Stability in Chip-Firing Games

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Abstract

In my thesis, I studied the stability of divisors on fixed-energy sandpiles (FES). On a FES, a divisor will either continue firing forever or stabilize within a finite number of steps. For any alive divisor D, there always exists a cycle with firing vector $\vec{1}$ that D fires into. The minimal alive divisors are closely related to minimal recurrent divisors. Following the discussion on generic graphs, I examined the stability of divisors on three types of graphs: complete graphs, circles and trees. I explored a question of the probability of life of the divisor obtained by randomly dropping chips onto the vertices of a graph for the range of degrees for which the question is interesting.

Introduction

0.1 Background

Sandpiles have been introduced by Bak, Tang, and Wiesenfeld (BTW) as an example of self-organized criticality [P. Bak and Wiesenfeld, 1987, 1988]. The phenomenon of self-organized criticality appears in a wide range of contexts, such as earthquakes, electrical circuits and fractal structures. In a sandpile model, as grains of sand, or what I will use in this thesis to save paper, chips, on a vertex grow to a critical threshold, that pile of chips will collapse and one chip will be sent to each adjacent vertex. This action is called a firing or a toppling. The BTW sandpile models can be described in terms of abelian groups. There are different alterations on the BTW sandpile models. In this thesis, I studied one of these alterations known as the fixed-energy sandpiles (FES). The total energy or the total number of chips is conserved in a FES.

Studies on FESs take on both a statistical mechanics approach, and an algebraic, combinatorial approach. The critical behavior of driven dissipative systems differs from that of energy conserving systems for $\mathbf{Z}^2$, the complete graph K_n, the Cayley tree, the ladder graph, the bracelet graph, or the flower graph [Anne Fey, 2010]. For FES on complete graphs, the activity of the system traces out a "devil's Staircase" as the number of chips increases in large n limit, where n is the number of vertices [Levine]. An exact solution was found by Janowsky and Laberge for the steady-state probability distribution of avalanche sizes on complete graphs [Janowsky and Laberge, 1993]. The period of a parallel chip-firing game on a bipartite graph with n vertices is at most n. In fact, all possible periods of a parallel chip-firing FES on a c-partite ($c \geq 2$) graph are characterized [Jiang, 2010]. Exact solutions were found for the cycle structures of the circle graph by Asta [Dall'Asta, 2006]. For parallel chip-firing game on trees, periods are of lengths either one or two. For generic graphs, periods can depend on the graph size [Bitar and Goles, 1992].

0.2 Preliminary

Let Γ be an undirected, connected graph with vertices V and edges E. In this thesis, I will use $e_{v_iv_j}$ or e_{ij} to denote the edge between vertex v_i and v_j. All graphs concerned are undirected, connected graphs. We allow loops and multiple edges between any given pair of vertices. For $v \in V$, define the *degree* of v,

$$d_v = \text{number of edges connected to } v.$$

A *divisor* is an integer-valued linear function on the vertices of the graph:

$$D = \sum_{v \in V} D(v)\, v.$$

An *effective divisor* is a divisor with non-negative coefficients. For effective divisor D, coefficient $D(v)$ denotes the number of chips on the vertex v. The vector representation of D is $(D(v_1), D(v_2), D(v_3), ...)$.

Assumption: All divisors are effective in this thesis unless otherwise specified.

The *degree* of D is defined as

$$\deg D = \sum_{v \in V} D(v).$$

A vertex v is *unstable* if $D_v \geq d(v)$. Otherwise, it is *stable*. If a vertex v of D is unstable, we can fire or topple the vertex to get a new divisor D'. Define $r(w, v) = |\{e \mid e = \{w, v\}\}|$. Then the divisor D' is defined by

$$D'(w) = \begin{cases} D(v) - d_v + r(v, v) & \text{if } w = v, \\ D(w) + r(w, v) & \text{if } w \neq v. \end{cases}$$

A *parallel firing* is firing all unstable vertices at one firing step. It is equivalent to a sequence of single firings that fires these vertices one by one.

The *Laplacian* Δ of Γ is defined as the matrix

$$\Delta_{ij} = \begin{cases} d_{v_i} - r(v_i, v_i) & \text{if } i = j \\ -r(v_i, v_j) & \text{if } i \neq j \end{cases}$$

A *legal firing sequence* is a sequence of vertex firings $f_\sigma = (u_1, u_2, ...)$ such that at the ith vertex firing, vertex u_i is unstable for all i. We call σ the corresponding firing vector: $\sigma = (i_1, i_2, ...)$ where i_j is the number of times that v_j is fired in f_σ.

The *linear system* of a divisor D, called $|D|$, is $\{D' \mid D \sim D'\}$. The equivalence relation $\sim$ is defined as

$$D \sim D' \Leftrightarrow D - D' = \sum_i k_i \Delta e_i,$$

where e_i's are unit vectors. A *sink* is a vertex with the property that it has an infinite capacity in holding chips and it can not be fired. A sandpile graph with a sink always stabilizes to a configuration that is indifferent to the order of firing [Holroyd et al., 2008]. For this thesis, graphs do not have a sink unless specified otherwise.

0.3 Recurrent Configuration

In this section, we consider an undirected, connected graph Γ that has a sink.

A *configuration* c is a vector of non-negative integers indexed by the non-sink vertices of a graph Γ, where $c(v)$ is the number of chips on vertex v.

On Γ, a configuration c is *recurrent* if for any configuration b, there exists a configuration a such that

$$(a+b)^o = c,$$

where $(a+b)^o$ is obtained by stabilizing $a+b$, that is, firing vertices until all vertices are stable.

A *reduced Laplacian matrix* Δ' is a Laplacian matrix with its row and column corresponding to the sink removed. Suppose $|V(\Gamma)| = n$. Then each equivalence class of $\mathbf{Z}^{n-1} \bmod \Delta'(\Gamma)$ has one and only one recurrent configuration of Γ [Holroyd et al., 2008].

The *maximal stable configuration* on Γ is defined as

$$c_{\max} = \sum_{v \in V} (d_v - 1)\, v.$$

A configuration c is recurrent if and only if there exists a configuration c' such that

$$c = (c' + c_{\max})^o\,,$$

where $(c' + c_{\max})^o$ is the stabilization of $c' + c_{\max}$ [Holroyd et al., 2008].

Define

$$\beta = \sum_{v \in N} v,$$

where N is the set of non-sink neighboring vertices to the sink s. Thus β is the configuration corresponding to firing the sink.

Burning Algorithm. *A configuration c is recurrent if and only if $(c+\beta)^o = c$. If configuration c is recurrent, then any firing sequence of the stabilization has firing vector $\vec{1}$.*

A proof of the theorem can be found here [Holroyd et al., 2008].

Chapter 1

Foundations

This chapter is devoted to generic results on questions of stability that do not depend on the the specific structures of graphs. Many of these results can be found in the literature. Mentioned above, all divisors are assumed effective in this thesis.

A divisor D is *alive* if there does not exist a legal firing sequence f_σ such that divisor $D - \Delta\sigma$ is stable. A divisor D is *dead* if there exist a legal firing sequence f_σ such that divisor $D - \Delta\sigma$ is stable.

A *cycle* is any legal firing sequence that results in the same divisor as the one it starts from. Because there are only finite different divisors of a certain degree on a given finite graph, any alive divisor will eventually fire into a cycle. We call a divisor that appears in a cycle *cyclic*.

Theorem 1. *On a graph Γ, the firing vector σ for any cycle $\mathcal{C}$ is a scalar multiplication of $\vec{1}$.*

Proof. Let v_0 be a vertex such that $\sigma(v_0) = \max_v \{\sigma(v)\}$. In a cycle $\mathcal{C}$, v_0 sends $d_{v_0}\sigma(v_0)$ number of chips in total to its neighbors. In order for v_0 to receive the same number of chips during $\mathcal{C}$, by the pigeon-hole principle and the maximality of $\sigma(v_0)$, each of the neighbors of v_0 must also fire $\sigma(v_0)$ times during $\mathcal{C}$. By the same argument, all neighbors of the neighbors of v_0 must also fire $\sigma(v_0)$ times during $\mathcal{C}$. Apply the argument enough times. Since Γ is connected and finite, eventually all vertices must fire $\sigma(v_0)$ times during $\mathcal{C}$. □

Theorem 2. *A dead divisor will stabilize in a finite number of firings regardless of the order of firing.*

Proof. Let $D' = D - \Delta\sigma$ for $f_\sigma = \{u_1, u_2, ..., u_m\}$ such that D' is stable. Then to prove the theorem, it suffices to prove that no legal firing sequence starting with D can be longer than m. Suppose there is a legal firing sequence $f_{\sigma'} = \{u'_1, u'_2, ..., u'_n\}$ with $n > m$. Then since u'_1 is unstable in D, it has to appear in the firing sequence f_σ, say as u_i. Change the firing sequence to $\{u'_1, u_1, ..., u_{i-1}, u_{i+1}, ..., u_m\}$, which is a legal firing sequence whose resulting configuration is still D'. With the same argument, bring u'_2 $(= u_j)$ to the second firing place and we have $\{u'_1, u'_2, u_1 ..., u_{i-1}, u_{i+1}, ..., u_{j-1}, u_{j+1}, ..., u_m\}$. Repeat the procedure to rearrange the order of firing. Eventually the firing sequence becomes $\{u'_1, u'_2, ..., u'_m\}$. Since $\{u'_1, u'_2, ...u'_m\}$ is just a rearrangement of $\{u_1, u_2, ..., u_m\}$, the resulting configuration is also D', which is stable. However, since $f_{\sigma'} = \{u'_1, u'_2, ..., u'_m\}$ is a legal firing sequence, u'_{m+1} is an unstable vertex in D'! By contradiction, no legal firing sequence starting with D can be longer than m. $\square$

Therefore, a divisor is either alive or dead.

Lemma 1. *For any cyclic divisor D and a legal firing sequence f_σ, there always exists a legal firing sequence $f_{\sigma'}$ such that $D - \Delta(\sigma + \sigma') = D$.*

Proof. Let $n = \max_v \sigma(v)$, the maximum number of firings of vertices in f_σ. Consider a cycle of D with firing vector $k\vec{1}$. Concatenate m such cycles where $km \geq n$. Let this concatenated firing sequence be f_{σ_m}. Now create a firing sequence $f_{\sigma'}$ in the following way. Take the first vertex to fire in f_σ, say v_0. Eliminate v_0 from f_{σ_m} where v_0 first appears. Then eliminate the second vertex to fire in f_σ from f_{σ_m}. Continue until all vertices in f_σ disappear from f_{σ_m}. Call the firing sequence left $f_{\sigma'}$. Thus contructed, $f_{\sigma'}$ is a legal firing sequence starting at $D - \Delta\sigma$, and satisfies $D - \Delta(\sigma + \sigma') = D$. $\square$

Lemma 2. *Let b and c be sandpile configurations on any undirected graph, and suppose $b \neq 0$. Suppose that $c + b$ stabilizes to c. Then c is recurrent.*

Proof. Note that for any integer $k > 0$, we have that $c + kb$ stabilizes to c. Choose k large enough so that we can selectively fire vertices, starting at $c + kb$ and arrive at a configuration a such that $a \geq c_{\max}$. Then, since a stabilizes to c, we see that c is obtained by adding chips to the maximal stable configuration and stabilizing. Hence c is recurrent. $\square$

Theorem 3. *For any cyclic divisor D, there exists a single firing cycle $\mathcal{C}$ with firing vector $\vec{1}$ such that $D \in \mathcal{C}$.*

Proof. Starting from D, fire a maximum sequence f_σ of unstable vertices $F = \{v_i\}$ where $v_i \neq v_j$ for all i, j. Let E be the resulting configuration. Let U be the set of vertices left. Since F is maximum, all vertices from U must be stable in configuration E. Let $D_U = \sum_{v \in U} D(v)v$ and $E_U = \sum_{v \in U} E(v)v$. Let $\tilde{\Gamma}$ be the graph obtained from Γ by shrinking F to a single vertex. In the rest of the proof, I will argue that D_U is recurrent with respect to $\tilde{\Gamma}$. Since D is cyclic, by Lemma 1, there exists a legal firing sequence such that $E - \Delta\sigma' = D$. Call the cycle from D to E then back to D, cycle $\mathcal{C}$. Let b be the configuration on U consisting of all the chips added by firing vertices from F during one repetition of the cycle C. Then we have that $D_U + b$ stabilizes to D_U. By Lemma 2, D_U is recurrent. Since configuration E is obtained from D by firing the "sink" F, by the Burning Algorithm, E_U must be unstable. This gives a contradiction. Then U must be an empty set. Therefore, $D - \Delta\sigma$ completes a single firing cycle with firing vector $\vec{1}$. $\square$

Corollary 1. *For any cyclic divisor D on graph Γ and any subgraph Γ' of Γ such that D restricted to Γ' is stable, let $\widetilde{\Gamma}$ be the graph obtained by shrinking $\Gamma \setminus \Gamma'$ to a point and regard that vertex as the sink. Then $D_{\Gamma'}$ is a recurrent configuration on $\widetilde{\Gamma}$.*

Proof. Let U be the support of c. Let F be the set containing the rest of the vertices. Again let $\tilde{\Gamma}$ be the graph obtained from Γ by shrinking F to a single vertex. Using the same logic as in the proof of Theorem 3, c must be recurrent with respect to $\tilde{\Gamma}$. $\square$

If the degree of a divisor exceeds $\sum_v (d_v - 1)$, by pigeonhole principle the divisor never stabilizes. Its probability of life is 1. On the other hand, the minimum degree of an alive divisor is the number of edges in the graph.

Theorem 4. *For a graph Γ, the minimum degree of an alive divisor is $|E|$, the number of edges of Γ.*

Proof. By Theorem 3, any alive divisor D can enter a cycle with firing vector $\vec{1}$. Since every vertex is fired once in the cycle, one can associate a chip a_{ij} to each edge (v_i, v_j) by sending a_{ij} from v_i to v_j when v_i was fired and from v_j to v_i when v_j is fired, assuming that v_i fires before v_j. Thus the total number of chips, i.e., the degree of an alive divisor, is at least $|E|$.

Given $|E|$ chips, an alive divisor can be constructed through the following method. Let G be an arbitrary graph with a spanning tree T. Then pick a leaf and label the leaf vertex v_1. Label its neighboring vertex v_2. Each time pick one vertex from the neighbors of $\{v_1, v_2, ..., v_i\}$. Label it v_{i+1}. Continue the labeling process until T is fully labeled. Define $l_i = \left|\left\{e_{v_i v_j} \in G \mid 1 \leq j \leq i\right\}\right|$. Define $f_n = d_{v_n} - l_n$ for all $v_n \in V$. Let $D = \sum f_i v_i$. Then D is an alive divisor since firing in the order $v_1, v_2, ..., v_{|V|}$ restores D. Moreover,

$$|E| = \left|\left\{e_{v_i v_j} \in G \mid j \leq i, 1 \leq i, j \leq |V|\right\}\right| = \sum_{i=1}^{|V|} \left|\left\{e_{v_i v_j} \in G \mid 1 \leq j \leq i\right\}\right| = \sum_{i=1}^{|V|} l_i.$$

Hence,

$$\deg D = \sum_{t=1}^{|V|} f_t = \sum_{t=1}^{|V|} d_{v_t} - l_t = \sum_{t=1}^{|V|} d_{v_t} - \sum_{t=1}^{|V|} l_t = 2\,|E| - |E| = |E|\,.$$

Thus the minimum degree of an alive divisor is at most $|E|$. Therefore, the minimum degree of an alive divisor is equal to $|E|$. $\square$

A divisor is *minimal alive* if removing any chip from it will result in a dead divisor. Similarly, a configuration c is *minimal recurrent* if c is recurrent and reducing any chip from c will result in a non-recurrent configuration.

Notice that every alive divisor can be reduced to a minimal alive divisor by removing chips from it.

Corollary 2. *For any ordering of the vertices $v_1, v_2, ..., v_{|V|}$, there exist a minimal alive divisor D such that firing $v_1, v_2, ..., v_{|V|}$ results in a cycle.*

Proof. Notice that the construction of a minimal alive divisor in the proof of Theorem 4 does not depend on which underlying spanning tree is picked or v_{i+1} being a neighbor of $\{v_1, v_2, ..., v_i\}$. Thus for any order of the vertices, an alive divisor D can be defined in the same way. The resulting divisor has degree $|E|$ and hence is minimal. $\square$

Theorem 5. *For any minimal recurrent configuration c on an undirected graph with the sink s, $c + d_s s$ is minimal alive and cyclic.*

Proof. Any recurrent configuration c follows the Burning Algorithm. Hence $c + d_s s$ is a cyclic divisor.

Let $f_\sigma = \{s\,(= v_0)\,, v_1, ..., v_k\}$ be a legal firing sequence of a cycle of $c + d_s s$. Then $\sigma = \vec{1}$. Let l_i be defined in the same way as in Theorem 4. Then $D\,(v_i) \geq d_{vi} - l_i$ for $i = 0, 1, 2, ..., k$. If $D\,(v_i) = d_{v_i} - l_i$ for all i, then

$$\deg D = \sum_{k}^{i=1} D\,(v_i) = |E|\,.$$

Since the minimum degree of an alive divisor is $|E|$, it follows that $D = c + d_s s$ must be minimal alive. Otherwise, there must exist j such that $D\,(v_j) > d_{v_j} - l_i$. In this case, as many as $D\,(v_j) - \left(d_{v_j} - l_i\right)$ chips can be reduced from the vertex v_i while f_σ remains a legal firing sequence. Hence $c - v_i$ satisfies the Burning Algorithm. Therefore $c - v_i$ is recurrent. But this contradicts c being minimal recurrent. Therefore, $D = c + d_s s$ is minimal alive and cyclic. □

Theorem 6. *For any alive divisor D on a graph Γ, there exists a legal firing sequence f_σ such that $D - \Delta\sigma = c + ts$, where c is a recurrent configuration and $t \geq d_s$. If D is minimal alive, then $t = d_s$.*

Proof. Choose a vertex s in Γ and stabilize D with respect to s. In the stabilization of D, vertex s must be unstable. Fire s and stabilize again with respect to s. This procedure can be repeated for an arbitrary number of times.

$$\underbrace{(...((c' + \beta)^o + \beta)^o + ...)^o}_{k \text{ times}} = (c' + k\beta)^o =: c,$$

where just like before β is the configuration added by firing the sink. By taking k sufficiently large, $k\beta$ can be fired to $c_{\max} + c'$ for some configuration c'. Stabilizing from the configuration $c_{\max} + c'$, the configuration c is recurrent. Thus there exists a firing sequence f_σ such that $D - \Delta\sigma = c + ts$, where c is a recurrent configuration and $t \geq d_s$.

Now let D be a minimal alive divisor. To show $t = d_s$, it suffices to prove $c + (t-1)s$ is dead. The firing sequence f_σ consists of two kinds of firing steps: the first kind is the stabilization of divisor D with respect to the sink s, the second kind is firing the sink s plus stabilization with respect to s. Divisor $c + ts$ is obtained from D by performing the first kind of firing step once and the second kind of firing step as many times as needed. I will prove below that the divisor remains minimal alive after both kinds of firing steps.

First, I will prove that if $D^{'}$ is obtained by stabilizing D with respect to s, then divisor $D^{'} - s$ is dead. Let f_σ be the firing sequence such that $D - \Delta\sigma = D'$. Then $(D - s) - \Delta\sigma = D' - s$ since s is not fired in the firing sequence f_σ. Since D fires to a cyclic divisor via a legal firing sequence, without loss of generality we can assume

$D(s) > 0$ so that both $D - s$ and $D' - s$ are effective divisors. Then D being minimal alive implies that $(D - s)$ is dead, which implies that $(D' - s)$ is dead.

Next, let D_1 be any alive divisor on Γ obtained by stabilizing with respect to s. Then consider the following firing sequence $f_{\sigma'}$: fire $s \in D_1$ to obtain D_s, then stabilize D_s with respect to s. Call the resulting divisor D_2.

$$D_1 - \Delta\sigma' = D_2.$$

I will prove that if $D_1 - s$ is dead, then $D_2 - s$ is dead. Since D_1 is alive and stabilized with respect to s, it follows that $D_1(s) \geq d_s$. If $D_1(s) > d_s$, then $f_{\sigma'}$ is a legal firing sequence for $D_1 - s$ such that

$$(D_1 - s) - \Delta\sigma' = D_2 - s.$$

In this case,

$$D_1 - s \text{ is dead } \Rightarrow D_2 - s \text{ is dead.}$$

If $D_1(s) = d_s$, it suffices to prove $D_2(s) = d_s$ since then $D_2 - s$ is dead. Again let the firing sequence be $f_{\sigma'}$. Suppose there exist vertices that fired twice or more in $f_{\sigma'}$. Let F be the set of vertices that are the first to fire for the second time in the paralleling firing sequence that is equivalent to $f_{\sigma'}$. Pick a vertex $v \in F$. Then

$$D_1(v) < d_v \Rightarrow D_1(v) + d_v < 2d_v.$$

In order for v to send out $2d_v$ chips, at least one of the neighbouring vertices of v must have already fired at least twice. This contradicts with $v \in F$. Hence vertices of Γ fire at most once in $f_{\sigma'}$. Therefore, $D_2(s) \leq d_s$. Since D_2 is alive, it follows that $D_2(s) = d_s$. With these two conclusions for both kinds of firing steps, we have

$$D - s \text{ is dead } \Rightarrow (c + ts) - s \text{ is dead.}$$

Therefore, $t = d_s$.

□

Define the directed *single firing graph* $f\Gamma$ of a linear system $|D|$ by

$$V = \{D' \mid D' \in |D|\}, \text{ and}$$
$$E = \{\vec{e}_{D_1 D_2} \mid D_2 \text{ can be obtained from } D_1 \text{ by firing a single vertex, for all } D_1, D_2 \in V\}.$$

Lemma 3. *For any divisor E, if $E \sim c + ks$ where configuration c is recurrent and $k \geq d_s$, then divisor E is alive.*

Proof. The proof is from John Wilmes' thesis Lemma 2.7. By the Burning Algorithm, there exists a legal firing cycle $(v_1, v_2, ..., v_n)$ for $c + ks$ with firing vector $\vec{1}$. Let σ be the firing vector such that $c + ks - \Delta\sigma = E$. By adding or subtracting firing vector $\vec{1}$'s, we may assume that $\sigma > 0$, and that $\sigma(v_i)$ is 0 for some i. Then choose the minimal k such that $\sigma(v_k) = 0$. It follows that v_k is unstable in E since all vertices with smaller index are fired at least once in f_σ and $(v_1, v_2, ..., v_k)$ is a legal firing sequence. So any divisor linearly equivalent to $c + ks$ must have some unstable vertex. Therefore, E can not be a dead divisor and thus E is alive. □

Theorem 7. *For any alive divisor D, the single firing graph $f\Gamma$ of $|D|$ is connected.*

Proof. By Theorem 6, there exists a legal firing sequence f_σ such that $D - \Delta\sigma = c + ks$ where configuration c is recurrent and $k \geq d_s$. Let E be any divisor in $|D|$. It follows that $E \sim c + ks$. Hence by Lemma 3, E is alive. Again by Theorem 6, there exists a legal firing sequence $f_{\sigma'}$ such that $E - \Delta\sigma' = c' + k's$ where configuration c' is recurrent and $k' \geq d_s$. Then,

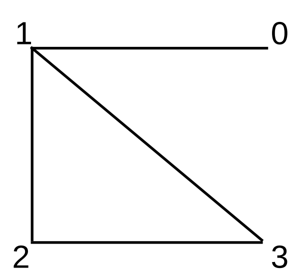
$$D \sim E \Leftrightarrow c + ks \sim c' + k's \Rightarrow c \sim' c'$$

where the last equivalence is with respect to the reduced Laplacian matrix of the graph. The last implication can be checked by performing matrix algebra and using the property of the Laplacian Δ that $\Delta\vec{1} = \vec{0}$. Since there is only one recurrent configuration in an equivalence class, we have $c = c'$. The relationship $D \sim E$ implies that the degree of D is equal to that of E. Then necessarily $k = k'$. Therefore,

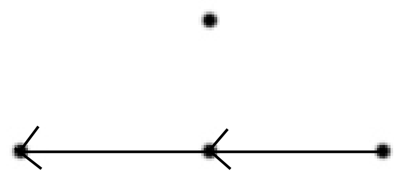
$$c + ks = c' + k's.$$

So every divisor in $|D|$ fires to $c + ks$ via a legal firing sequence. Therefore, the single firing graph $f\Gamma$ is connected. □

The single firing graph for a dead divisor is not necessarily connected. Here's a counterexample. Let $D = \{0, 0, 1, 1\}$ on the following graph. Then D is stable.



The linear system of D contains $\{2, 0, 0, 0\}, \{0, 0, 1, 1\}, \{1, 1, 0, 0\}, \{0, 2, 0, 0\}$. The corresponding single firing graph is



It is not connected.

Theorem 7 does not apply to multiple firing graphs either, where two divisors are connected by an edge if one can be obtained from the other by a parallel firing step. Here is a counterexample:

Let $D = \{29, 0, 0, 0, 0\}$ on the complete graph on five vertices, K_5. Since $\deg D \geq \sum d_v - 1 + 1 = 5(5-2)+1 = 16$, by the pigeonhole principle, D is alive. The multiple firing graph of its linear system is

It is not connected.

Chapter 2

Complete Graphs, Circles and Trees

2.1 Complete Graphs

For a complete graph with n vertices, denoted K_n, the minimum degree of divisors that ensures life is $\sum(d_v - 1) + 1 = n(n-2) + 1 = (n-1)^2$. The minimum degree of possible alive divisors is $|E| = \frac{n(n-1)}{2}$.

Theorem 8. *Any minimal alive divisor on K_n is a permutation of $(n-1, n-2, ..., 2, 1, 0)$ and cyclic. Any cyclic divisor $D = \sum_{i=1}^{n} f_i v_i$ satisfies $f_i \geq i - 1$ after a permutation of vertices if necessary.*

Proof. Let D be any minimal alive divisor on K_n. By Theorem 3, there exists a cycle $\mathcal{C}$ with firing vector $\vec{1}$ such that D appears in $\mathcal{C}$. Let $v_1, v_2, ..., v_n$ be the order of firing for cycle $\mathcal{C}$ starting at D. Let $D = \sum_{i=1}^{n} f_i v_i$. For v_i to be fired in $\mathcal{C}$, it follows that $f_i \geq d_{v_i} - (i-1) = n - i$ for $i = 1, 2, ..., n$. Then $\deg D$ reaches its minimum when $f_i = n - i$ for all i, that is, $D = (n-1, n-2, ..., 2, 1, 0)$ up to permutation. Let $D(v_1) = 0$. Suppose there exists a divisor E on K_n such that D can be obtained by firing a single vertex of E. Then the vertex fired must be v_1 since firing any other vertex adds a chip onto v_1 and as a result $D(v_1)$ can not be 0. Reverse-fire v_1 in D, that is, sending one chip from each neighbor of v_1 to v_1. The resulting divisor E is a permutation of $(n-1, n-2, ..., 2, 1, 0)$. Hence, any divisor that fires to obtain D appears in cycle $\mathcal{C}$. Therefore, any minimal alive divisor is also cyclic. $\square$

Randomly putting k chips on the vertices of a complete graph K_n, what is the probability that the divisor is alive? There are three versions of the question. I tried to answer this question in its first version.

- Drop k chips, one after another, randomly onto the vertices of G with no firing allowed. So each vertex has the same probability of getting that chip being dropped. When done, start the chip-firing game. What is the probability that the resulting configuration never stabilizes?

- Suppose that each (effective) divisor of a certain degree has the same probability of occurring. What is the probability of life of a random divisor of degree k?
- Drop k chips randomly onto the vertices of G, but drop a chip only if the configuration formed by all previous chips completely stabilizes. What is the probability that the chip-firing graph never stabilizes with k or fewer chips dropped?

For the first version of the question, the probability that the resulting configuration never stabilizes for a total of k chips dropped is,

$$P(k) = \sum_{\left\{\vec{k}\text{ alive}, k_1 \leq \ldots \leq k_n\right\}} \frac{\frac{k!}{k_1!\ldots k_n!} \cdot \frac{n!}{n_1!\ldots n_{k_n}!}}{n^k},$$

where n_i is the number of vertices that have i chips on them. So $n_1+n_2+\ldots+n_{k_n} = n$.

I calculated the probabilities of life according to this fomula for a range of degrees of alive divisors on complete graphs. Please see Appendix A for probabilities of divisors of degrees from $\frac{n(n-1)}{2}$ to $(n-1)^2$ for $n = 4, 5, 6, 7, 8, 9, 10$.

A following question is: how many chips does one expect to use to obtain an alive divisor on K_n?

Let $\mathbb{P}$ be the probability space of life with the random variable being the degree of the divisor. Then the expectation for the degree of an alive divisor is,

$$\begin{aligned}
\mathbf{E} &= \sum_{i=1}^{\infty} i\,\mathbb{P}(i) = \sum_{i=1}^{\infty}\sum_{j=1}^{i} \mathbb{P}(j) = \sum_{j=1}^{\infty}\sum_{i=j}^{\infty} \mathbb{P}(j) = \sum_{j=1}^{\infty} \left(1 - \mathbb{P}(j-1)\right) \\
&= \sum_{j=1}^{\frac{n(n-1)}{2}} \left(1-\mathbb{P}(j-1)\right) + \sum_{j=\frac{n(n-1)}{2}+1}^{(n-1)^2} \left(1-\mathbb{P}(j-1)\right) + \sum_{j=(n-1)^2+1}^{\infty} \left(1-\mathbb{P}(j-1)\right) \\
&= \frac{n(n-1)}{2} + (n-1)^2 - \frac{n(n-1)}{2} - \sum_{j=\frac{n(n-1)}{2}+1}^{(n-1)^2} \left(\mathbb{P}(j-1)\right) \\
&= (n-1)^2 - \sum_{j=\frac{n(n-1)}{2}}^{(n-1)^2-1} \left(\mathbb{P}(j)\right).
\end{aligned}$$

I do not have a closed formula for the expectation function on K_n, but here are two approximating functions of the expectation function. Call them $\mathbf{E_1}$ and $\mathbf{E_2}$. Define

$$e_i = \frac{\mathbf{E_i} - \frac{n(n-1)}{2}}{(n-1)^2 - \frac{n(n-1)}{2}},$$

for $i = 1, 2$.

I.

$$\mathbf{E_1}(n) = \frac{(n-1)(2n-1)}{3}.$$

The approximation $\mathbf{E_1}(n)$ is a third of the distance $\left((n-1)^2 - \frac{n(n-1)}{2}\right)$ away from $\frac{n(n-1)}{2}$. So $e_1 = \frac{1}{3}$.

II.

$$\mathbf{E_2}(n) = \frac{(n-1)(3n-1)}{2} - \frac{n(n-1)\phi}{2},$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

$$\begin{aligned}
\mathbf{E_2}(n) &= \frac{n-1}{2}(3n-1-n\phi) \sim n(n-1)\frac{3-\phi}{2}. \\
e &= \frac{\mathbf{E_2} - \frac{n(n-1)}{2}}{(n-1)^2 - \frac{n(n-1)}{2}}, \\
&\sim \frac{n(n-1)\frac{3-\phi}{2} - \frac{n(n-1)}{2}}{(n-1)^2 - \frac{n(n-1)}{2}}, \\
&\sim 2-\phi,
\end{aligned}$$

which is approximately 0.382. $\mathbf{E_2}$ was found by rounding $\mathbf{E}(n)$ and looking at the encyclopedia of integer sequences [Sloane, 2008].

The following comparison table runs from $n = 4$ to 10:

$\mathbf{E}(n)$	$\mathbf{E_1}(n)$	$\mathbf{E_2}(n)$
6.97558593750000,	7.00000000000000,	6.79179606750063,
11.9227801600000,	12.0000000000000,	11.8196601125011,
18.1980179985134,	18.3333333333333,	18.2294901687516,
25.8380990775113,	26.0000000000000,	26.0212862362522,
34.8830260927871,	35.0000000000000,	35.1950483150029,
45.3693334946985,	45.3333333333333,	45.7507764050038,
57.3273883125771	57.0000000000000,	57.6884705062547

2.2 Circles

On a circle graph with n vertices, $\mathcal{C}_n$, the minimum degree of an alive divisor is n. If $\deg D \geq n+1$, then by pigeonhole principle D is always alive.

Theorem 9. *The number of dead divisors of degree n on $\mathcal{C}_n$ is the number of n-subsets $\{a_1, a_2, ..., a_n\}$ of the set $\{1, 2, ..., 2n-1\}$ such that $\sum_{i=1}^{n} a_i \equiv 0 \,(mod\ n)$.*

Proof. Present a divisor D as a unique sequence of n chips and $(n-1)$ dividers in the following way. Let the number of chips between two adjacent dividers, say the ith and the $(i+1)$st, be the number of chips on the vertex v_{i+1} for $1 \leq i \leq n-2$. The number of chips before the first divider is v_1 and that after the last divider is v_n. Call the ith divider the left divider of v_{i+1} and the $(i+1)$st divider the right divider of v_{i+1} for $1 \leq i \leq n-2$. Let the first divider be the right divider of v_1 and the last divider the left divider of v_n. So the sequence representation of a divisor has length $n+(n+1)-2 = 2n-1$. For a divisor of degree n, let the ith chip be at the a_ith position in its sequence representation. For example, on $\mathcal{C}_7$, the sequence representation of $D = (1,1,0,1,2,1,1)$ is illustrated below. The top row is the sequence representation of D; the bottom row is the position index. The symbol "p_i" denotes the ith chip and the symbol "$|$" denotes a divider.

$$\begin{array}{ccccccccccccc} p_1 & | & p_2 & | & | & p_3 & | & p_4 & p_5 & | & p_6 & | & p_7 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$

The corresponding set $\{a_i\}$ is $\{1,3,6,8,9,11,13\}$ with the ith chip p_i occupying the a_ith position for $i = 1,2,...,7$. The set of divider positions is $\{2,4,5,7,10,12\}$, which is the complement set of $\{a_i\}$.

Since $\sum_{i=1}^{2n-1} i \equiv 0 \,(\mathrm{mod}\ n)$, if a subset of $\{1,2,...,2n-1\}$ has a sum divisible by n, so does its complement set. Now consider the complement set of $\{a_1,a_2,...,a_n\}$ containing positions of the $n-1$ dividers in the sequence.

The only stable divisor of degree n on $\mathcal{C}_n$ is $(1,1,...,1)$. It is easy to check that $\sum_{i=1}^{n} a_i \equiv 0 \,(\mathrm{mod}\ n)$. Hence the sum of divider positions for divisor $(1,1,...,1)$ is also divisible by n. Starting from $(1,1,...,1)$, consider the reverse operation of a firing, that is, the vertex being reverse-fired obtains one chip from each of its neighbors. All the vertices being reverse-fired are divided into three cases for consideration: v_1 being reverse-fired, v_n being reverse-fired and any other vertex being reverse-fired. For the last case, reverse-firing $v_i\,(i \neq 1,n)$ changes the sequence representation of a divisor by moving v_i's left divider one position to the left and its right divider one position to the right. Hence the sum of divider positions remains unchanged in this case. In the first case, a reverse-firing of v_1 moves the first divider, which is the right divider of v_1, by two positions to the right and all other dividers by one position to the right. Together the reverse-firing increases the sum of divider positions by n. In a similar situation, reverse-firing v_n decreases the sum of divider positions by n. For all cases of reverse-firing, the sum of divider positions remains divisible by n. Therefore, if any divisor stabilizes to $(1,1,...,1)$, the sum of divider positions in its sequence representation must be divisible by n. Thus each dead divisor of degree n corresponds to an n-subset $\{a_1,a_2,...,a_n\}$ of the set $\{1,2,...,2n-1\}$ such that $\sum_{i=1}^{n} a_i \equiv 0 \,(\mathrm{mod}\ n)$.

On the other side, any alive divisor of degree n can fire to one of the following two forms up to rotation: $(0,2,0,2,...)$ or a concatenation of $(0,2,1,1,...,1)$'s of various lengths [Dall'Asta, 2006]. Starting with a divisor A in either of the two forms, choose the minimal k such that $A\,(v_k) = 2$. Fire v_k. Choose and fire in the same way as many times as needed until a permutation of $(2,0,1,1,...,1)$ is obtained. Rotate the vertices to make v_1 have two chips and call the resulting divisor D. Then up to rotation, every alive divisor can fire to D via a legal firing sequence. Let $D = (D_1, D_2, ..., D_n)$ where $D_1 = 2$, $D_i = 0$ for some $i \neq 1$ and $D_k = 1$ for $k \neq 1, i$. Notice that the ith divider in the sequence representation of D is at position $\sum_{j=1}^{i} (D_j + 1)$. Hence the sum of divider positions (call it $\mathcal{S}$),

$$
\begin{aligned}
\mathcal{S} &= \sum_{i=1}^{n} \left(\sum_{j=1}^{i} (D_j + 1) \right), \\
&= \sum_{j=1}^{n} (n + 1 - j) D_j + (1 + 2 + 3 + ... + n), \\
&= n + (1 + 2 + 3 + ... + n) - (n + 1 - i) + (1 + 2 + 3 + ... + n), \\
&= n + n (n + 1) - (n + 1 - i), \\
&\equiv n + 1 - i \,(\text{mod } n), \\
&\not\equiv 0 \,(\text{mod } n),
\end{aligned}
$$

since $i \neq 1$. The calculation shows that the sum of divider positions of D is not divisible by n. I showed reverse-firing of any vertex does not change the sum of divider positions mod n. Notice that a rotation of vertices does not change the sum of divider positions either. Therefore, the sum of divider positions for any alive divisor of degree n is not divisible by n. Thus, the sum of divider positions is divisible by n if and only if the corresponding divisor is dead. Considering the complement set of the set of divider positions, we have that the sum of chip positions is divisible by n if and only if the corresponding divisor is dead. Since divisors have a one-to-one correspondence to the sets of chip positions $\{a_1, a_2, ..., a_n\}$, the number of n-subsets $\{a_1, a_2, ..., a_n\}$ of the set $\{1, 2, ..., 2n - 1\}$ such that $\sum_{i=1}^{n} a_i \equiv 0 \,(\text{mod } n)$ is the number of dead divisors of degree n on graph $\mathcal{C}_n$. □

2.3 Trees

For a tree on n vertices, divisors with degree greater than or equal to $|E| = n - 1$ are always alive. This is simply because

$$\sum_{v \in G} (d_v - 1) + 1 = 2|E| - n + 1 = |E|.$$

A study by Bitar and Goles on the cycle structures of trees shows that all non-trivial cycles on trees are of period two [Bitar and Goles, 1992].

Appendix A

Probability Table for Complete Graphs

Dropping k chips onto vertices of K_n one by one without firing, each vertex has the same probability of getting the chip being dropped. Start firing after all k chips have been dropped. The following table lists the probabilities that a resulting divisor of degree k is alive for $\frac{n(n-1)}{2} \le k \le n(n-2)$. For $k < \frac{n(n-1)}{2}$, the probability is always 0; for $k > n(n-2)$, the probability is always 1.

$n = 4$	$n = 5$	$n = 6$	$n = 7$
0.3515625	0.154828800000000,	0.0579414845086284,	0.0185271824152743,
0.73828125	0.430848000000000,	0.204665264856933,	0.0803359215284532,
0.9345703125	0.684288000000000,	0.399805010509890,	0.188907915661729,
	0.862750720000000,	0.596909230019520,	0.331346817876961,
	0.953753600000000,	0.757896547772922,	0.484450374325662,
	0.99075072	0.872360631037167,	0.629184483955316,
		0.942032563813919,	0.751775649062990,
		0.977842531447553,	0.845724421852806,
		0.993596060731446,	0.911787858814649,
		0.99893267678857	0.954068109352954,
			0.978498897970897,
			0.991167029107214,
			0.996994700502023,
			0.999240115054074,
			0.99989144500772

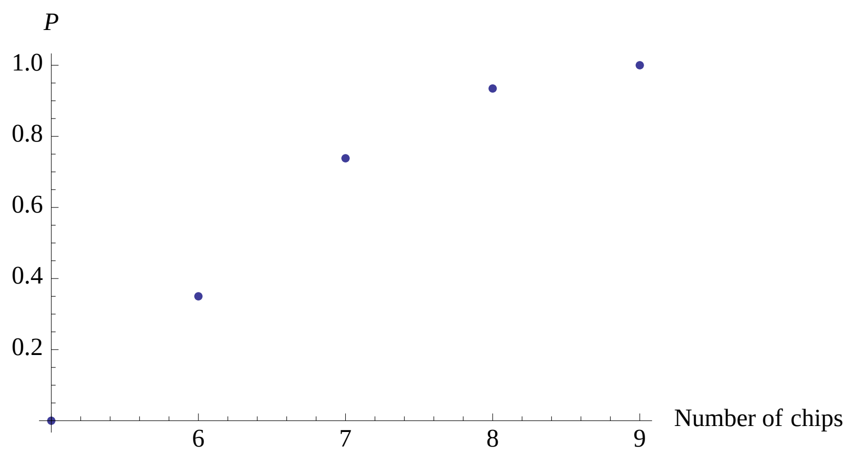
$n = 8$	$n = 9$	$n = 10$
0.00506763052292704,	0.00118497872206685,	0.000236567222540293,
0.0262755081655314,	0.00719522148040845,	0.00165411995946712,
0.0729276414094939,	0.0231749689238607,	0.00609434033072679,
0.148501289722390,	0.0541171778922327,	0.0161352805700888,
0.248575096748981,	0.102834114247736,	0.0344966521695637,
0.364556240854447,	0.169515404236564,	0.0635041059780071,
0.485761091713079,	0.251613492317011,	0.104553033278314,
0.602028714899604,	0.344475826599487,	0.157782766632050,
0.706002143094565,	0.442602890939955,	0.222135902727840,
0.793066517161597,	0.540345262689234,	0.295472241243248,
0.861502029998555,	0.632666881404964,	0.374858223066575,
0.912145953233039,	0.715774730223783,	0.456997235887567,
0.947463311720624,	0.787328911865324,	0.538619981372679,
0.970564065728625,	0.846314978339156,	0.616752414989677,
0.984667295434990,	0.892900373869423,	0.688959788325807,
0.992662806419971,	0.928168362664896,	0.753509212111594,
0.996840285222755,	0.953759481321762,	0.809406169981370,
0.998818999676057,	0.971530400413598,	0.856330271739209,
0.999638593698881,	0.983304496018087,	0.894523279542401,
0.999918837143807,	0.990723015400760,	0.924664116817278,
0.99998985464298	0.995151202251594,	0.947723891312782,
	0.997641977709434,	0.964819119890667,
	0.998950558618244,	0.977086821474233,
	0.999582985745504,	0.985594219860808,
	0.999857327550492,	0.991283344805823,
	0.999960373411558,	0.994943173692117,
	0.999991999399893,	0.997201402690843,
	0.99999911104443	0.998532596085867,
		0.999277936935217,
		0.999670888465787,
		0.999863506240416,
		0.999949749038306,
		0.999984167487470,
		0.999995976861225,
		0.999999262395794,
		0.99999992623958

Appendix B

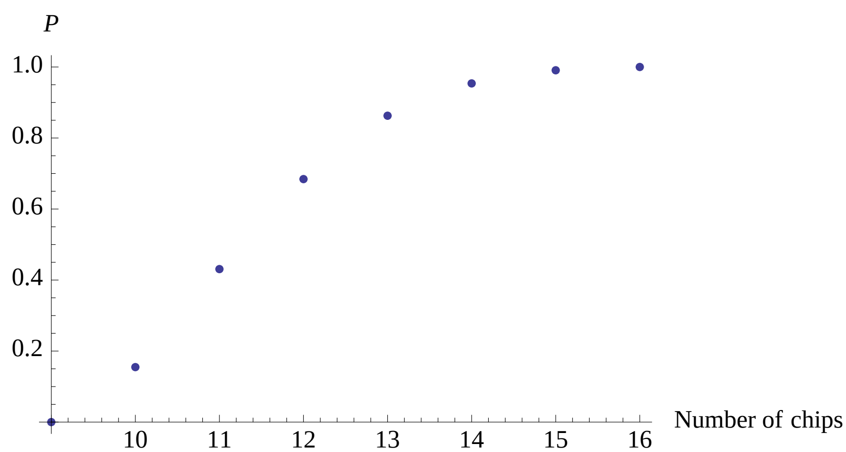
Probability Plots for Complete Graphs

The following plots are based on the probability table for complete graphs:

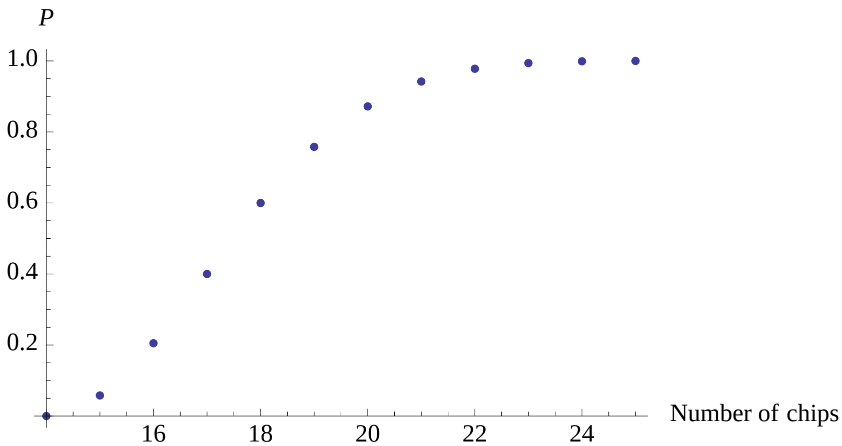
$n = 4$:



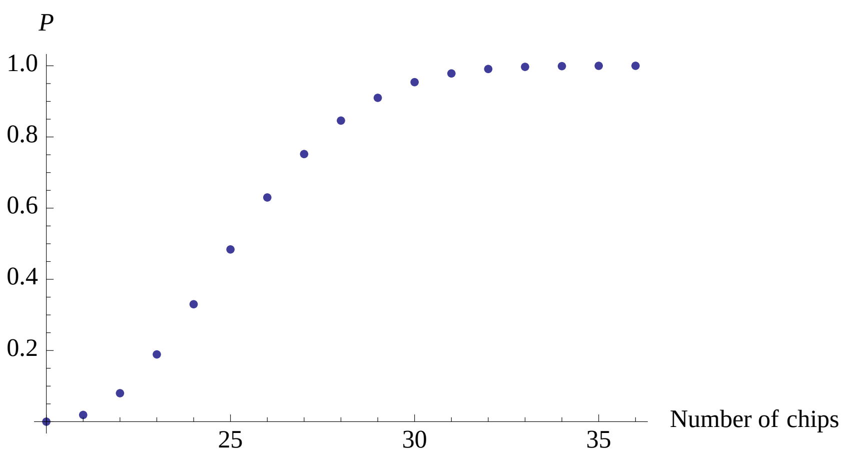
$n = 5$:



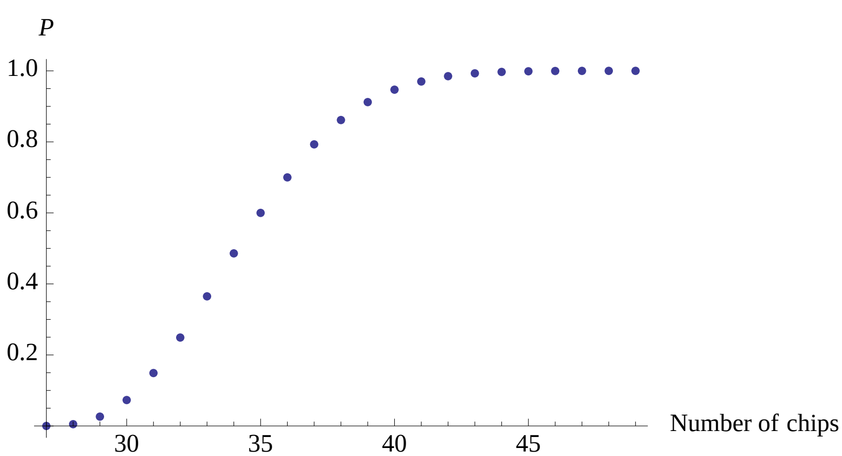
$n = 6$:


P
1.0
0.8
0.6
0.4
0.2
16
18
20
22
24
Number of chips

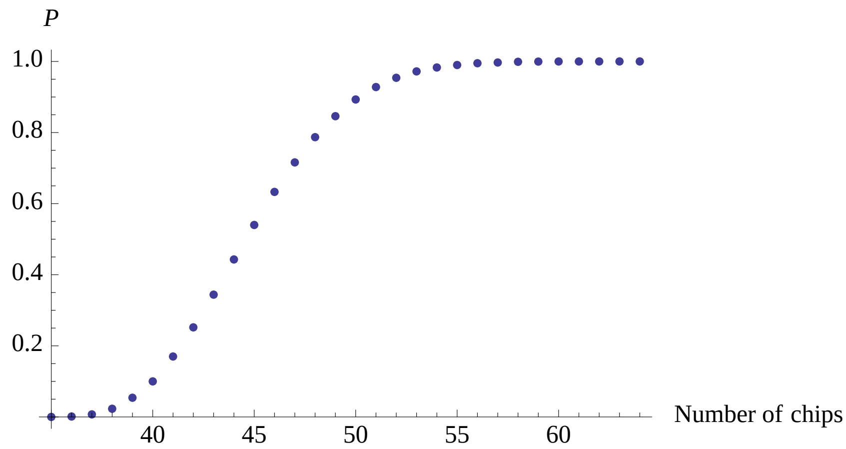
$n = 7$:


P
1.0
0.8
0.6
0.4
0.2
25
30
35
Number of chips

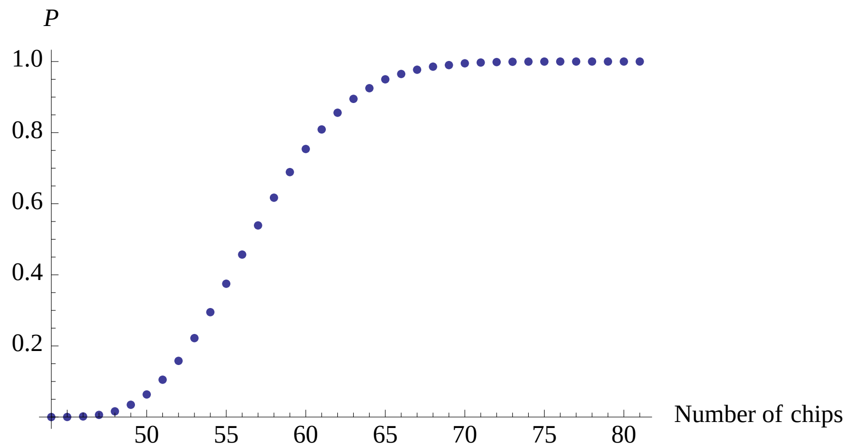
$n = 8$:


P
1.0
0.8
0.6
0.4
0.2
30
35
40
45
Number of chips

$n = 9$:


P
1.0
0.8
0.6
0.4
0.2
40
45
50
55
60
Number of chips

$n = 10$:


P
1.0
0.8
0.6
0.4
0.2
50
55
60
65
70
75
80
Number of chips

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