end{cases}$$

which forces $1 > a_1 = 0$ by definition of the coparking function. We present an algorithm for finding the required σ.

Start with $\sigma = [g]$ and $X = \emptyset$. While $\mathcal{C} \neq \{\ell\}$, choose $e \in C_j \cap \mathcal{C}_\sigma$ for some $j \in \sigma$ such that $j \neq 1$. Remove j from σ and add e to set X.

We show that the loop halts successfully and we eventually reach σ such that $\mathcal{C}_\sigma = \{\ell\}$. Suppose it does not; then at some iteration of the loop we find some σ such that $1 \in \sigma$ and $\mathcal{C}_\sigma \subseteq C_1$. Let M' be the matroid formed by deleting the elements of X in the order they were added to X (we can delete these elements since they are non-bridges by Proposition 3.3) and let $\mathcal{C}' = \mathcal{C} \setminus \{C_j : j \notin \sigma\}$ the cycle system for M'. Then Proposition 3.4 implies $S = \mathcal{C}'_\sigma \cap C_1 = \mathcal{C}_\sigma$ is contained in a circuit of C_1. But since $\ell \in S$, then $S = \{\ell\}$, contradicting our supposition that our algorithm crashed. $\square$

Lemma 3.13. *Let $\mathcal{C} = \{C_1, ..., C_g\}$ be a cycle system for M with corresponding set of coparking functions P^*. Let $e \in \mathcal{C}_{[g]} \cap C_i$ for some i, and let $\mathcal{C}', \mathcal{C}''$ be the cycle systems for $M \setminus e, M/e$ as in Theorem 3.10.*

1. *There is an injective mapping $P^*_{M\setminus e} \hookrightarrow P^*_M$ given by*

$$(a_1, ..., a_{i-1}, a_{i+1}, \dots, a_g) \mapsto (a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_g).$$

   The image is the set $\{(a_1, \dots, a_g) \in P^* : a_i = 0\}$. If e is a loop, then this mapping is bijective.

2. *Suppose e is a non-loop. Then there is a injective mapping $P^*_{M/e} \hookrightarrow P^*_M$ given by*

$$(a_1, \dots, a_i, \dots, a_g) \mapsto (a_1, ..., a_{i-1}, a_i + 1, a_{i+1}, \dots, a_g).$$

   The image is the set $\{(a_1, ..., a_g) \in P^* : a_i > 0\}$.

Proof. Without loss of generality, let $i = g$. For part 1, let $a' = (a_1, ..., a_{g-1}) \in P^*_{M\setminus e}$ and let $a = (a_1, \dots, a_{g-1}, 0)$. We will show that $a \in P^*$ as claimed. Take some non-empty $\sigma \subseteq [g]$ and suppose $g \in \sigma$; then it follows $a_g = 0 < 1 \leq |\,\mathcal{C}_\sigma \cap C_g|$, and so $a \in P^*$. On the other hand, suppose $g \notin \sigma$: then $\mathcal{C}'_\sigma = \mathcal{C}_\sigma$. We have then that for some $i \in \sigma$,

$$a_i < |\,\mathcal{C}'_\sigma \cap C'_g| = |\,\mathcal{C}_\sigma \cap C_g|$$

and so $a \in P^*$ as desired. Here we see that if e is a loop, then it must be the only element in $\mathcal{C}_\sigma \cap C_g$ by Corollary 3.5, and since coparking functions have only non-negative integer entries, $a_g = 0 < |\mathcal{C}_\sigma \cap C_g| = 1$. If e is a loop, the injection $P^*_{M\setminus e} \hookrightarrow P^*$ is a bijection as proved in Lemma 3.12.

For part 2, recall that in Theorem 3.10 we identified for $i \in [g]$,

$$C_i = \begin{cases} C''_i & \text{if } i \neq g, \text{ and} \\ C''_i \cup \{e\} & \text{if } i = g. \end{cases}$$

Thus, for all non-empty $\sigma \subseteq [g]$,

$$\mathcal{C}_\sigma = \begin{cases} \mathcal{C}''_\sigma & \text{if } g \notin \sigma, \text{ and} \\ \mathcal{C}''_\sigma \cup \{e\} & \text{if } g \in \sigma. \end{cases}$$

It follows, with the addition of another element, that therefore

$$|\,\mathcal{C}_\sigma \cap C_i| = \begin{cases} |\,\mathcal{C}''_\sigma \cap C''_i| & \text{if } i \neq g \text{ or } g \notin \sigma, \text{ and} \\ |\,\mathcal{C}''_\sigma \cap C''_i| + 1 & \text{if } i = g \text{ and } g \in \sigma. \end{cases}$$

from which the resulting injection follows. $\square$

Remark. Suppose M has exactly g loops and b bridges. Then for $\ell = |E| - g - b$, the h-vector for M will be of the form

$$h(M) = (h_0, \ldots, h_\ell, 0, \ldots, 0)$$

with exactly b zeros at the end.

Theorem 3.14. *If $\mathcal{C} = \{C_1, ..., C_g\}$ is a cycle system for a matroid M with coparking functions P^*. Then the degree vector $\deg(P^*)$ is equal to $h(M)$ the h-vector of M.*

Proof. We proceed by induction on the number of non-bridges. Suppose that M is a matroid consisting of bridges only; then the cycle system $\mathcal{C}(M) = \emptyset$, implying $g = 0$. Since $\mathbb{N}^g = \mathbb{N}^0 = \{()\}$, we get $()$ to be the only element in our multicomplex. The degree of $()$ is 0, and we have one element of this degree, so $\deg(P^*_M) = (1)$. Meanwhile, if M is only bridges, $T_M(x, 1) = x^{\mathrm{rank}(M)}$, meaning that the h-vector of M is also (1).

Now suppose e is a non-bridge in M, so that $g > 0$. Moreover, let $e \in \mathcal{C}_{[g]} \cap C_i$ for some $i \in [g]$. We first consider the case where e is a loop. By definition of the Tutte polynomial, $T_M(x, y) = yT_{M\setminus e}(x, y)$ and hence $T_M(x, 1) = T_{M\setminus e}(x, 1)$. Because e is a loop, there is a degree-preserving bijection $P^*(M \setminus e, \mathcal{C} \setminus C_i) \to P^*(M, \mathcal{C})$ as shown in Lemma 3.13. Thus, our result follows by induction.

Now consider the case where e is not a loop or bridge, and suppose our theorem holds for $M' = M \setminus e$ and $M'' = M/e$. Let $\ell = |E| - g - b$, and denote the h-vectors for M, M', M'' like so:

$$h = (h_0, h_1, \ldots, h_\ell) \qquad h' = (h'_0, h'_1, \ldots, h'_\ell) \qquad h'' = (h''_0, h''_1, \ldots, h''_{\ell-1})$$

and coparking function degree sequences

$$\deg(P^*) = (d_0, d_1, \ldots, d_\ell) \quad \deg(P^*_{M\setminus e}) = (d'_0, d'_1, \ldots, d'_\ell) \quad \deg(P^*_{M/e}) = (d''_0, d''_1, \ldots, d''_{\ell-1}).$$

By Lemma 3.13, it follows

$$\begin{aligned}\deg(P^*) = (d_0, d_1, \ldots, d_\ell) &= (d'_0, d'_1, \ldots, d'_\ell) + (d''_0, d''_1, \ldots, d''_{\ell-1}) \\ &= (d'_0 + d''_0, \ldots, d'_{\ell-1} + d''_{\ell-1}, d'_\ell).\end{aligned}$$

Recall from our definition of the Tutte polynomial that

$$T_M(x, y) = T'_{M\setminus e}(x, y) + T''_{M/e}(x, y)$$

which can be expanded as

$$\sum_{i=0}^{\ell} h_{\ell-i} x^i = \sum_{i=0}^{\ell} h'_{\ell-i} + \sum_{i=0}^{\ell-1} h''_{\ell-1-i} x^i$$

and equivalently written

$$(h_0, \ldots, h_{\ell-1}, h_\ell) = (h'_0, \ldots, h'_{\ell-1}, h'_\ell) + (h''_0, \ldots, h''_{\ell-1}, 0).$$

We assumed inductively that $h' = \deg(P'_{M\setminus e})$ and $h'' = \deg(P''_{M/e})$. Combining this with our work thus far, we see

$$\begin{aligned} h &= (h_0, \dots, h_{\ell-1}, h_\ell) = (h'_0, \dots, h'_{\ell-1}, h'_\ell) + (h''_0, \dots, h''_{\ell-1}, 0) \\ &= (h'_0 + h''_0, \dots, h'_{\ell-1} + h''_{\ell-1}, h'_\ell) = (d'_0 + d''_0, \dots, d'_{\ell-1} + d''_{\ell-1}, d'_\ell) = \deg(P^*). \end{aligned}$$

□

Corollary 3.15. *The h-vector conjecture holds for matroids with cycle systems.*

Example 3.16. Recall our calculation of the Tutte polynomial for the diamond graph with loop in Example 2.1.3. We extract h-vector of the cycle matroid from its coefficients:

$$T_M(x,1) = x^3 + 2x^2 + 3x + 2 \implies h(M) = (1,2,3,2).$$

Let M have the cycle system from Example 3.7. Starting from the bottom of the deletion-contraction tree, we can build the associated coparking functions as described by Lemma 3.13 for each matroid. The maximal elements are $(2,1,0)$ and $(1,2,0)$, giving rise to the following multicomplex:

$$\begin{tikzcd}[arrows=dash]
 & (2,1,0) \arrow[dl] \arrow[dr] & & (1,2,0) \arrow[dl] \arrow[dr] & \\
(2,0,0) \arrow[dr] & & (1,1,0) \arrow[dl] \arrow[dr] & & (0,2,0) \arrow[dl] \\
 & (1,0,0) \arrow[dr] & & (0,1,0) \arrow[dl] & \\
 & & (0,0,0) & &
\end{tikzcd}$$

which has degree sequence $(1,2,3,2)$. Thus, $h(M) = \deg(P^*(M))$ as desired.

There are several classes of matroids, for which the h-vector conjecture has been confirmed, that we can prove have cycle systems. We present here some examples of such classes.

Cographic matroids. Let M be the cycle matroid for a connected multigraph $G = (V, E)$. The bases for M are the spanning trees of G, each of which contains $|V| - 1$ elements. Hence, $g(M) = |E| - |V| + 1$. Since a basis for the dual matroid is by definition the complement of a basis for M, it follows that the rank of the dual matroid M^* is $|E| - \operatorname{rank}(M)$.

Let U be a proper non-empty subset of V: the *cut set* of G corresponding to U, denoted C^*_U, is the set of edges with one vertex in U and the other in $V \setminus U$. In particular, if U is comprised of a single vertex v for some $v \in V$, then $C^*_v := C^*_U$ is called a *vertex cut set*. If a cut has no nonempty proper subsets of edges that are cuts (i.e. if it is minimal with respect to inclusion), it is called a *bond* or *cocircuit* of G. Moreover, every cut is a disjoint union of bonds and the bonds of M are exactly

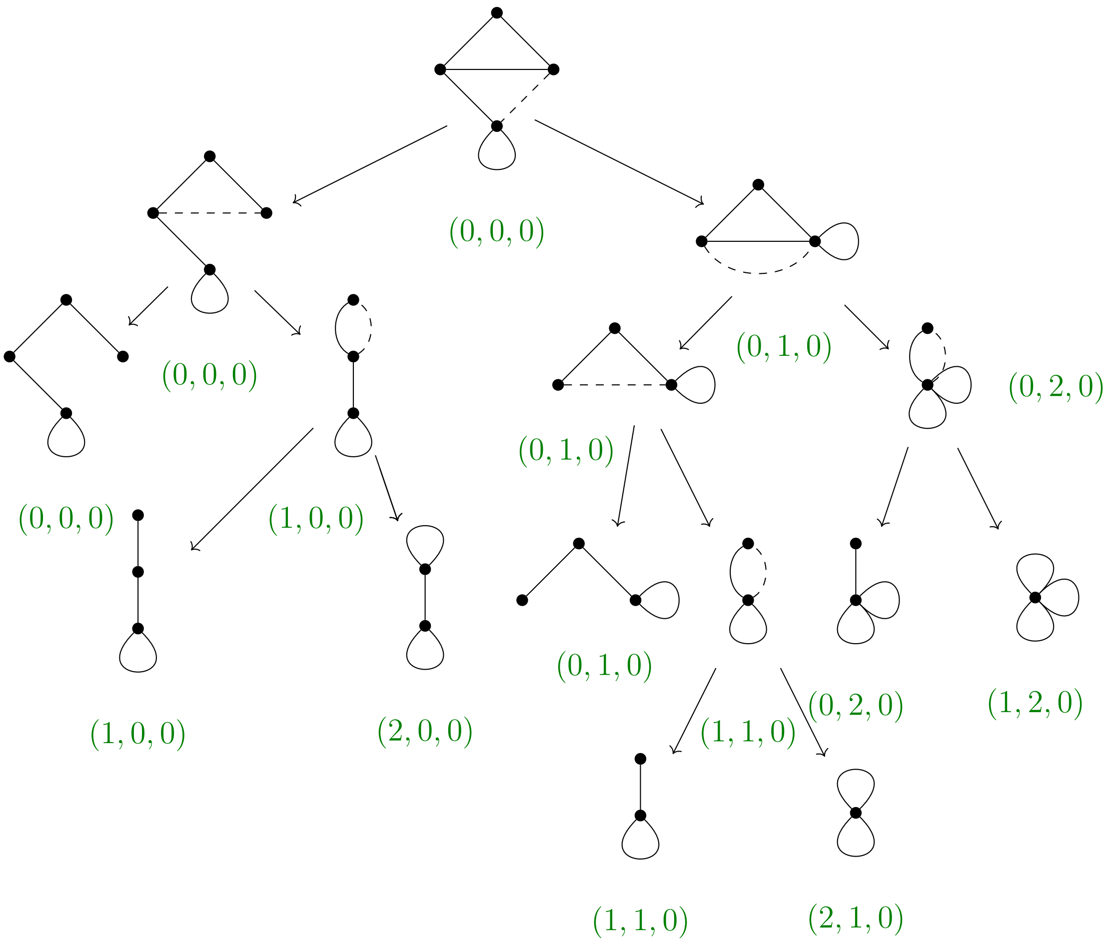
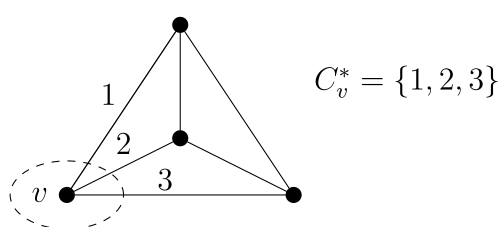


Figure 3.1: The deletion-contraction tree for the diamond graph with loop, where each graph is associated with a green coparking function.

the circuits of M^* (Bondy & Murty (2008)). Below, we see one example of a vertex cut set for the complete graph on 4 vertices.



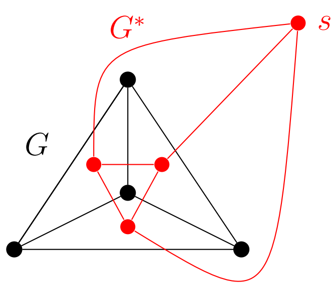
Fix a vertex $s \in V$. We claim that $\mathcal{C} = \{C_v^* \colon v \in V \setminus \{s\}\}$ is a cycle system for M^*. First, we verify that indeed the set contains the correct number of elements $g(M^*) = |V| - 1$. Let U be a nonempty subset of $V \setminus \{s\}$. We claim the unique union of vertex cuts $\{C_v^* \colon v \in U\}$ must itself contain a cut: moreover, the unique union of vertex cuts $\{C_v^* \colon v \in U\}$ is C_U^*. Thus we will show that $e \in C_U^*$ if and only if $e \in C_v^*$

for a unique $v \in U$. To show this, let e be in the vertex cut set for some $v \in U$. Then e is incident to v and w for some $w \in V$. If $w \in U$, then $e \in C_v^*$ and $e \in C_w^*$, and thus is not in the unique union of vertex cut sets, and since $w \notin V \setminus U$ it is also not in C_U^*. If $w \in V \setminus U$, then $e \in C_U^*$ by definition and $e \notin C_u^*$ for any other $u \in U$.

Then since every cut is a disjoint union of bonds, and a bond is by definition dependent in M^*, it follows that C_U^* contains a dependent set.

Planar graphs. Let M be the cycle matroid for a connected planar multigraph $G = (V, E)$. We claim the faces of the graph, the regions bounded by edges, comprise a cycle system for M.

This is because the dual matroid, M^*, is isomorphic to the cycle matroid for the graph dual G^* of G. Hence, M is cographic. If we let s be the vertex of G^* corresponding to the unbounded face (the region "outside" of the graph) of G, it follows from our proof above that the dual of M^* has a cycle system made from the cut sets of all non s vertices, which correspond to the bounded faces of G. See K_4 and K_4^* below for example.



Thus, the cycle system described above corresponds to the set of cycles forming the faces of G.

Cones over graphs. Let $G = (V, E)$ define a graph, and let s be a point not in V. The *cone over* G *with cone point* s, denoted $\operatorname{cone}(G, s)$, is the graph formed from G by adding an edge from s to each $v \in V$. For each non-loop $e \in E$ connecting u, v of G, let C_e be the cycle consisting of e, the edge from u to s and the edge from s to v. If $e \in E$ is a loop of G, let C_e be that loop. We claim $\mathcal{C} = \{C_e : e \in E\}$ forms a cycle system for $\operatorname{cone}(G, s)$.

We first verify that $\mathcal{C}$ has the correct cardinality; since the cone has an extra edge for each vertex in G,

$$\begin{aligned} g(\operatorname{cone}(G, s)) &= |E(\operatorname{cone}(G, s))| - |V(\operatorname{cone}(G, s))| + 1 \\ &= (|E| + |V|) - (|V| + 1) + 1 \\ &= |E| \\ &= |\mathcal{C}|. \end{aligned}$$

Now let X be a proper, non-empty subset of E and let $U = \{C_e : e \in X\}$ be the unique union of the set of cycles of $\operatorname{cone}(G, s)$. Note $X \subseteq U$. Then if X contains a cycle of G, it follows U contains a cycle. Otherwise, if X is a forest, we may pick a path $P \subseteq X$ connecting two distinct leaves u, v of the tree. It follows that U contains the cycle path from s to u to v back to s.

Complete graphs. Consider K_3 the complete graph with 3 vertices. Since this is a planar graph, it follows from our work above K_3 has a cycle system. Then since K_4 is essentially a cone on K_3 and K_5 a cone on K_4, it follows by induction that each complete graph has a cycle system.

Chapter 4

Cycle system-to-DC diagram

Thus far, we have seen several examples of how cycle systems lead to "deletion-contraction trees," a data structure that illustrates the recursive process of deletion and contraction. In this chapter we see that the addition of a groundset ordering allows us to use these trees to create bijections between bases and coparking functions.

Definition 4.1. A *binary tree* is a planar tree recursively defined as a tuple (L, S, R), where L and R are binary trees (or the empty set) and S is a singleton containing the *root* node. Each node in the binary tree has at most two *child* nodes whose path distance from the root is one more than their parent. Each child of a node is designated as its left or right child, and the *leaves* of the binary tree are those nodes with no children.

In our case, the 'root' or topmost node of the tree represents the initial matroid M, while the left child $M \setminus e$ is a deletion of non-bridge element e and the right child M/e is a contraction of that same element (for example, see Fig. 1.4).

Definition 4.2. A *deletion-contraction* or *DC diagram* of matroid M, written $D(M)$, is an edge-labeled binary tree defined recursively such that $D(M) = M$ if M is comprised of bridge and loop elements only. Otherwise, $D(M) = (D(M'), M, D(M''))$ where $M' = M \setminus e$ and $M'' = M/e$ for some non-loop, non-bridge $e \in E(M)$.

However these diagrams do more than illustrate deletion and contraction on a given element: they also provide the underlying structure for a multicomplex of coparking functions, which we can relate to bases, terms within the Tutte polynomial, or other aspects of a matroid that we wish to investigate. Corollary 1.2.2 demonstrated a relationship between the deletion and contraction of $e \in E(M)$ with the bases of M, and Theorem 3.10 showed us that these two children inherit cycle systems from M. We put this information together to find that if M has a cycle system, the leaves of the deletion-contraction tree are in bijection with the set of coparking functions and the set of bases of M.

DC diagram construction. Given a matroid M with cycle system $\mathcal{C}$ and a fixed groundset ordering ξ, we can build a uniquely determined DC diagram. The process for this goes as follows: begin by labeling the root of the DC diagram M. Let $\sigma = [g]$, let e be the maximal element in $\mathcal{C}_\sigma$ with respect to ξ, and let i be the index of the uniquely determined element $C_i \in \mathcal{C}$ such that $e \in \mathcal{C}_\sigma \cap C_i$. If e is a loop, we may replace σ with $\sigma \setminus \{i\}$ and redetermine the maximal element, repeating until e is a non-loop or $\sigma = \emptyset$. By Lemma 4.4, the latter occurs when M consists only of bridges and loops, in which case the DC diagram consists solely of the root vertex. Otherwise, there exists some non-loop element $e \in \mathcal{C}_\sigma$ for non-empty σ, which by Proposition 3.3 is also a non-bridge. The deletion-contraction tree is formed by joining its root vertex to a DC diagram for $M \setminus e$ on the left and a DC diagram for M/e on the right with cycle systems $\mathcal{C}'$ and $\mathcal{C}''$ respectively, following the notation of Theorem 3.10. Thus, the indices for $\mathcal{C}'$ are $[g] \setminus \{i\}$. The matroids $M \setminus e$ and M/e also inherit the total ordering on the ground set from M.

Next label the right edges of the tree with the pair (e, i) to keep track of the element that was contracted and the cycle to which it belonged. Finally, we complete the construction of the edge-labeled DC diagram by recursion, so that our leaves are matroids that contain bridges and loops only.

Definition 4.3. The *DC diagram with respect to cycle system $\mathcal{C}(M)$ and groundset ordering* ξ is the binary tree with labeled right edges, whose construction is described above.

Lemma 4.4. *There must exist some $\sigma \subseteq [g]$ such that the maximal element of C_σ is a non-loop, or else E consists only of bridges and loops.*

Proof. Let $\sigma = [g]$ and let e be the maximal element of $\mathcal{C}_{[g]}$. If e is a loop, we may replace σ with $\sigma \setminus \{i\}$ where $e \in C_i$. Suppose we repeat the process for every maximal element $e \in \mathcal{C}_\sigma$, until eventually $\sigma = \emptyset$. Since we started with $|\sigma| = g$ and removed one cycle at each step, we removed exactly g loops from the ground set E. Each base for our matroid consists of exactly $r = |E| - g$ elements, and no base can contain a loop. Therefore, if $\sigma = \emptyset$, the remaining r elements constitute the unique base of M, and each element in the base is a bridge. Thus, in this case, our original matroid consists entirely of loops and bridges. □

4.1 Leaf-to-Coparking Function

The process described so far is everything needed to associate a coparking function with each leaf of the corresponding DC diagram. Let M be a matroid with cycle system $\mathcal{C}$ and groundset ordering ξ. Recall that each contraction was labeled with a cycle index i; taking the path from the leaf to the root, we may then assign to each leaf a coparking function a such that

$$a = (\#\{1s \text{ in path}\}, \ldots, \#\{gs \text{ in path}\}).$$

This is to say the ith component of a records the number of times an element of C_i was contracted to reach the leaf corresponding to a. By Lemma 3.13, a is a coparking function and every coparking function arises uniquely in this way. To verify uniqueness we can compare the coparking functions of two leaves by examining their most recent common ancestor N. At this point in the path from root to N, both leaves have the same associated coparking function. However, deleting an element $e \in C_j$ will mean that j is removed from σ, thus ensuring that no more will be added to the jth entry of the coparking function. However, contracting this same element $e \in C_j$ adds one to the jth entry of the coparking function. Thus, any leaf to the left of N will have a different value for a_j than any leaf to the right of N.

4.2 Basis-to-Coparking Function

We have just seen that fixing a cycle system and a groundset ordering of the matroid M determines a DC diagram, and we saw that leaves of that diagram are in bijection with both the bases and coparking functions of M. Therefore, we may use the DC diagram to create a bijection between bases and coparking functions.

First, note that we can associate each leaf of the deletion-contraction tree to a basis for M. Start with the set of bridges in the leaf, which is a unique basis for the leaf. Then we can complete a basis for M by adding every edge that was contracted in the creation of that leaf. It is straightforward to show that the constructed set is a basis, by our definitions of independence under deletion and contraction from Chapter 1. The fact that each leaf is uniquely associated with a basis follows from Corollary 1.2.2, which also tells us that the bases are in bijection with the leaves. Then since we also have a bijection between the leaves and coparking functions by Lemma 3.13, the algorithm for the basis-to-coparking function is bijective.

The algorithm for obtaining a specific coparking function from a basis is outlined in Algorithm 2, and is described as follows. Given a basis B, we start by letting $\sigma = [g]$ and $a = \vec{0} \in \mathbb{N}^g$. Let $T = E \setminus B$. Find the elements of the unique union $\mathcal{C}_\sigma$, and let $e = \max(\mathcal{C}_\sigma)$. We take $\ell(e)$ to be the index of the cycle such that $e \in \mathcal{C}_\sigma \cap C_{\ell(e)}$ uniquely. If e is not a part of the basis, we essentially delete it by replacing σ by $\sigma \setminus \ell(e)$. Then if $e \in B$ our basis, we contract it by reassigning $C_{\ell(e)}$ to $C_{\ell(e)} \setminus \{e\}$ and adding one to the $\ell(e)$th entry of a. We continue with this process until σ is empty, meaning that we have run into g elements that were not a part of our basis but were uniquely occurring in a cycle. Thus, we have added to our coparking function only when elements of the original basis were contracted, paralleling how we got to the basis from the leaf.

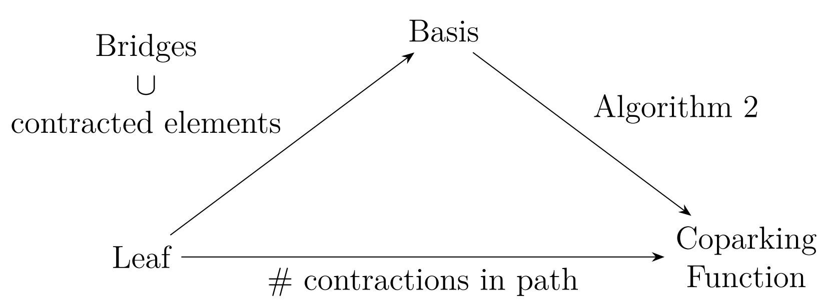
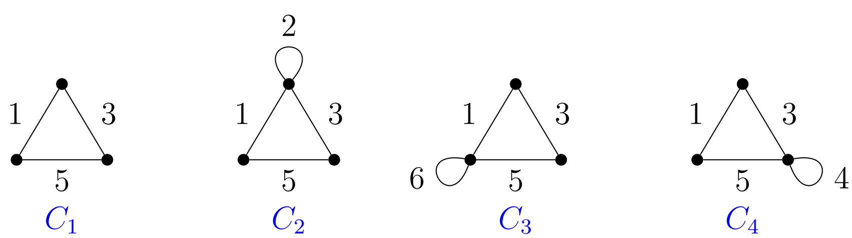


Figure 4.1: A commutative diagram illustrating the functions that take us from one construction to another.

Algorithm 2: Base-to-Coparking Function

input: cycle system $\mathcal{C} = \{C_1, \ldots, C_g\}$ for matroid M, total ordering $\prec$ on the n elements of groundset E, basis $B \subset E$;
output: coparking function a with respect to $\mathcal{C}$, function $\ell \colon E \to [g]$ labeling a groundset element with the index of an element of $\mathcal{C}$;
$\sigma \leftarrow [g]$;
$T \leftarrow [n] \setminus B$; `/* the non-base elements */`
$a \leftarrow \vec{0} \in \mathbb{N}^g$;
while $\sigma \neq \emptyset$ **do**
  $C_\sigma = \{i \in [n] :$ there exists unique $j \in \sigma$ such that $i \in C_j\}$;
  $e \leftarrow \max(C_\sigma)$;
  $\ell(e) \leftarrow j \in \sigma$ such that $e \in C_j$;
  if $e \in T$ **then**
    $\sigma \leftarrow \sigma \setminus \{\ell(e)\}$;
  end
  else
    $a_{\ell(e)} \leftarrow a_{\ell(e)} + 1$;
    $C_{\ell(e)} \leftarrow C_{\ell(e)} \setminus \{e\}$;
  end
end
return a, ℓ

Example 4.1. Let $\mathcal{C}$ be a cycle system for the cycle matroid M of a triangle with a loop appended to each vertex, defined by



.

The corresponding DC diagram appears in Fig. 4.2. For each leaf, we trace the path up to the root and construct the coparking function a by starting with the zero vector in $\mathbb{N}^g$, and adding one to the jth entry of a whenever an edge $e \in C_j$ is contracted. Similarly, we can arrive at a basis by taking the union of contracted elements with the bridges in the leaf. We can finally verify that this relationship is bijective using our base-to-coparking function algorithm and coparking function-to-base algorithm, described below, to go back and forth. We furthermore note that, as expected, $T_M(x, y) = x^2 + x + 1$ and $\deg(P^*(M)) = (1, 1, 1)$.

To arrive at a base from a coparking function, we set basis $B = [n]$ and $\sigma = [g]$, and observe the maximal element in the unique union. If that element is from a cycle with index i and $a_i = 0$, we remove that element from the unique union and basis, and set $\sigma = \sigma \setminus \{i\}$. If $a_i > 0$, we remove that element from the unique union and decrease a_i by one. The elements that remain in the basis when $\sigma = \emptyset$ are the elements of the basis that correspond with the coparking function.

Remark. Algorithm 3 may raise some flags concerning loops; however, we needn't worry because while it is possible that a loop is the maximum ordered element of $\mathcal{C}_\sigma$, we recall that by Lemma 3.12 if a loop is in the unique union for some σ and some cycle i, then $a_i = 0$ in the associated coparking function.

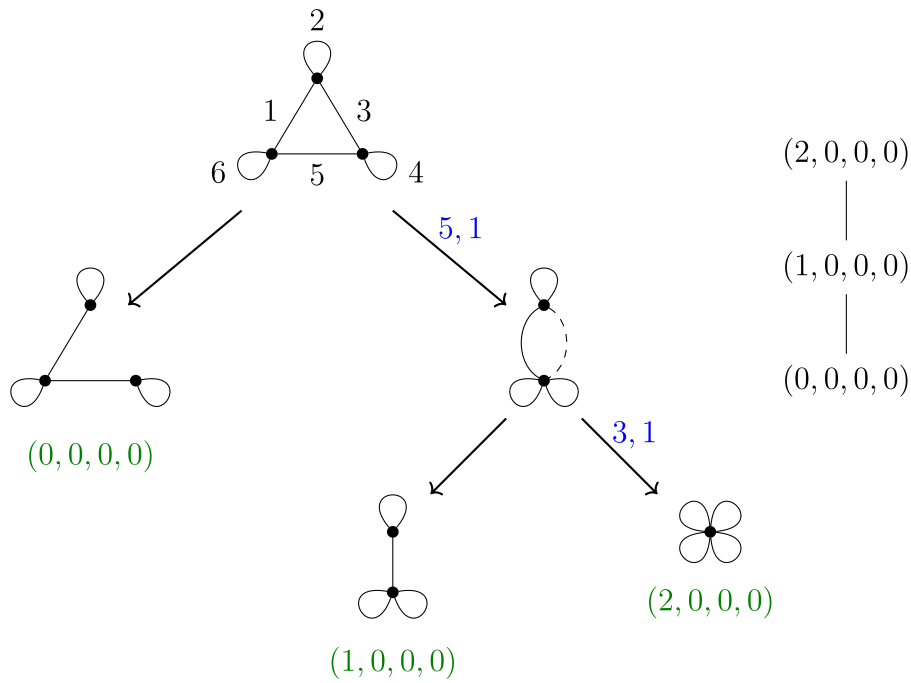


Figure 4.2: (Left) A DC diagram for the cycle matroid of the triangle with three loops. (Right) The resulting multicomplex with three elements.

Example 4.2. Consider Example 3.16 of the diamond graph with loop, with the cycle system from Example 3.2 and with edge-ordering $\xi : 1 < 2 < 3 < 4 < 5 < 6$. We will walk through the application of Algorithm 3, the coparking function-to-basis algorithm, to obtain bases from coparking functions in the associated multicomplex, and note that they align with the leaves associated with each coparking function.

(a) **(0,0,0).** Set $\sigma = [g]$ and $B = [6]$; then $\mathcal{C}_\sigma = \{1, 2, 4, 5, 6\}$. The maximal element is $6 \in C_3$. Immediately, since $a_3 = 0$, we know 6 is not in our basis so replace B with $B \setminus \{6\}$. Yet σ is not empty, so we must continue. Now $\mathcal{C}_\sigma = \{1, 2, 4, 5\}$, and $5 \in C_2$ is maximal. Again, since $a_2 = 0$, we know 5 cannot be a part of our basis. Thus we remove 5 from B. Finally $\mathcal{C}_\sigma = \{1, 2, 3\}$, hence $3 \in C_1$ is maximal. However, $a_1 = 0$ so we know 3 cannot be in our basis. Remove 3. Then since $\sigma = \emptyset$, our basis is $\{1, 2, 4\}$. We note these are the bridges of the leaf with coparking function $(0, 0, 0)$ in Example 3.16.

Algorithm 3: Coparking Function-to-Base

input: cycle system $\mathcal{C} = \{C_1, \ldots, C_g\}$ for matroid M, total ordering $\prec$ on the n elements of groundset E, coparking function a with respect to $\mathcal{C}$;
output: basis B of M, function $\ell \colon E \to [g]$ labeling a groundset element with the index of an element of $\mathcal{C}$;

```
σ ← [g];
B ← [n];
while σ ≠ ∅ do
    C_σ = {i ∈ [n] : there exists unique j ∈ σ such that i ∈ C_j};
    done = false;
    while done = false do
        e ← max(C_σ);
        ℓ(e) ← j ∈ σ such that e ∈ C_j;
        𝒞_σ ← 𝒞_σ \ {e};
        a_ℓ(e) ← a_ℓ(e) − 1;
        if a_ℓ(e) < 0 then
            σ ← σ \ {ℓ(e)};
            B ← B \ {e}; /* non-contracted element removed from B */
            done=true;
        end
    end
end
return B
```

(b) **(2,0,0).** Since $a_3 = 0$ and $a_2 = 0$, the same process follows from above in this example for the first two steps. However, when we get to $\sigma = \{1\}$ and $\mathcal{C}_\sigma = \{1, 2, 3\}$, this time we subtract 1 from a_1 and 3 remains in B. Now $\mathcal{C}_\sigma = \{1, 2\}$, but a_1 is still positive, so we subtract again. Finally $\sigma = \{1\}$ and $\mathcal{C}_\sigma = \{1\}$, but $a_1 = 0$ so we remove 1 from B. Then the elements left in B are $\{2, 3, 4\}$, which is the basis corresponding to the leaf in Example 3.16.

(c) **(1,2,0).** Remove $6 \in C_3$ as before. Now when we get to maximal element $5 \in C_3$, we subtract one from a_2 and 5 remains in B. The next maximal element, $4 \in C_2$, also remains in B and a_2 is decreased again. When $\sigma = \{1, 2\}$ and $\mathcal{C}_\sigma = \{1, 2\}$, we get $2 \in C_1$ maximal and so we decrease a_1. Finally, $\mathcal{C}_\sigma = \{1\}$ but $a_1 = 0$, so we remove 1 from B. Thus our remaining elements are $\{2, 4, 5\}$, corresponding to the leaf in the DC diagram where every maximal element was contracted.

The bijections between leaves, bases, and coparking functions allow us insight into why the proof of the h-vector conjecture for matroids with cycle systems works. As long as we can correctly identify what groundset elements to recurse on, the DC diagram simultaneously has the structure of the recursion defining the Tutte polynomial and a pure multicomplex.

Chapter 5

Future work

Having shown that the h-vector conjecture holds for those matroids with cycle systems, and that a choice of cycle system and groundset ordering gives a uniquely determined DC diagram, our attention now turns to some related questions that inspire future work.

Question 1. If a matroid has a cycle system, does this imply the existence of a cycle system consisting of circuits?

Evidence would seem to support this: for several examples of matroid classes with cycle systems, we have been able to make cycle systems of strictly circuits (see Chapter 3 on planar graphs and cones). We also have as a result of Proposition 3.4 that elements in the unique union of $\mathcal{C}_\sigma \cap C_i$ belong to a single circuit $C \subseteq C_i$. Similarly, if e is a loop, at most one loop uniquely belongs to a $C_i \in \mathcal{C}$ by Corollary 3.5. This would seem to suggest that we could restrict cycle C_i to the circuit containing the unique union, or the unique loop if it has one. Doing so for each $C_i \in \mathcal{C}$ might result in a cycle system of circuits.

However, issues could potentially arise in trying to capture every necessary circuit; for instance, if there exist σ and σ' for which C_i has different unique elements, we would need to figure out which circuit to restrict to so that the rest of the cycle is still covered. There is also no guarantee that the DC diagram would continue to uniquely generate the bijections between leaves, coparking functions, and bases. Thus far, neither a proof nor counterexample has so far been derived to conclude if the existence of a cycle system $\mathcal{C}(M)$ implies the existence of a "circuit system".

Question 2. How do we characterize matroids with cycle systems? Perhaps there is a list of forbidden subgraphs or minors preventing a graph from having one.

Those familiar with Kuratowski's theorem may recognize the concept of *forbidden subgraphs* K for which, if $K \subseteq G$, then the graph G is necessarily non-planar. One might then consider if something similar exists for matroid cycle systems: a list of forbidden minors such that if K is obtainable by some series of deletion-contraction operations on M, then M has no cycle system.

The *Kuratowski minors* are the non-planar graphs $K_{3,3}$ and K_5; any subgraph that is a subdivision of either $K_{3,3}$ or K_5 is thus non-planar, along with the graph containing it[1]. Yet as we saw in Chapter 3, the complete graph on 5 vertices K_5 does have a cycle system. By computer search we find that $K_{3,3}$ has no cycle systems consisting solely of circuits; however, it does have a *quasi-cycle system*, a construction which we will define at the end of this chapter in Question 5.

Question 3. Is there an example of a non-linear matroid with a cycle system?

In Albers (2023), cycle systems are built for linear matroids only; though this thesis generalizes the construction to not depend on linear algebra, we have yet to find any examples of non-linear matroids with cycle systems. The search has not been given up, however. The code used thus far in SageMath tests for arbitrary circuit systems, that is cycle systems comprised entirely of circuits, which could be generalized to arbitrary cycle systems.

While there are some examples of non-graphic matroids with cycle systems, such as K_5, there are also some very simple linear matroids with no cycle system, including the following.

Proposition 5.1. *If $1 < k < n - 1$ the matroid $U_{k,n}$ has no cycle system.*

Proof. We first show that $U_{2,4}$ has no cycle system. We note $g(U_{2,4}) = 2$ and since $\mathrm{rank}(U_{2,4}) = 2$, any cycle system would contain two cycles, each of which contains at least 3 elements. It is straightforward to check that there is no way to assign 3 elements to 2 cycles such that their unique union has more than 2 elements.

Now suppose we have $U_{k,n}$ such that $k \geq 2$ and $n \geq 4$, and $n-1 > k$. Furthermore suppose $U_{k,n}$ has some cycle system $\mathcal{C}$ and so there exists non-loop, non-bridge $e \in \mathcal{C}_\sigma$ (by definition of the uniform matroid, there are loops only if $k = 0$ and bridges only if $k = n$). We will show that $U_{2,4}$ is always achieved via a sequence of deletion-contraction operations. We have $U_{k,n}/e \cong U_{k-1,n-1}$ and by Theorem 3.10 $U_{k-1,n-1}$ inherits a cycle system $\mathcal{C}''$. Contract $k-2$ elements; the resulting matroid is $U_{2,n-k+2}$. Furthermore, $U_{k,n} \setminus e \cong U_{k,n-1}$ which also inherits a cycle system. If we delete $n-k-2$ elements from $U_{2,n-k-2}$, the result is a matroid isomorphic to $U_{2,4}$ which we showed has no cycle system. Then because $U_{2,4}$ was supposed to inherit a cycle system and didn't, we contradict our supposition that $\mathcal{C}$ was a cycle system for $U_{k,n}$. □

Question 4. If M has a cycle system, does there exist an ordering of the groundset such that the basis-to-coparking function algorithm sends each basis to a coparking function with degree equal to the *internal passivity* of the basis?

Let $M = (E, \mathcal{I})$ be a matroid with groundset ordering ξ and basis B.

Definition 5.2. Keeping B fixed, an element $e \in E \setminus B$ is *externally active* if it is the smallest ordered element in its fundamental circuit with B. An element $e \in B$ is *internally active* if it is the smallest element in its fundamental cut with B. An element $e \in E \setminus B$ is *externally passive* if it is not externally active. An element $e \in B$ is *internally passive* if it is not internally active. The *internal passivity* $\mathrm{ip}(B)$ is the number of internally passive elements in B.

[1]For more insight on subdivisions and Kuratowski's theorem, see Bondy & Murty (2008), Chapter 10.5.

The Tutte polynomial may also be defined $T_M(x,y) = \sum_B x^{i(B)}y^{\epsilon(B)}$, where $\epsilon(B)$ is the number of externally active elements in $E \setminus B$ and $i(B)$ is the number of internally passive elements $e \in B$. A result "dual" to this conjecture is known in the theory of parking functions (Cori & Le Borgne (2003)). An investigation of deletion-contraction proof for the activity definition of the Tutte polynomial may provide insight into this question.

Question 5. Is there a further generalization of the definition of cycle systems?

The definition of a cycle system has changed as we expand the theorem to include a broader variety of matroids. Thus, it is entirely possible that we have yet to discover the optimal definition for the cycle system construction.

Towards this end, consider the following definition due to Lixing Yi.

Definition 5.3. Let $M = (E, \mathcal{I})$ be a matroid and $\hat{\mathcal{C}}(M) = \{C_1, \ldots, C_g\}$ a collection of cycles of M. We call $\hat{\mathcal{C}}$ a *quasi-cycle system* if it satisfies the following property: for all non-empty $\sigma \subseteq [g] = \{1, \ldots, g\}$, the set

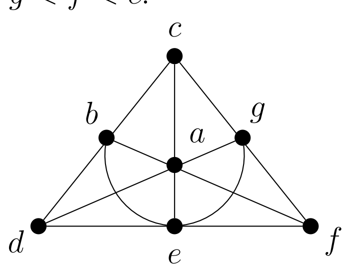
$$\hat{\mathcal{C}}_\sigma := \left\{ e \in \bigcup_{i\in\sigma} C_i : e \text{ appears in exactly one } C_i \text{ for } i \in \sigma \right\}$$

is dependent **or** is a basis for $\bigcup_{i\in\sigma} C_i$.

Accepting this alternative property for a cycle system allows us to apply our same deletion-contraction process as before, for a larger class of matroids for which our original definition would not apply. Rather than requiring the elements we delete or contract to be dependent in relation to each other, the new definition has more lenient requirements for the elements we can recurse on that hopefully are sufficient.

Conjecture 5.4. *The h-vector conjecture holds for matroids with quasi-cycle systems.*

Example 5.5. Recall the Fano matroid from Example 1.1.5. There exists a quasi-cycle system for M the Fano matroid, which allows us to construct a DC diagram using the cycle system described below and the groundset ordering $\xi : e < d < b < a < g < f < c$.



$\hat{\mathcal{C}}(M)$	a	b	c	d	e	f	g
C_1	-	b	c	d	-	-	-
C_2	-	-	-	d	e	f	-
C_3	-	b	-	-	e	-	g
C_4	a	b	-	d	e	-	-

The associated DC diagram appears on the following pages.

Computation via the Tutte polynomial reveals that the h-vector for the Fano matroid is $h(F_7) = (1, 4, 10, 13)$. The maximal elements of the coparking multicomplex derived from the DC diagram are

$$\{(1,1,0,1),(1,0,1,1),(2,1,0,0),(1,2,0,0),(1,0,2,0),(2,0,1,0),(2,0,0,1),$$
$$(0,1,1,1),(0,2,1,0),(0,1,2,0),(0,2,0,1),(0,0,2,1),(0,0,0,3)\}$$

and there are 13, which create the pure multicomplex with degree vector $(1, 4, 10, 13)$.

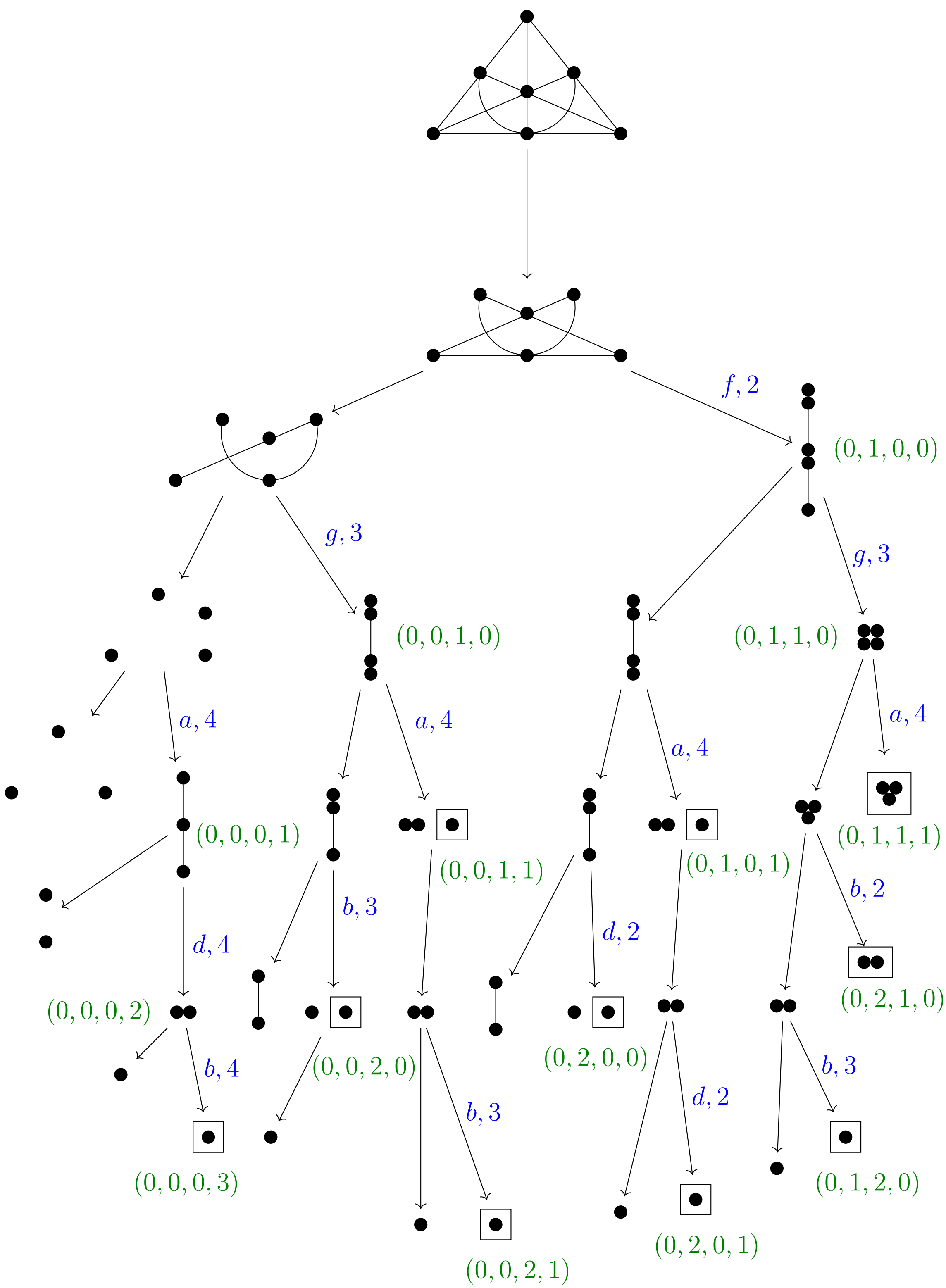


Figure 5.1: The DC diagram for $M \setminus c$ with cycle system $\hat{\mathcal{C}}'$. The associated Tutte polynomial is $T_{M\setminus c}(x, 1) = x^3 + 3x^2 + 6x + 6$.

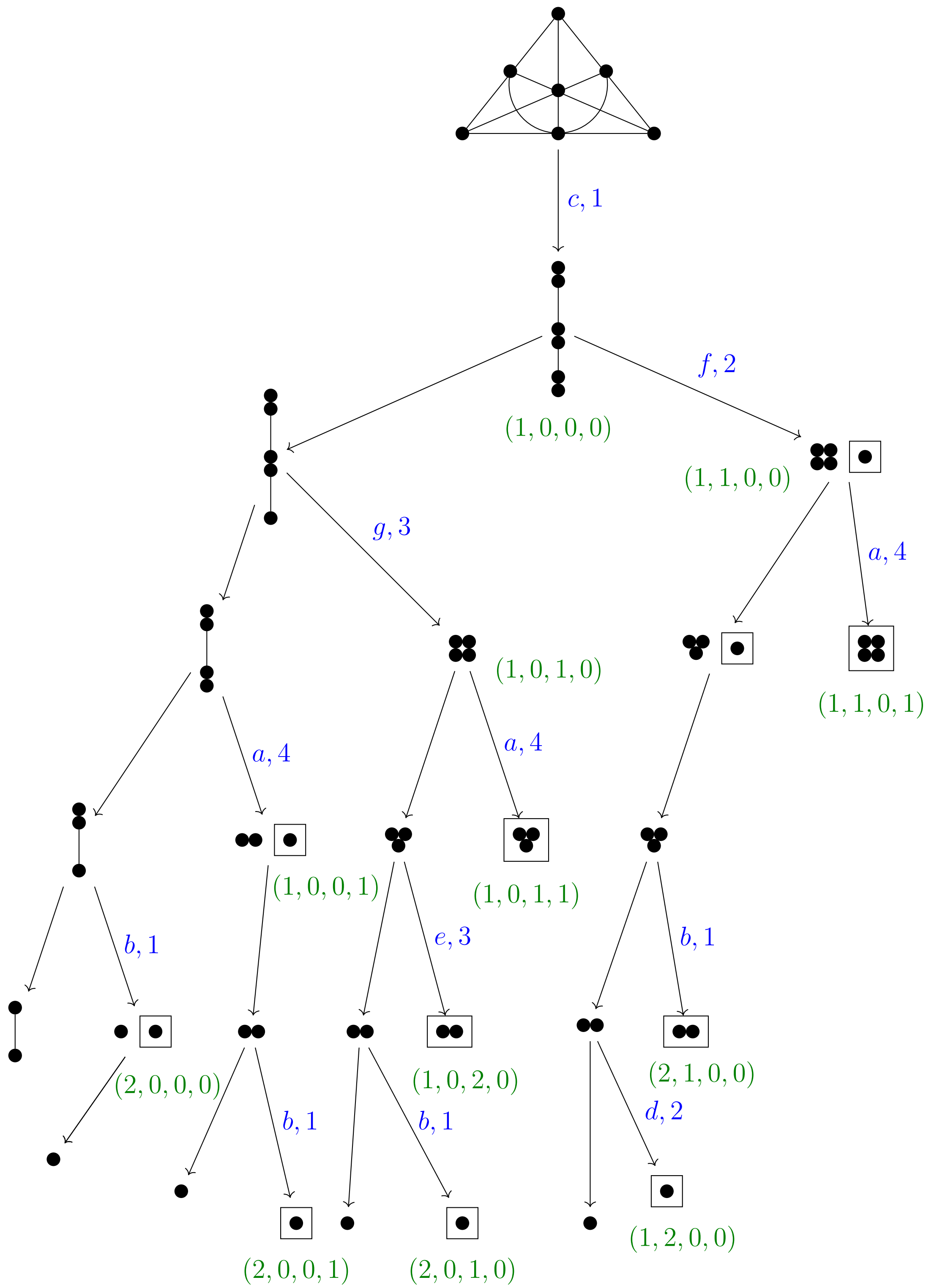


Figure 5.2: The DC diagram for M/c with cycle system $\hat{\mathcal{C}}''$. The associated Tutte polynomial is $T_{M/c}(x, 1) = x^2 + 4x + 7$.

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