Frameworks and Tensegrities

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Abstract

A study of the rigidity of frameworks and tensegrity frameworks.

Chapter 1

Frameworks

We begin by introducing the concept of frameworks with the interest of determining when frameworks are rigid. Once we have gained a thorough understanding of how to find the rigidity of frameworks we will then study rigidity in the case of cross products of frameworks. In the final section we'll introduce the idea of stresses on frameworks to facilitate our understanding of tensegrities in the next chapter.

1.1 Examples of Frameworks

Let us begin with a discussion of frameworks and the questions that arise naturally from their study. Our main resources for information about frameworks and rigidity are from [5] and [1]. Consider a square where each vertex is a freely pivoting joint and each edge is a rigid bar.

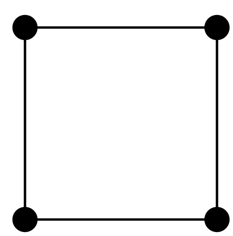

Figure 1.1: A square

One of the questions which arises as we consider this framework is whether it is rigid. One can imagine that if held, this framework would collapse and move freely. Now let's consider embedding this framework in the plane, since it is itself a two-dimensional object. Even with this restriction in movement the framework can be moved in the plane without changing the lengths of the edges, as is illustrated in Figure 1.2. However, a flexing of the framework is not the only movement to be considered for our square. We can also slide the framework freely through the plane with out changing its current shape. Since these movements of the framework are not the flexings we would like to consider, let's begin by fixing the lower two corners of the square at the points $d = (0, 0)$ and $e = (1, 0)$. Let's call the two upper

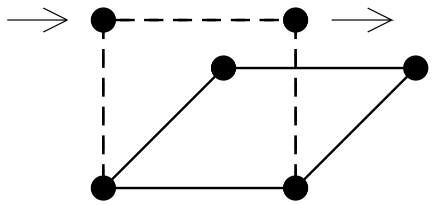

Figure 1.2: A flexing of the square

vertices $x_1 = (x_{11}, x_{12})$ and $x_2 = (x_{21}, x_{22})$, and the whole framework A. In the starting position of the square, x_1 is at $(0,1)$ and x_2 is at$(1,1)$. We can find all the

x_1 x_2

$d = (0,0)$ $e = (1,0)$

Figure 1.3: Framework A

positions reachable by the points x_1 and x_2 by solving a system of equations that describe the length constraints of the edges. Letting $|\ |$ be the usual absolute value, the system of equations is as follows:

$$|x_1 - x_2|^2 = 1, \quad |x_1 - d|^2 = |x_1|^2 = 1, \quad |x_2 - e|^2 = |x_2 - (1,0)|^2 = 1.$$

One family of solutions is $x_1(t) = (t, \sqrt{1-t^2})$ and $x_2(t) = (t+1, \sqrt{1-t^2})$ for $t \in [0,1]$. In the solutions given here we allow edges to cross and vertices and edges to lie on top of one another, even though this is not mechanically possible. This family of solutions gives the flexing of the square shown in Figure 1.2.

Let us now insert a new bar into the framework, one which connects x_2 and d. To see if this new framework, which we'll call B, is flexible, we can solve the system of edge equations which describe this framework. These equations are the same three previous equations, as well as the following:

$$|x_2 - d|^2 = |x_2|^2 = 2.$$

The only solutions to this system of equations are

$$(x_1, x_2) \in \{((0,1),(1,1)), ((1,0),(1,1)), ((0,-1),(1,-1)), ((1,0),(1,-1)).$$

These solutions are illustrated in Figure 1.5 This means that our framework is not

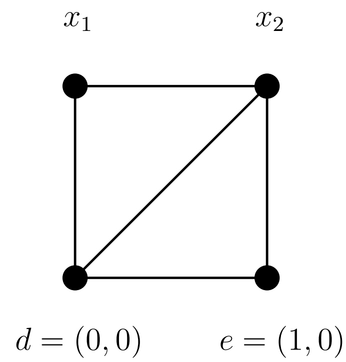

Figure 1.4: Framework B

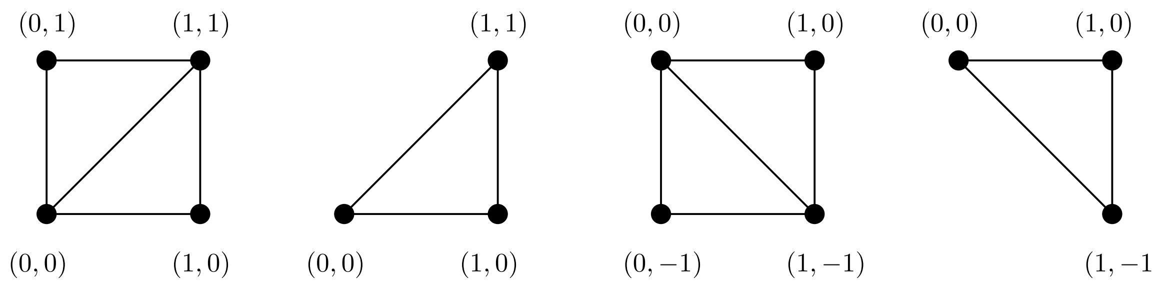

Figure 1.5: Solutions

smoothly flexible, but instead has only a few positions which preserve edge lengths.

We will now consider the edge function. The edge function is a listing of the squared edge lengths. For this particular framework B, the edge function is defined as follows:

$$f(x_1, x_2) = (|x_1 - x_2|^2, |x_1|^2, |x_2 - (1,0)|^2, |x_2|^2).$$

Consider the Jacobian matrix of $f(p)$ where $p = ((0,1), (1,1))$ is the starting position of x_1 and x_2.

$$Jf(p) = \begin{bmatrix} p_1 - p_2 & p_2 - p_1 \\ p_1 - p_3 & 0 \\ 0 & p_2 - p_4 \\ 0 & p_2 - p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The rank of $Jf(p)$ is 4, which means $Jf(p)$ is invertible at the point p.

Theorem 1. *Inverse Function Theorem*

Suppose $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on an open set containing $p \in \mathbb{R}^n$ and the rank of $Jf(p)$ is n. Then there exist neighborhoods U of p and V of $f(p)$ such that $f\colon U \to V$ is invertible, with a differentiable inverse.

For our framework B, the inverse function theorem implies that there exists an open neighborhood of p such that $f^{-1}(f(p)) \cap U = \{p\}$. Therefore there is no smooth flexing of framework B, i.e., it is rigid.

Now let's consider framework A at the same point p. The edge function for A is

$$g(x_1, x_2) = (|x_1 - x_2|^2, |x_1|^2, |x_2 - (1,0)|^2)$$

This edge function gives the following Jacobian matrix.

$$Jg(p) = \begin{bmatrix} p_1 - p_2 & p_2 - p_1 \\ p_1 - p_3 & 0 \\ 0 & p_2 - p_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rank of $Jg(p)$ is 3.

Theorem 2. *Implicit Function Theorem*

Suppose $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable on an open set containing (a,b) and $F(a,b) = c$. Let M be the $m \times m$ matrix

$$\left(\frac{\partial F_i}{\partial x_{n+j}}(a,b) \right)_{1 \le i,j \le m}$$

If M has rank equal to m, there is an open set $A \subset \mathbb{R}^n$ containing a and an open set $B \subset \mathbb{R}^m$ containing b. For each $x \in A$ there is a unique $g(x) \in B$ such that $F(x, g(x)) = c$ and the function $g\colon A \to B$ is differentiable.

To fully understand this theorem, let's see what it means in terms of our framework A. The mapping we are interested in is the edge function for the framework,

$$g =\colon \mathbb{R}^4 \simeq \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3;$$

in this case $m = 3$, $a = 0$, and $b = (1,1,1)$ since $p = ((0,1),(1,1))$. Considering the Jacobian matrix above, we see that M is the following matrix.

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rank of M is 3, thus implying that there exists a differentiable function g defined on an open interval, I, about $0 \in \mathbb{R}$, and an open neighborhood U of p, such that

$$\{(t, g(t)) \mid t \in I\} = f^{-1}(f(p)) \cap U.$$

Since g is continuous, this gives us an entire family of solutions $x \in U$ to the equation $f(x) = f(p)$. This means framework A is flexible, as expected. So we have found that the Jacobian matrix of the edge function gives us a simple way to determine the rigidity of a framework. Namely, if the Jacobian matrix has full rank, the framework is rigid; if it does not, the framework is flexible.

1.2 The Formal Language of Frameworks

In order to fully understand and further discuss this topic it will be useful to formalize the concepts we have been talking about. The examples of frameworks we discussed earlier are types of graphs, as we can see from the following definitions.

Definition 3. An *abstract framework* is a graph $G = (V, E)$, where $V = \{1, 2, \ldots, v\}$ is the set of vertices and E is a collection of pairs of elements of V, i.e., the edges.

Definition 4. A *framework* in $\mathbb{R}^d$, denoted $G(p)$, is a pair (G, p), where G is an abstract framework and $p \in \mathbb{R}^d \times \ldots \times \mathbb{R}^d = \mathbb{R}^{dv}$, i.e., p is a set of v d-tuples.

A framework $G(p)$ is equivalent to an embedding of the graph G into $\mathbb{R}^d$ where the i-th vertex is located at p_i.

Although we only considered the edge function for a two-dimensional framework, it can be defined for frameworks in any dimension.

Definition 5. Given a framework $G(q)$ in $\mathbb{R}^d$ with $|V| = v$ and $|E| = e$, we can define the *edge function* in the following manner.

$$f : \mathbb{R}^{dv} \longrightarrow \mathbb{R}^e$$

$$(q_1, \ldots, q_v) \longrightarrow (\ldots, |q_i - q_j|^2, \ldots) \quad i, j \in E, i < j$$

We are also going to use the Jacobian matrix of the edge function. Consider a famework $G(p)$ in $\mathbb{R}^d$ with $|E| = e$ and $|V| = v$. Then the edge function is

$$f(p) = (\ldots, |p_i - p_j|^2, \ldots).$$

Then $Jf(p)$ will be a $e \times dv$ matrix with e rows and dv columns. The number of columns arise because each of the v vertices has d partial derivatives. Often we will denote an element of $Jf(p)$ as $(q_i - q_j)$, but it is important to remember that it is a vector in $\mathbb{R}^d$. Take $G(p)$ and f to be as described above. Then one half of the Jacobian matrix of f is the following $e \times dv$ matrix.

$$\begin{array}{cc} & \begin{array}{ccccccccc} p_1 & \cdots & p_i & \cdots & p_j & \cdots & p_v \end{array} \\ \text{edge}\{i,j\} & \begin{bmatrix} \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & (p_i - p_j) & \cdots & (p_j - p_i) & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \end{bmatrix} \end{array}.$$

Now let's consider framework A using the definitions just given. Then framework A is the graph $G(p)$ where $p = (p_1, p_2, p_3, p_4) = (0, 0, 1, 0, 0, 1, 1, 1)$, $V = \{p_1, p_2, p_3, p_4\}$, and $E = \{\{p_1, p_2\}, \{p_1, p_3\}, \{p_2, p_4\}, \{p_3, p_4\}\}$. Let's look at the rigidity of $G(p)$ more formally by understanding the set

$$f^{-1}(f(p)) = \{q \in \mathbb{R}^{dv} \mid G(p) \text{ and } G(q) \text{ have corresponding edge lengths equal}\}.$$

We can see from our prior discussion of the flexibility of framework A that $f^{-1}(f(p))$ includes the circular flexing of the framework described earlier. However, this is not the whole picture. Also included in $f^{-1}(f(p))$ are all the rigid motions of $G(p)$.

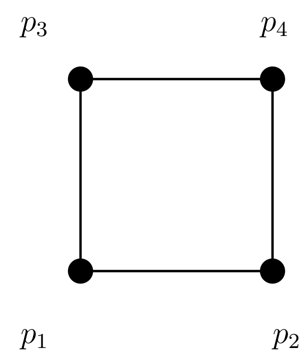

Figure 1.6: Framework G(p)

Definition 6. A *rigid motion* of $\mathbb{R}^d$ is a mapping $T\colon \mathbb{R}^d \to \mathbb{R}^d$ which preserves distances. In other words, the following condition holds.

$$|Tx - Ty| = |x - y| \quad \text{for all} \quad x, y \in \mathbb{R}^d.$$

One way to think of the rigid motions of a d-dimensional framework $G(p)$ is to think of all the ways you can move $G(p)$ through $\mathbb{R}^d$ without changing the shape given by our original p, i.e., the set of all translations and rotations. However, rigid motions will also include reflections about a hyperplane.

Definition 7. Two points $p, q \in \mathbb{R}^d$ are *congruent*, denoted $p \sim q$ if there exists a rigid motion T such that $T(p_i) = q_i$ for all i.

Now let's consider the set $M = \{m \in \mathbb{R}^d \mid m \sim p\}$ of all points m congruent to p. If $p_1, \ldots, p_v$ don't lie on a hyperplane, then M is isomorphic to the manifold of rigid motions of p. The dimension of M can be shown to be $\frac{d(d+1)}{2}$.

It will be helpful in our study of frameworks to understand what a manifold is, so a brief introduction to the subject will be given. One can think of a manifold as a blob in space, or more particularly, as a blob which 'looks like' a piece of $\mathbb{R}^d$ near any particular point. For a detailed examination of manifolds, refer to [8] and [4]. An example of a manifold is the shell of a sphere in $\mathbb{R}^3$, as shown in Figure 1.7. At any point on the sphere pictured, such as x_1, there is an open neighborhood U of x_1 isomorphic to a disk in $\mathbb{R}^2$. An example of something that is not a manifold is shown

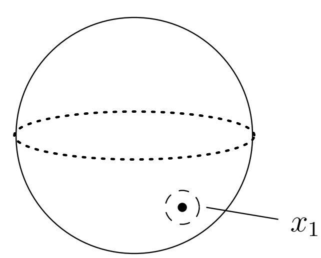

Figure 1.7: Manifold

in Figure 1.8. The point which causes the trouble is x_1. No open neighborhood of x_1 is isomorphic to $\mathbb{R}^d$ for any d. For a more formal understanding of manifolds we

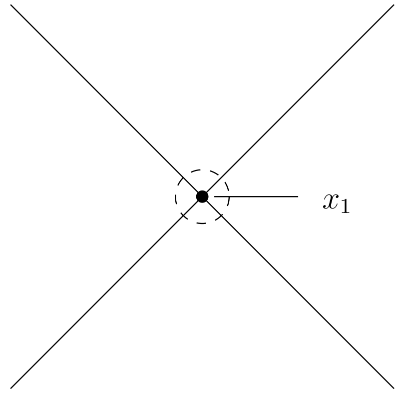

Figure 1.8: Not a Manifold

will need the following definitions.

Definition 8. Let X and Y be subsets of $\mathbb{R}^d$ and $F\colon X \to Y$ be a continuously differentiable map. If X and Y are not both open, we assume F can be extended to open sets containing X and Y. If F is one-to-one and onto and $F^{-1}\colon X \to Y$ is continuously differentiable, then X and Y are *diffeomorphic*.

If two sets are diffeomorphic, we think of them as being equivalent. Consider the sphere discussed above. A blob with no holes is diffeomorphic to a sphere, but a torus is not diffeomorphic to a sphere. We now have sufficient terminology for the

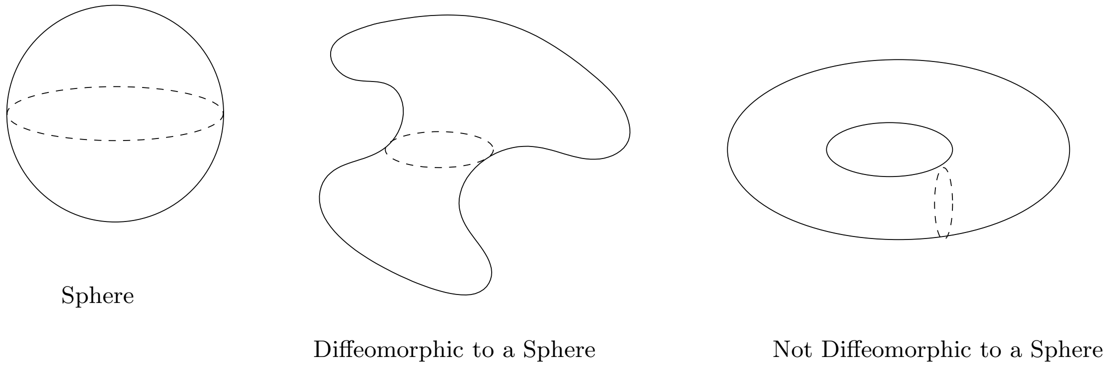

Figure 1.9: Diffeomorphic to Sphere

definition of a manifold.

Definition 9. Let X be a subset of $\mathbb{R}^d$. The set X is a *k-dimensional manifold* if it is locally diffeomorphic to $\mathbb{R}^k$. This means that for all $x \in X$ there exists a neighborhood $U \subseteq X$ of x which is diffeomorphic to an open set $V \subseteq \mathbb{R}^k$.

Now we can begin to discuss the flexibility of frameworks in greater detail.

Definition 10. Let $G(p)$ be a framework in $\mathbb{R}^d$ with edge function f. Then $G(p)$ is *flexible* if there exists continuous $x\colon [0,1] \to \mathbb{R}^{dv}$ such that the following conditions hold:

1. $x(0) = p$
2. $x(t) \in f^{-1}(f(p))$ for all t
3. $x(t)$ is not congruent to p for $t \in (0,1]$.

The function x is a *flexing* of $G(p)$. The framework $G(p)$ is *rigid* if it is not flexible.

Suppose we have a flexing $x(t) \in f^{-1}(f(p))$ for all $t \in \mathbb{R}$, where $x(0) = p$. Then for all $\{i,j\} \in E$ and $t \in \mathbb{R}^d$, we have

$$|x_i(t) - x_j(t)|^2 = |p_i - p_j|^2$$

When we take the derivative at $t = 0$ we get

$$(x_i(0) - x_j(0)) \cdot (x_i'(0) - x_j'(0)) = (p_i - p_j) \cdot (x_i'(0) - x_j'(0)) = 0.$$

Letting $x_i'(0) = \mu_i$ for $i = 1, \ldots, v$, we get a vector of vertex velocities $\mu = (\mu_1, \ldots \mu_v)$ where the following condition holds for all $\{i,j\} \in E$:

$$(p_i - p_j) \cdot (\mu_i - \mu_j) = 0$$

A vector $\mu = (\mu_1, \ldots, \mu_v) \in \mathbb{R}^{dv}$ satisfies the previous condition if and only if μ is an element of the kernel of $Jf(p)$. Showing this is not difficult and will be left as an exercise for the reader. Now lets consider a curve $x(t)$ on the manifold M. We can see that $x(t)$ satisfies

$$|x_i(t) - x_j(t)|^2 = |p_i - p_j|^2$$

which implies $x(0) = (\mu_1, \ldots, \mu_v)$ also satisfies

$$(p_i - p_j) \cdot (\mu_i - \mu_j) = 0.$$

Hence, $x'(0)$ is an element in the kernel of $Jf(p)$.

Definition 11. The *tangent space*, denoted T_p, of M at p is the collection of all such $x'(0)$ as x varies over curves in M.

We have just seen that $T_p \subseteq \ker Jf(p)$, which motivates the following definition of rigidity.

Definition 12. Let $G(p)$, where $p \in \mathbb{R}^{dv}$, be a framework with an edge function f. Then the framework $G(p)$ is *infinitesimally rigid* in $\mathbb{R}^v$ if $T_p = \ker Jf(p)$ and *infinitesimally flexible* otherwise. The elements of the set $\ker J(p) \setminus T_p$ are called *infinitesimal flexings* of $G(p)$.

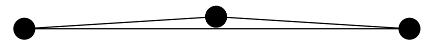

Figure 1.10: Degenerate Triangle

It is easy to suppose, and true, that if a framework is flexible this implies it is infinitesimally flexible. However, the converse is not true. We will illustrate this with the following example of a framework which is infinitesimally flexible, but not flexible. The framework we will be working with is a degenerate triangle, where all three vertices are collinear. For our example, $p = ((0, 0), (1, 0), (2, 0))$ and $G(p)$ is illustrated in Figure 1.10.

The edge function for $G(p)$ is

$$f(x_1, x_2, x_3) = (|x_1 - x_2|^2, |x_1 - x_2|^2, |x_2 - x_3|^2)$$

We get the Jacobian matrix at the point p as follows.

$$Jf(p) = 2 \begin{bmatrix} p_1 - p_2 & p_2 - p_1 & 0 \\ p_1 - p_3 & 0 & p_3 - p_1 \\ 0 & p_2 - p_3 & p_3 - p_2 \end{bmatrix} = 2 \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

After performing some row operations on $Jf(p)$ we get the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the row reduced matrix we see that the rank of $Jf(p)$ is 2 and by the Rank-Nullity theorem the dimension of the kernel of $Jf(p)$ is 4. Earlier we found that $\dim M = \frac{d(d+1)}{2}$ as long as the vertices of the framework did not lie on a hyperplane, which does not apply to our degenerate triangle. Instead we have, in this case, that $\dim M = 2$. We can see this from the following explanation. Each configuration congruent to $G(p)$ in a rigid motion may be obtained by first rotating $G(p)$ about the origin by θ and then translating along the ray at the origin determined by $G(p)$, as shown in Figure 1.11.

Therefore, there are elements in $\ker Jf(p) \setminus T_p$ and the degenerate triangle is infinitesimally flexible. With a little thought, it is easy to see that $G(p)$ is rigid, and so we have a framework which is both rigid and infinitesimally flexible.

Recall the manifold $M = \{q \in \mathbb{R}^{dv} \mid q \sim p\} \subseteq f^{-1}(f(p))$ of rigid motions we discussed previously. Algebraic geometry implies that $G(p)$ is rigid if and only if there exists an open set $U \subseteq \mathbb{R}^{dv}$ with $p \in U$ such that $U \cap f^{-1}(f(p)) = U \cap M$. For more information about this please see [5].

More can be said about rigidity and flexibility if we introduce the concept of regular points.

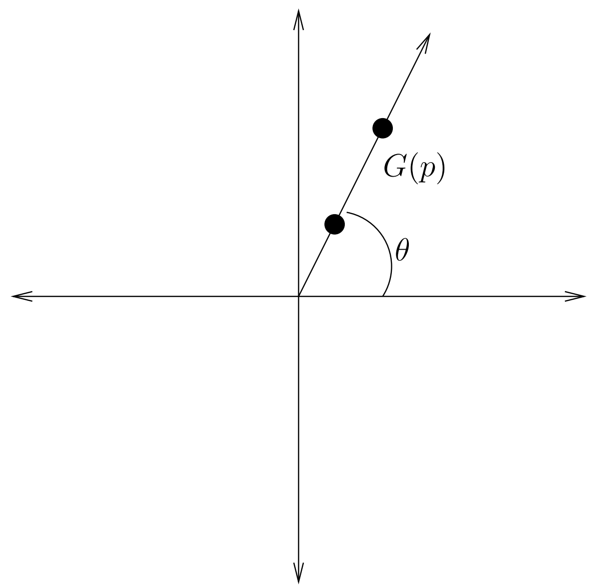

Figure 1.11: Finding $\dim M$

Definition 13. Let $f\colon \mathbb{R}^{dv} \to \mathbb{R}^e$ be an edge function. Also, let $k = \max\{\operatorname{rank} Jf(p) \mid p \in \mathbb{R}^{dv}\}$. If $\operatorname{rank} Jf(p) = k$, then p is a *regular point* for f.

The following is a standard theorem from the theory of manifolds.

Theorem 14. If p is a regular point, then $f^{-1}(f(p))$ is a manifold of dimension $dv - k$, where $k = \operatorname{rank} Jf(p)$.

So we have that if p is a regular point then both M and $f^{-1}(f(p))$ are manifolds with $M \subseteq f^{-1}(f(p))$. Recall that if $p_1, \ldots, p_v$ do not lie on a hyperplane, then $\dim M = \frac{d(d+1)}{2}$, and that if $G(p)$ is rigid then $U \cap f^{-1}(f(p)) = U \cap M$ for some open set U containing p. From this we can see that $G(p)$ is rigid if and only if $\dim M = \dim f^{-1}(f(p))$, i.e., $\frac{d(d+1)}{2} = nv - k$. We know by the Rank-Nullity theorem that $\dim\ker Jf(p) = nv - k$. So by rearranging the equation we find

$$\operatorname{rank} Jf(p) = k = dv - \frac{d(d+1)}{2}.$$

Similarly, we can see that if $G(p)$ is flexible, then $\dim M < \dim f^{-1}(f(p))$ implies

$$\operatorname{rank} Jf(p) < dv - \frac{d(d+1)}{2}.$$

We would like to show that a framework $G(p)$ is infinitesimally rigid if and only if $G(p)$ is rigid and p is regular. Before we state and prove this theorem it would be helpful to have the following lemmas.

Lemma 15. The complement of the set of solutions to a system of polynomial equations is either empty or an open dense set.

We will not prove this here, but accept it as a fact from algebraic geometry and use it to prove the following two lemmas.

Lemma 16. The set of regular points of the edge function f is an open dense set.

Proof. Let $r = \max_q\{\operatorname{rank} Jf(q)\}$ and let $R = \{q \mid \operatorname{rank} Jf(q) = r\}$ be the set of regular points of f. The complement of this set is the set

$$\begin{aligned} R' &= \{q \mid \operatorname{rank} Jf(q) < r\} \\ &= \{q \mid \text{the determinants of all } r \times r \text{ submatrices of } Jf(q) \text{ vanish}\}. \end{aligned}$$

So R' is the solution set to a system of polynomial equations. By Lemma 15 we have that R is an open dense set. □

Lemma 17. Suppose $v \geq d$. Then the set

$$P = \{p = (p_1, \ldots, p_v) \in \mathbb{R}^{dv} \mid p_1, \ldots, p_v \text{ do not lie on a hyperplane}\}$$

is open and dense in $\mathbb{R}^{dv}$.

Proof. If $v \geq d$, it is clear that the set P is nonempty. Suppose $p_1, \ldots, p_v$ lie on a hyperplane defined by $a_0 + \sum_i a_i x_i = 0$. This is equivalent to $a_0 + \sum_i a_i p_{ji} = 0$ for all j. This is true if and only if

$$\begin{bmatrix} 1 & p_1 \\ 1 & p_2 \\ \vdots & \vdots \\ 1 & p_v \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix} = 0$$

for some $a \neq 0$, thinking of each p_j as a row vector. This holds if and only if

$$\operatorname{rank} \begin{bmatrix} 1 & p_1 \\ 1 & p_2 \\ \vdots & \vdots \\ 1 & p_v \end{bmatrix} < d + 1.$$

Equivalently we have the determinants of all $d + 1 \times d + 1$ submatrices of the displayed matrix vanish. By Lemma 15 we have that all points p which do not lie on a hyperplane form an open dense set. □

Theorem 18. Let $G(p) \in \mathbb{R}^d$ be a framework and suppose $p_1, \ldots, p_v$ don't lie on a hyperplane. Then $G(p)$ is infinitesimally rigid if and only if p is a regular point and $G(p)$ is rigid.

Proof. Suppose p is a regular point and $G(p)$ is rigid. Then $\dim \ker Jf(p) = \frac{d(d+1)}{2}$ and therefore $\ker Jf(p) = T_p$ which implies $G(p)$ is infinitesimally rigid. By Lemmas 16 and 17 we know there exists a point $q \in \mathbb{R}^{dv}$ which is regular and $q_1, \ldots, q_v$ don't lie in a hyperplane. So $G(p)$ is infinitesimally rigid if and only if the kernel of

$Jf(p) = T_p$. Equivalently, we have that the dimension of $Jf(p)$ equals the dimension of T_p. So we have

$$dv - \operatorname{rank} Jf(p) = \frac{d(d+1)}{2}.$$

By rearranging the equation we get

$$\operatorname{rank} Jf(p) = dv - \frac{d(d+1)}{2}.$$

Since q is regular we know $\operatorname{rank} Jf(p) \leq \operatorname{rank} Jf(q)$ and $\operatorname{rank} Jf(q) \leq dv - \frac{d(d+1)}{2}$. Putting this all together we find

$$dv - \frac{d(d+1)}{2} = \operatorname{rank} Jf(p) \leq \operatorname{rank} Jf(q) \leq dv - \frac{d(d+1)}{2}.$$

So $\operatorname{rank} Jf(p) = \operatorname{rank} Jf(q)$ which implies p is a regular point. It has been established previously that infinitesimal rigidity implies rigidity so we are done. □

1.3 Frameworks and the Cross Product

We will now begin to study frameworks which are formed from polytopes in the following way. Define $G(p)$ to be the framework whose edges and vertices are the edges and vertices of a polytope. To begin with, it will be helpful to formally define some concepts relating to polytopes. Our basic reference is [9].

Definition 19. A *V-polytope* is the convex hull of a finite number of points in $\mathbb{R}^d$.

Definition 20. An *H-polytope* is the bounded intersection of finitely many closed halfspaces in $\mathbb{R}^d$.

It can be shown that H-polytopes and V-polytopes are equivalent and shall be referred to as *polytopes* throughout this paper.

Definition 21. The *dimension* of a polytope is the dimension of its affine hull.

Definition 22. Let $P \subseteq \mathbb{R}^d$ be a polytope. A linear inequality $cx \leq c_0$ is *valid* for P if it is satisfied for all points $x \in P$.

Definition 23. A *face* of a polytope P is any set of the form

$$F = P \cap \{x \in \mathbb{R}^d \colon cx = c_0\}$$

where $cx \leq c_0$ is a valid for P.

Definition 24. The *proper faces* of a polytope are all the faces except for the whole polytope and the empty face.

Definition 25. The *vertices* of a polytope are the 0-dimensional faces; the *edges* of a polytope are the 1-dimensional faces.

Definition 26. The *facets* of a polytope are the faces of dimension one less than that of the entire polytope.

We are now going to study the rigidity of frameworks formed from polytopes. For ease of discussion we will refer to the Jacobian matrix of the edge function of a framework as the *rigidity matrix.* The rigidity matrix of a framework F will be denoted $R(F)$. Given two polytopes, P and Q, there are several ways to construct a new polytope. A natural question which arises from considering the rigidity matrix of a polytopal framework P is how the rigidity matrix of a construction involving P is related to $R(P)$.

The construction we are interested in is the cross product. As we shall see, the rank of the rigidity matrix of the cross product can be expressed elegantly in terms of the ranks of the original frameworks.

Definition 27. Let P be a polytope of dimension d and Q be a polytope of dimension d'. The *cross product* of P and Q is the following

$$P \times Q = \{(p, q) \mid p \in P, q \in Q\}.$$

The cross product $P \times Q$ has dimension $d + d'$.

An example of a simple cross product is as follows. Suppose we have two polytopal frameworks P and Q which are line segments of equal length; then the cross product is a square, as illustrated in Figure 1.12.

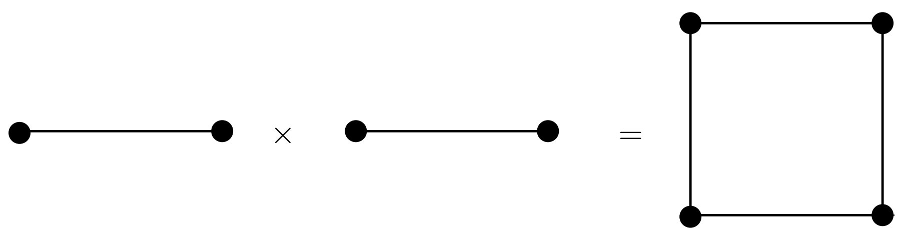

Figure 1.12: Cross Product

Theorem 28. Suppose P and Q are polygonal frameworks and let v_P and v_Q denoted the number of vertices of P and Q, respectively. Then

$$\operatorname{rank} R(P \times Q) = v_Q \operatorname{rank} R(P) + v_P \operatorname{rank} R(Q)$$

Proof. We will begin by finding the rigidity matrix of $P \times Q$. It is helpful to note that the edges of $P \times Q$ are formed either from an edge of P cross a vertex of Q or from an edge of Q cross a vertex of P. Let v_i, $0 \leq i \leq v_P$, be the vertices of P and w_i, $0 \leq i \leq v_Q$, be the vertices of Q. Let us first consider the entries of $R(P \times Q)$ which correspond to the edges formed from an arbitrary edge of P and a fixed vertex w of Q.

Then we have

$$\begin{array}{cc} & \begin{array}{ccccccc} (v_1,w) & \cdots & (v_i,w) & \cdots & (v_j,w) & \cdots & (v_{v_P},w) \end{array} \\ \text{edge } P\times w & \left[\begin{array}{ccccccc} \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & (v_i,w)-(v_j,w) & \cdots & (v_j,w)-(v_i,w) & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \end{array}\right]. \end{array}$$

Note that when we subtract $(v_i, w) - (v_j, w)$ we get $(v_i - v_j, 0)$. Remember that we are writing the entries of $R(P \times Q)$ in this manner for ease of typesetting, what $(v_i - v_j, 0)$ really is a vector with $d + d'$ elements: from $v_i - v_j$ we get d of the elements and the other d' are zeros. With some permutation of the columns, which doesn't affect the rank, we derive the matrix $R(P)$ with several columns of zeros appended. Repeating the process for each vertex of Q we have a copy of $R(P)$ for each vertex w_i, giving us v_Q copies of $R(P)$. A similar process, which the reader can work through if desired, gives us v_P copies of $R(Q)$ in $R(P \times Q)$. The matrix we now have after permuting the columns, $R(P \times Q)$, is shown below. The zeros in this matrix are zero vectors.

$$\begin{bmatrix} R(P) & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & R(P) & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \ddots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & R(P) & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & R(Q) & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & R(Q) & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & R(Q) & 0 \end{bmatrix}$$

With the matrix in this form it is clear that $\operatorname{rank} R(P \times Q) = v_Q \operatorname{rank} R(P) + v_P \operatorname{rank} R(Q)$, however we should understand why it is possible to arrange the columns of $R(P \times Q)$ in this manner. Recall that when considering the row of $R(P \times Q)$ indexed by the edge $(v_i, v_j) \times w_k$ we have an entry $(v_i - v_j, 0)$, where 0 is a d'-vector in the column indexed by v_i in $R(P \times Q)$. Also in this column we have an entry for the edge $v_i \times (w_k, w_l)$, which turns out to be $(0, w_k - w_l)$, where 0 is a d-vector. This means that in all the columns with entries of $R(P)$ there are zeros in all entries except those with an edge of P involved. Similarly, for columns with entries of $R(Q)$ there are all zeros except for entries with edges of Q involved. Therefore we can rearrange the matrix the way shown. □

This proof is rather difficult to parse, so the reader is encouraged to use the following example to clarify any confusion. Let $Q = G(q)$ be the unit square with $q = (0, 0, 1, 0, 0, 1, 1, 1)$. Let $P = G(p)$ be the unit interval with $p = (0, 1)$. To find $p \times q$ we will consider q to be an element of $\mathbb{R}^{2^4} = \mathbb{R}^8$, namely that $q = ((0, 0), (1, 0), (0, 1), (1, 1))$. By taking the cross product of first 0 and then 1 from p with each pair of q we find $p \times q$. Then $P \times Q = G(p \times q)$ is the unit cube with

$$p \times q = (0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1).$$

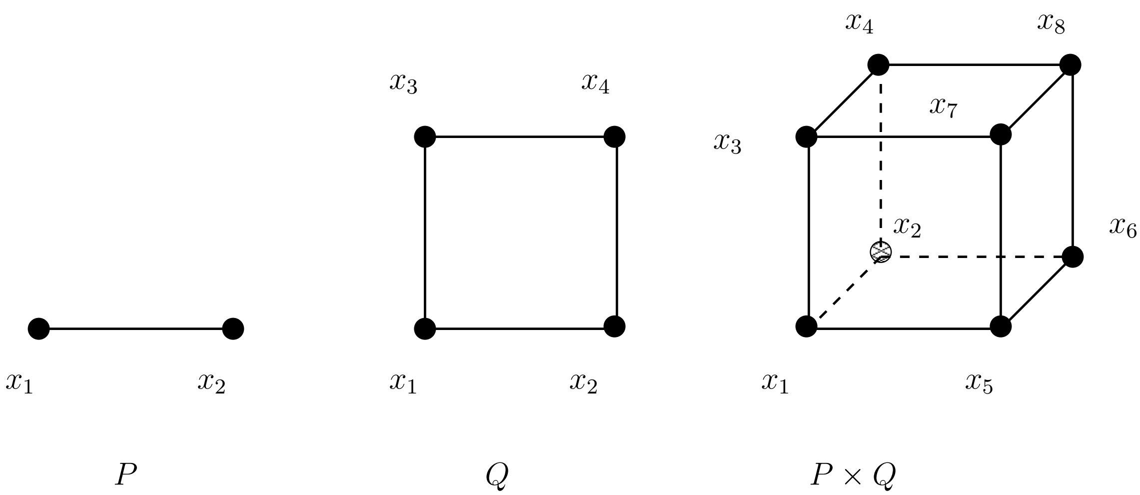

Figure 1.13: P,Q, and $P \times Q$

These frameworks are shown in Figure 1.13.

The edge function for Q is

$$f_Q(x_1, x_2, x_3, x_4) = (|x_1 - x_2|^2, |x_1 - x_3|^2, |x_2 - x_4|^2, |x_3 - x_4|^2).$$

From this we get the rigidity matrix:

$$R(Q) = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

Clearly, $\operatorname{rank} R(Q) = 4$. Now consider P. The edge function in this case is $f_P(x_1, x_2) = |x_1 - x_2|^2$ which gives us the matrix

$$R(P) = \begin{bmatrix} -1 & 1 \end{bmatrix}.$$

So we have $\operatorname{rank} R(P) = 1$. In our example $v_Q = 4$ and $v_P = 2$, so

$$\operatorname{rank} R(P \times Q) = v_Q \operatorname{rank} R(P) + v_P \operatorname{rank} R(Q) = 4(1) + 2(4) = 12.$$

Let's find $\operatorname{rank} R(P \times Q)$ by hand to make sure we understand the whole process. In this case the edge function is

$$\begin{aligned} &f_{P \times Q}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \\ &\quad (|x_1 - x_2|^2, |x_1 - x_3|^2, |x_1 - x_5|^2, |x_2 - x_4|^2, |x_2 - x_6|^2, |x_3 - x_4|^2, \\ &\quad |x_3 - x_7|^2, |x_4 - x_8|^2, |x_5 - x_6|^2, |x_5 - x_7|^2, |x_6 - x_8|^2, |x_7 - x_8|^2). \end{aligned}$$

From this we find $R(P \times Q)$ to be the following matrix.

$$\begin{bmatrix}
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1
\end{bmatrix}$$

As we can see, the rank of $R(P \times Q)$ is 12.

1.4 Stresses of Frameworks

It will now be convenient to introduce the idea of stresses on a framework. We would like the reader to have some familiarity with stresses in the simpler context of frameworks before we introduce stresses of tensegrities in the next chapter. Let's picture the edges as springs that are either stretched or compressed. Consider a vertex p_i and an edge connected to that vertex,$\{p_i, p_j\}$. Then there is a force exerted on the vertex p_i along the edge $\{p_i, p_j\}$, say $\omega_{ij}(p_i - p_j)$, where $\omega_{ij} \in \mathbb{R}$. The force exerted on the vertex p_j is $\omega_{ij}(p_j - p_i)$ and is clearly equal and opposite to the previous force. To continue with the spring analogy, $\omega_{ij} > 0$ is a compression of the edge, while $w_{ij} < 0$ is a stretching of the edge. Now let us consider a framework for which the sum of all the stresses on each vertex is zero. This gives us the following definition.

Definition 29. Let $G(p)$ be a framework with $p = (p_1, \dots, p_v)$. Let $a(i)$ denote the set of vertices sharing an edge with vertex p_i. Then the collection $\{w_{ij}\}$ is a *self stress* on $G(p)$ if the following condition holds.

$$\sum_{j \in a(i)} \omega_{ij}(p_i - p_j) = 0.$$

We can describe the self stresses of a framework using the Jacobian matrix of the edge function. A self-stress is a linear relation on the rows of $Jf(p)$ or, equivalently, an element of the transposed matrix $Jf(p)^t$. To fully understand self stresses, let's do an example using the framework depicted in Figure 1.14. We'll call this framework $G(p)$ and let $p = (0, 0, 1, 0, 0, 1, 1, 1)$.

Then we have the edge function

$$f(x_1, x_2, x_3, x_4) = (|x_1 - x_2|^2, |x_1 - x_3|^2, |x_1 - x_4|^2, |x_2 - x_3|^2, |x_2 - x_4|^2, |x_3 - x_4|^2).$$

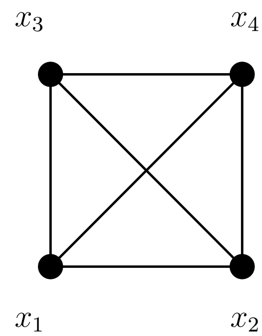

Figure 1.14: $G(p)$

From this we get the Jacobian matrix at the point p which looks like the following.

$$\frac{1}{2}Jf(p) = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

Then we can find the transpose of the Jacobian matrix.

$$\frac{1}{2}Jf(p)^t = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

To find the kernel we perform Gausian elimination to get the following matrix.

$$\frac{1}{2}Jf(p)^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So we find that

$$\ker Jf(p)^t = \text{ span} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

Now we need to understand how this vector tells us the self stress of our framework. When we originally found the Jacobian matrix of $G(p)$ we set up the matrix in the following manner.

$$Jf(p) = \begin{array}{c} \\ p_1, p_2 \\ p_1, p_3 \\ p_1, p_4 \\ p_2, p_3 \\ p_2, p_4 \\ p_3, p_4 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \begin{bmatrix} p_1 - p_2 & p_2 - p_1 & 0 & 0 \\ p_1 - p_3 & 0 & p_3 - p_1 & 0 \\ p_1 - p_4 & 0 & 0 & p_4 - p_1 \\ 0 & p_2 - p_3 & p_3 - p_2 & 0 \\ 0 & p_2 - p_4 & 0 & p_4 - p_2 \\ 0 & 0 & p_3 - p_4 & p_4 - p_3 \end{bmatrix} \end{array}$$

The order in which we placed the rows of the Jacobian matrix determines which entry in the vector corresponds to which edge. In our case we find that

$$(\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34}) = (-1, -1, 1, 1, -1, -1).$$

It will help to visualize how this self stress acts upon $G(p)$. Please refer to Figure 1.15 for the duration of the following discussion.

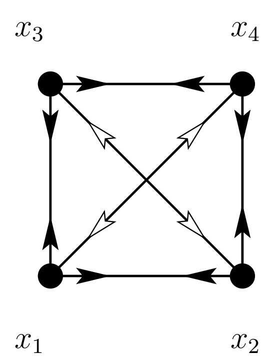

Figure 1.15: A stress on $G(p)$

It's easily verified that

$$\sum \omega_{ij}(p_i - p_j) = 0.$$

holds at each vertex.

Suppose that our framework was instead the one shown in Figure 1.16, which we'll call $\tilde{G}(p)$ with edge function $\tilde{f}$. The vector p will remain the same. By similar computations as we performed for $G(p)$ we find that $J\tilde{f}(p)^t$ has full rank, and

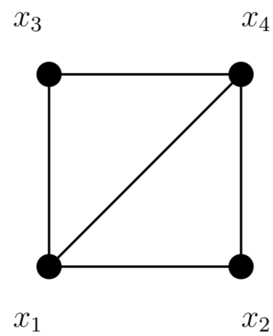

Figure 1.16: $\tilde{G}(p)$

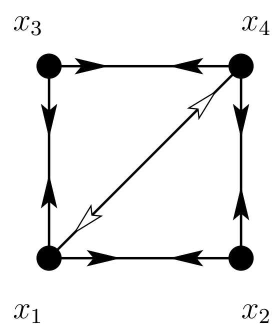

Figure 1.17: There is no stress of $\tilde{G}(p)$

therefore no self stresses. We can see that this must be true if we look at how the stresses must be arranged on the framework.

It is clear that in order for the stresses on vertices x_1 and x_4 to sum to zero, we must add stresses which force the sums at x_2 and x_3 to be nonzero.

Chapter 2

Tensegrity Frameworks

In the summer of 1948 a young student of art and architecture at Black Mountain College by the name of Kenneth Snelson attended a summer session taught by R. Buckminster Fuller. Intrigued by geometry learned in that course, Snelson created a sculpture during the following winter of a completely new type. It was a sculpture built of rods and string which had no apparent weight bearing elements and yet maintained its shape. The following summer Snelson showed his sculpture, which he called a 'floating compression', to Fuller. Fuller was very excited by it. He popularized it by incorporating it in many of his lectures and gave it the name *tensegrity*, which is a compound of tensional integrity. Because Fuller named and popularized tensegrities it is a common misconception that he invented them. For more information about Snelson's role in the discovery of tensegrities please see [6] and [7].

With the discovery of tensegrities came the need to answer the basic question of what a tensegrity is in a mathematical sense. Once that was accomplished through careful definitions (presented later in this chapter), mathematicians now had the language with which to analyze and classify tensegrities.

2.1 Tensegrities

Before we can begin defining what a tensegrity is we need to define some preliminary concepts relating to graphs. Our main sources for this section are [3] and [2].

Definition 30. A *graph* is the pair $G = (V, E)$ where V is a finite set of points called the vertices of G and E is a collection of pairs $\{i, j\}$ of vertices $i, j \in V$ called the edges.

A graph is basically a set of points connected by edges. There are no restrictions on the edges, meaning they cross, loop, or form double edges. Please see Figure 2.1 for an example.

Definition 31. A *signed graph* is a graph with a partition of the edges E into three classes, denoted $G_{\pm} = (V; E_-, E_0, E_+)$.

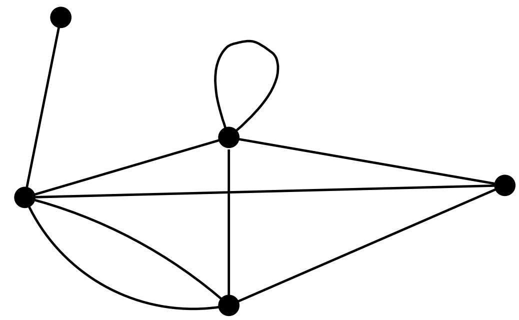

Figure 2.1: A Graph

Definition 32. A *tensegrity framework* in $\mathbb{R}^d$, denoted $G_{\pm}(p)$, is the pair $(G_{\pm}, p)$ where $p \in \mathbb{R}^d \times \ldots \times \mathbb{R}^d = \mathbb{R}^{dv}$, i.e., p is a set of v d-tuples.

Definition 33. A *cable* is a member of E_{-}; a *bar* is a member of E_0; a *strut* is a member of E_{+}.

The elements of E can be thought of as constraints on pairs of vertices in the tensegrity. A cable is an edge which can get shorter but not longer. If you think of two vertices connected by a cable, they can be pushed closer together, but there is a maximum distance between them determined by the length of the cable. A strut is in some sense the opposite of a cable; it is an edge which can get longer, but not shorter. In this case, our two vertices have a minimum distance between them determined by the length of the strut, but can be pulled apart. A bar is an edge which cannot get longer or shorter.

In the figures throughout this paper, I will use the same notation for cables, bars, and struts, as shown in Figure 2.2.

Figure 2.2:

An example of a simple tensegrity is given in Figure 2.3. As you can see, this is a tensegrity formed with two struts and four cables. To give the reader a sense of what a tensegrity really is, this tensegrity can be built quite easily from two pencils and a rubberband by holding the pencils in a cross and placing the rubberband around the ends. Clearly the elements used to build this small tensegrity are not struts and cables as they are defined to be. However, if desired, we can carefully replace the rubberband with string, which has the properties of a cable. Finding struts in the real world is a bit tricker, but for practical purposes we can build tensegrities using bars in the place of struts. The reason for this is when a tensegrity is rigid, its struts cannot get longer as there is inward pressure on the struts at the vertices.

We would like to be able to study the rigidity of tensegrities as we studied the rigidity of frameworks in the last chapter. The concepts of rigidity needed will

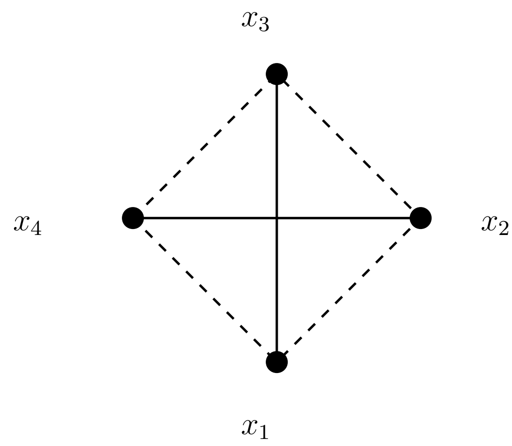

Figure 2.3: A Simple Tensegrity

take more thought to understand; they do not make as much intuitive sense as the concepts of rigidity for frameworks. One important fact to notice is that the frameworks defined in the last chapter are in fact tensegrity frameworks, just ones where $E = E_0$, i.e., all the edges are bars.

Definition 34. A tensegrity $G_\pm(p)$ *dominates* the tensegrity $G_\pm(q)$, denoted $G_\pm(p) \geq G_\pm(q)$, if the following conditions hold.

1. $|p_i - p_j| \geq |q_i - q_j|$ when $\{i, j\} \in E_-$.
2. $|p_i - p_j| = |q_i - q_j|$ when $\{i, j\} \in E_0$.
3. $|p_i - p_j| \leq |q_i - q_j|$ when $\{i, j\} \in E_+$.

If $G_\pm(p)$ dominates $G_\pm(q)$ then all of the cables of $G_\pm(p)$ are at least as long as those in $G_\pm(q)$, all of the bars are of equal length, and all the struts are no longer.

Now that we have the notion of dominance, we can define rigidity for a tensegrity.

Definition 35. A tensegrity $G_\pm(p)$ is *rigid* in $\mathbb{R}^d$ if for every continuous flex, $x(t) \in \mathbb{R}^{dv}$, with $x(0) = p$, and such that $G_\pm(p) \geq G_\pm(x(t))$ for all $0 \leq t \leq 1$, it follows that p is congruent to $x(t)$ for all $0 \leq t \leq 1$.

This definition is the converse of the definition of flexibility given for frameworks. The two definitions are similar enough that it may be helpful for the reader to review this definition given in the previous chapter.

The following conditions is equivalent to the condition given above for rigidity. There is an $\varepsilon > 0$ such that if $G_\pm(p) \geq G_\pm(q)$ and $|p - q| < \varepsilon$ then p is congruent to q.

The tensegrity shown in Figure 2.3 is rigid, where as it's dual, formed by switched struts and cables, is flexible. This is illustrated in Figure 2.4.

The notion of infinitesimal flex extends to tensegrities as we can see from the following definition.

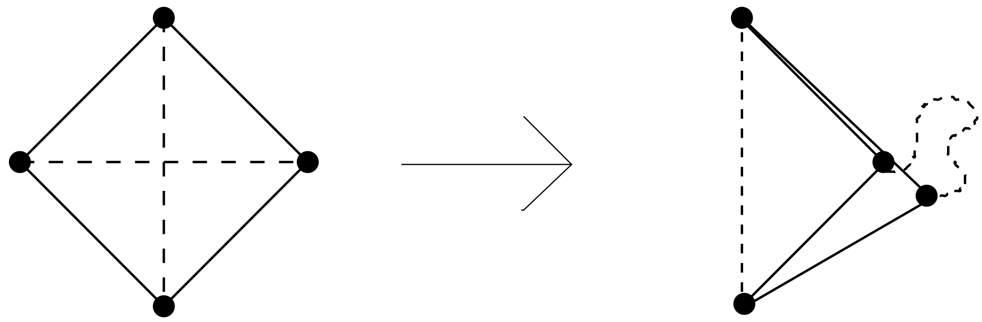

Figure 2.4: Flexing the Dual

Definition 36. An *infinitesimal flex*, also known as a *first order flex*, of a tensegrity $G_{\pm}(p)$ is an assignment $p' \colon V \to \mathbb{R}^d$, where $p'(v_i) =: p'_i$ for each $v_i \in V$, such that for each edge $\{i,j\} \in E$ the following conditions hold.

1. $(p_j - p_i) \cdot (p'_j - p'_i) \leq 0$ when $\{i,j\} \in E_-$ (cables)
2. $(p_j - p_i) \cdot (p'_j - p'_i) = 0$ when $\{i,j\} \in E_0$ (bars)
3. $(p_j - p_i) \cdot (p'_j - p'_i) \geq 0$ when $\{i,j\} \in E_+$ (struts)

Some examples of first order flexings are shown in Figure 2.5. It should be noted

Figure 2.5: First Order Flexings

that this definition holds for frameworks, as a framework can be thought of as a tensegrity where all the edges are elements of E_0, or bars.

Recall from algebra that a square matrix S is skew symmetric if $S + S^t = 0$. An example of a skew symmetric matrix is

$$S = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}.$$

Definition 37. An infinitesimal flex p' of a tensegrity $G_{\pm}(p)$ is *trivial* if there is a skew symmetric matrix S and a vector q such that $p'_i = Sp_i + q$ for all i.

For clarification, consider the following calculation. Given a skew symmetric matrix S and a (column) vector v, then

$$v^t S v = (v^t S v)^t = v^t S^t v = -v^t S v$$

This can only be true when $v^t S v$ is zero. So for a trivial flex:

$$
\begin{aligned}
(p_j - p_i) \cdot (p'_j - p'_i) &= (p_j - p_i) \cdot ((Sp_j + q) - (Sp_i + q)) \\
&= (p_j - p_i) \cdot S(p_j - p_i) \\
&= (p_j - p_i)^t \cdot S(p_j - p_i) \\
&= 0
\end{aligned}
$$

Now we have the terminology to define infinitesimally rigid.

Definition 38. A tensegrity $G_{\pm}(p)$ is *infinitesimally rigid*, also referred to as *first-order rigid*, if every infinitesimal flex is trivial.

The rigidity matrix of a tensegrity is defined the same way as for a framework.

Definition 39. The *rigidity matrix*, denoted $R(G_{\pm}(p))$ of a tensegrity $G_{\pm}(p)$ is the Jacobian matrix of the edge function of $G_{\pm}(p)$.

2.2 Stresses of Tensegrities

In this section we will study the stresses of tensegrities, which will be a generalization of the discussion of stresses of frameworks in the previous chapter.

Definition 40. Let $G_{\pm}(p)$ be a tensegrity with $p = (p_1, \ldots, p_v)$. Let $a(i)$ denote the set of vertices p_j sharing a common edge with vertex p_i. Then the collection $\{\omega_{ij}\}$ is a *self stress* $G_{\pm}(p)$ if the following equilibrium condition holds at each vertex i.

$$
\sum_{j \in a(i)} \omega_{ij}(p_j - p_i) = 0,
$$

Definition 41. A *proper self stress* is a self stress where the following conditions hold.

1. $\omega_{ij} \geq 0$ when $\{i, j\} \in E_-$ (cables)
2. $\omega_{ij} \leq 0$ when $\{i, j\} \in E_+$ (struts)

There is no condition for bars.

A tensegrity with a proper self stress means that at each of the cables, there is no inward force on that edge and at each strut there is no outward force on the edge. Let's give an example of a self stress and a proper self stress using the simple tensegrity introduced previously. We'll call this tensegrity $G_{\pm}(p)$ where $p = (0, 0, 1, 0, 0, 1, 1, 1)$, as shown in Figure 2.6. To begin with we need to find the kernel of $Jf(p)^t$. Since this calculation is the same as the one performed in Section 1.4, we will leave it out entirely. The reason these two calculations are exactly the same even though we were dealing with a framework and we are now dealing with a tensegrity

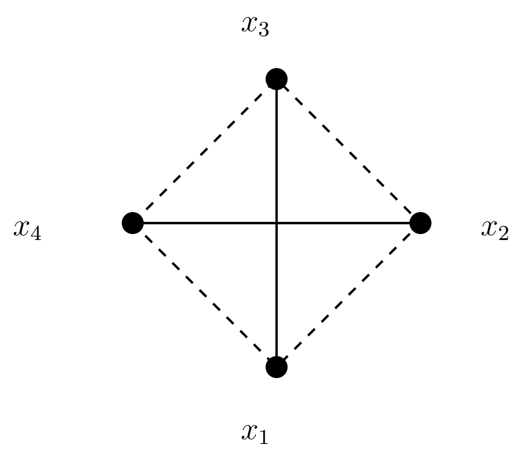

Figure 2.6: $G_{\pm}(p)$

is that the edge function is the same for both. So from our previous work we know the self stresses of $G_{\pm}(p)$ are in

$$\ker Jf(p) = \text{ span} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

If we first consider the self stress

$$\Omega = (-1, -1, 1, 1, -1, -1).$$

This is not a proper self stress, as the conditions given in Definition 41 do not hold. Now let

$$\Omega = (\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34}) = -1(1, 1, -1, -1, 1, 1) = (1, 1, -1, -1, 1, 1).$$

This self stress is a proper because

$$\omega_{14} = \omega_{23} = -1$$

and

$$\omega_{12} = \omega_{13} = \omega_{24} = \omega_{34} = 1.$$

This satisfies the condition that the stresses on struts be less than or equal to zero and the stresses on cables be greater than or equal to zero. In fact, since the inequalities are strict, this is a strict proper self stress, which is defined below.

Definition 42. A *strict proper self stress* is a proper self stress where the inequalities in Definition 41 are strict inequalities.

In this chapter we have learned the basic ideas surrounding tensegrities. This is, however, the tip of the iceberg as far as tensegrities are concerned. There are many more interesting concepts which can be understood once we understand these concepts.

2.3 Tensegrities and the Energy Function

In their paper *Second-order rigidity and prestress stability for tensegrity frameworks*, Connelly and Whiteley have refined the notion of rigidity farther, introducing the notions of second-order rigidity and prestress stability. We found in the previous chapter that infinitesimally rigid implied rigidity for frameworks. In a similar fashion for tensegrities, infinitesimally rigid implies prestress stability, which implies second-order rigidity, which in turn implies rigidity. However, none of these implications can be reversed. The definitions of second-order rigidity and prestress stability rely on the concept of energy functions of the tensegrity. This is studied in great detail in Connelly and Whiteley [3]; a short summary will be given here.

An energy function is a function on all of the edges of a tensegrity, where if an edge is changed in length, the energy in the edge changes. If a cable is lengthened, a strut is shortened, or a bar is changed in length, then the energy in that edge increases. If a cable is shortened or a strut is lengthened, then the energy decreases. This means that there is a general form for the energy functions of cables, struts and bars, which is illustrated in Figure 2.7.

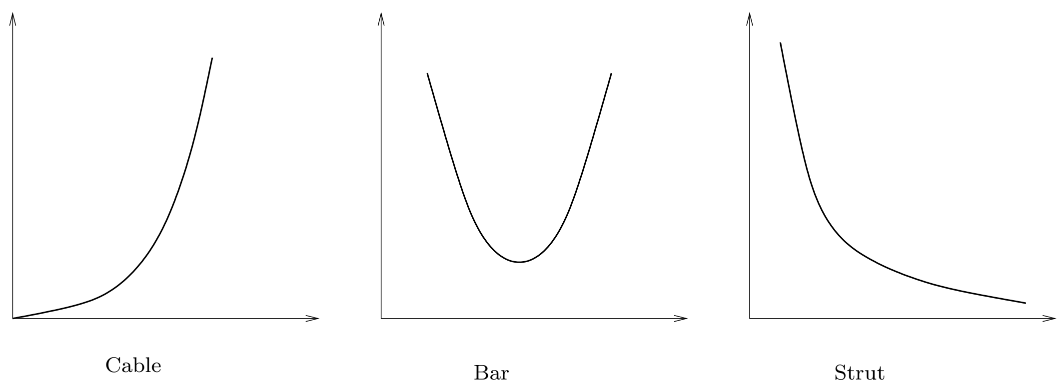

Figure 2.7: Energy Functions

Since a detailed explanation of energy functions is beyond the scope of this paper, further exploration of this topic will be left to the interested reader.

Bibliography

[1] Robert Connelly. Rigidity. In *Handbook of convex geometry, Vol. A, B*, pages 223–271. North-Holland, Amsterdam, 1993.

[2] Robert Connelly and Allen Back. Mathematics and tensegrity. *Amer. Sci.*, 86(1):142–151, 1998.

[3] Robert Connelly and Walter Whiteley. Second-order rigidity and prestress stability for tensegrity frameworks. *SIAM J. Discrete Math.*, 9(3):453–491, 1996.

[4] Victor Guillemin and Alan Pollack. *Differential topology*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1974.

[5] B. Roth. Rigid and flexible frameworks. *Amer. Math. Monthly*, 88(1):6–21, 1981.

[6] Kenneth Snelson. *Kenneth Snelson webpage*. (`http://www.kennethsnelson.net`).

[7] Kenneth Snelson. *letter to R. Moto*. (`http://www.grunch.net/snelson/rmoto.html`).

[8] Michael Spivak. *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*. W. A. Benjamin, Inc., New York-Amsterdam, 1965.

[9] Günter M. Ziegler. *Lectures on polytopes*. Springer-Verlag, New York, 1995.