

The Hilbert Function
of the
Permutahedron

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Abstract

Permuting the coordinates of the point $(1, 2, \dots, n)$ gives $n!$ points in \mathbf{C}^n . The convex hull of these points gives a well-known polytope called the permutahedron. The main result of thesis is a generating function for the Hilbert function of the vertices of the permutahedron, or, equivalently, the number of conditions placed on a generic polynomial of degree k in n variables by requiring that the above $n!$ points are zeroes. In addition, we explicitly compute, and produce a Gröbner basis for, the ideal of polynomials that are satisfied by these points.

Chapter 1

Introduction

This document is concerned with the Hilbert function of the permutahedron. In chapter 1, we present the Hilbert function and the permutahedron and introduce the problems addressed in later chapters. In chapter 2 we present the necessary machinery to crack these problems. Finally, in chapter 3 we present the results of this thesis.

1.1 The Hilbert function

Let R be a quotient of $S := \mathbf{C}[x_1, x_2, \dots, x_n]$ by an ideal, and let S_k denote those polynomials of S of degree less than or equal to k . Let R_k be the image of S_k under the natural map $S \rightarrow R$ sending a polynomial to its coset. Note that R_k is a vector space over \mathbf{C} .

Let $P \subset \mathbf{C}^n$ be a finite set of points. Define the ideal $I \subset S$ to be the set of polynomials that are satisfied by all elements of P :

$$I := \{f \in S : f(P) = 0\}.$$

Define the *(affine) Hilbert function of S/I* :

$$H_{S/I}(k) := \dim_{\mathbf{C}}((S/I)_k).$$

We will commonly refer to the *Hilbert series* of S/I , by which we mean simply $H_{S/I}(0), H_{S/I}(1), H_{S/I}(2), \dots$

It is well known that $\dim(S_k) = \binom{n+k}{k}$, so knowing the value of $H_{S/I}(k) = \dim(S_k) - \dim(I_k)$ is equivalent to knowing the dimension of I_k .

In order to calculate the Hilbert function, we may write out the generic degree k polynomial and plug in the various points of P to get linear equations in the coefficients. $H_{S/I}(k)$ is the rank of the corresponding matrix; the kernel is I_k . Therefore, we may interpret the value of $H_{S/I}(k)$ as the number of restrictions that P puts on the generic degree k polynomial.

For a proof of the following claim, see [4, pages 50–51]

Claim 1.1.1 *There exists a positive integer K such that $H_{S/I}(k) = |P| \forall k > K$.*

Example. Let $P := \{(0, 0), (1, 0), (2, 1), (2, 2), (1, 2), (0, 1)\} \subset \mathbf{C}^2$. By the method sketched above, we can compute the Hilbert series:

$$\begin{array}{ccc} \bullet & \bullet & \\ \bullet & \bullet & H_{S/I}(k) = 1, 3, 5, 6, 6, \dots \\ \bullet & \bullet & \end{array}$$

As we would expect, P puts 1 condition on the generic constant polynomial and 3 conditions on the generic linear polynomial in two variables. However, $H_{S/I}(2) = 5$, i.e., the six points put only five conditions on the generic degree 2 polynomial, so there is a one-dimensional space of degree 2 polynomials that are zero on P .

1.2 The permutahedron

Definition. The *permutahedron* is the polytope which is the convex hull of the orbit of the point $(1, 2, \dots, n) \in \mathbf{C}^n$ under the permutation representation of the symmetric group, \mathcal{S}_n .

The combinatorial structure of the permutahedron is well known. A good reference is [1].

Example. For $n = 3$, the permutahedron is the convex hull of the set of points:

$$\{(1, 2, 3), (2, 1, 3), (3, 1, 2), (1, 3, 2), (2, 3, 1), (3, 2, 1)\}$$

Note that the permutahedron always sits in the hyperplane $\sum_{i=1}^n x_i = \sum_{i=1}^n i$. In \mathbf{C}^3 , the permutahedron sits in a plane and is, up to a change of coordinates, the hexagon seen in the previous section. In \mathbf{C}^4 , it is a truncated octahedron.

1.3 The task at hand

The main problem solved in this thesis is the calculation of the Hilbert function of the permutahedron, or, more accurately, the set of its vertices: the $n!$ points obtained by permuting the coordinates of $(1, 2, \dots, n)$. This problem arose from the work of former Reed students Oliver Gugenheim [3] and John Mulliken [5] and of David Perkinson [6]. Briefly, let $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a monomial mapping, that is, let each coordinate function of f have the form $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for some natural numbers a_1, a_2, \dots, a_n . We can think of the vector of exponents, (a_1, a_2, \dots, a_n) , as a lattice point in \mathbf{R}^n . The function f is determined by m such points. Gugenheim, Mulliken and Perkinson have related the inflectional behavior of f with the Hilbert function of its set of exponents. In that context, the permutahedron naturally arises as the exponent set of a function with interesting inflectional behavior.

Furthermore, the problem of the Hilbert function of various orbitopes, i.e. polytopes whose vertices are the orbit of an initial point, is an inherently interesting one. Though it is beyond the scope of this document, we are interested also in the Hilbert functions of orbitopes other than the permutahedron. For further readings on orbitopes, consult [7].

Chapter 2

Machinery

Before approaching the problem directly, we need to build up a little machinery, specifically, that of symmetric polynomials and Gröbner bases.

2.1 Symmetric polynomials

As previously, let $S := \mathbf{C}[x_1, x_2, \dots, x_n]$.

Definition. A polynomial $f \in S$ is called *symmetric* if it is invariant under any permutation of the variables x_1, x_2, \dots, x_n .

Consider the expansion

$$\prod_{i=1}^n (z - x_i) = z^n - \sigma_1 x^{n-1} + \dots + (-1)^n \sigma_n.$$

With respect to x_1, \dots, x_n , the σ_i are symmetric:

$$\begin{aligned}\sigma_1 &= x_1 + \dots + x_n \\ \sigma_2 &= x_1 x_2 + x_1 x_3 + \dots + x_1 x_n + x_2 x_3 + x_2 x_4 + \dots + x_{n-1} x_n \\ &\vdots \\ \sigma_n &= x_1 x_2 \dots x_n.\end{aligned}$$

We will refer to σ_i as the i 'th *elementary symmetric polynomial* in S .

2.2 Gröbner bases

2.2.1 Monomial orderings

A polynomial $f \in S$ is simply a sum of terms, each term consisting of a complex coefficient and a monomial; we require a way to sort these terms. We will do so by imposing an ordering on the monomials.

Definition. A *monomial ordering* on S is a total order \succ on the monomials of S such that for monomials $m, m', m'' \in S$, $m \neq 1$,

$$m' \succ m'' \Rightarrow mm' \succ mm'' \succ m''.$$

Example. Let $m = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $p = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ be monomials in S . *Degree lexicographic order* is defined by: $m \succ p$ if $\deg(m) > \deg(p)$ or if $\deg(m) = \deg(p)$ and $a_i > b_i$ for the first i with $a_i \neq b_i$. For example,

$$x_1 \succ x_2 \qquad x_2 x_3 \succ x_1$$

Example. *Pure lexicographic order* is defined by: $m \succ p$ if $a_i > b_i$ for the first i with $a_i \neq b_i$. For example,

$$x_1 \succ x_2 \qquad x_1 \succ x_2 x_3$$

Definition. The *initial monomial* of a non-zero polynomial $f \in S$, $\text{init}(f)$, is the largest monomial with respect to a monomial ordering, \succ , that occurs with non-zero coefficient in f .

Now let $I \subset S$ be an ideal. By $\text{init}(I)$, we mean the ideal generated by $\{\text{init}(f) : f \in I\}$. For a proof of the next theorem, see [2, page 325].

Theorem 2.2.1 (Macaulay) $H_{S/I}(k) = H_{S/\text{init}(I)}(k)$.

2.2.2 Gröbner bases

Let G be an ideal in S and let \succ be a monomial ordering on S .

Definition. We say that $\{g_1, \dots, g_l\} \subset G$ is a *Gröbner basis* for G with respect to \succ iff $(\text{init}(g_1), \dots, \text{init}(g_l)) = \text{init}(G)$.

For a proof of the following theorem, see [8, pages 10-11].

Theorem 2.2.2 *If $\{g_1, \dots, g_l\}$ is a Gröbner basis for G , then $(g_1, \dots, g_l) = G$.*

Chapter 3

Results

We have two things in mind for this chapter. One is to identify precisely the ideal of polynomials, I , that are zero on the permutahedron, the other, to find a generating function for the Hilbert function of the permutahedron.

3.1 The ideal.

Using the notation of the previous chapter, we begin by defining a set of functions, the $\tilde{\sigma}_i$'s ($1 \leq i \leq n$):

$$\tilde{\sigma}_i := \sigma_i - \sigma_i(1, 2, \dots, n).$$

Clearly, since the σ 's are symmetric, $\tilde{\sigma}_i \in I$. In fact, we will find that $(\tilde{\sigma}_1, \dots, \tilde{\sigma}_n) = I$.

To that end, we will construct a Gröbner basis for I , employing purely lexicographic order. Of course, we will need to know $\text{init}(I)$.

Claim 3.1.1 $\text{init}(I) = (x_1, x_2^2, x_3^3, \dots, x_n^n)$.

Proof. Suppose the contrary: let $f \in I \setminus \{0\}$ such that $x_i^i \nmid \text{init}(f)$ for any $1 \leq i \leq n$. We will require that f is minimal in the sense that if $g \in I \setminus \{0\}$ such that $x_i^i \nmid \text{init}(g)$ for $1 \leq i \leq n$, then $\text{init}(g) \succ \text{init}(f)$ or $\text{init}(g) = \text{init}(f)$. Let k be the smallest integer such that $x_k \mid \text{init}(f)$. Since none of x_1, \dots, x_{k-1} divide $\text{init}(f)$, they neither divide any terms of f and we may write

$$f = \sum_{i=0}^{k-1} x_k^i g_i$$

where $g_i \in \mathbf{C}[x_{k+1}, \dots, x_n]$ for $i = 0, \dots, k-1$. Let p be the smallest index such that $g_p \neq 0$; it follows that $\text{init}(f) = x_k^p \text{init}(g_p)$. Since $x_k \mid \text{init}(f)$, we know that $p > 0$; hence $\text{init}(g_p) \prec \text{init}(f)$.

We will now show that each $g_i \in I$, contradicting the minimality of f . Let $[a_1, \dots, a_n]$ be some permutation of $1, \dots, n$. Since $f \in I$, it is zero on all permutations of $(1, 2, \dots, n)$. In particular, using the above expression for f and permuting a_1, a_2, \dots, a_k , we may write

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{k-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_k & a_k^2 & \cdots & a_k^{k-1} \end{pmatrix} \begin{pmatrix} g_0(a_{k+1}, \dots, a_n) \\ g_1(a_{k+1}, \dots, a_n) \\ \vdots \\ g_{k-1}(a_{k+1}, \dots, a_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The left-hand matrix is a Vandermonde matrix and therefore invertible. It follows that $g_i(a_{k+1}, \dots, a_n) = 0$ for $0 \leq i \leq k-1$ and for all distinct $a_{k+1}, \dots, a_n \in \{1, \dots, n\}$, i.e. $g_i \in I$ for $0 \leq i \leq k-1$. \square

In order to continue, will require some notation. Let

$$\vec{x}_m^k := (\overbrace{0, \dots, 0}^{k-m}, x_k, x_{k+1}, \dots, x_n) \quad (1 \leq m \leq k \leq n),$$

$$c_m := \sigma_m(1, 2, \dots, n),$$

$$\tau_1 := \sum_{i=1}^n x_i - c_1 = \tilde{\sigma}_1.$$

For $1 < m \leq n$, define

$$\tau_m := \sum_{i=m}^n x_i \tau_{m-1}(\vec{x}_{m-1}^i) - c_m.$$

Note that τ_m only contains the variables x_m, x_{m+1}, \dots, x_n .

Example.

$$\begin{aligned} \tau_3 &= \sum_{i=3}^n x_i \tau_2(\vec{x}_2^i) - c_3 \\ &= x_3 \tau_2(0, x_3, \dots, x_n) + x_4 \tau_2(0, 0, x_4, \dots, x_n) + \cdots + x_n \tau_2(0, \dots, 0, x_n) - c_3 \end{aligned}$$

Lemma 3.1.2

$$\tau_m(\vec{x}_m^k) = \tau_m - \sum_{i=m}^{k-1} x_i \tau_{m-1}(\vec{x}_{m-1}^i)$$

for $1 < m \leq k \leq n$.

Proof. We're interested in $\tau_m(\vec{x}_m^k) = \tau_m(0, \dots, 0, x_k, \dots, x_n)$, which is just τ_m with all terms containing x_1, x_2, \dots, x_{k-1} set to zero. We will simply subtract from τ_m all terms containing these variables:

$$\begin{aligned} \tau_m(\vec{x}_m^k) &= \sum_{i=m}^n x_i \tau_{m-1}(\vec{x}_{m-1}^i) - c_m - (\text{terms with } x_m, \dots, x_{k-1}) \\ &= \sum_{i=m}^n x_i \tau_{m-1}(0, \dots, 0, x_i, \dots, x_n) - c_m - (\text{terms with } x_m, \dots, x_{k-1}) \\ &= \sum_{i=m}^n x_i \tau_{m-1}(\vec{x}_{m-1}^i) - c_m - \sum_{i=m}^{k-1} x_i \tau_{m-1}(\vec{x}_{m-1}^i) \\ &= \tau_m - \sum_{i=m}^{k-1} x_i \tau_{m-1}(\vec{x}_{m-1}^i). \end{aligned}$$

□

Proposition 3.1.3 $\{\tau_1, \tau_2, \dots, \tau_n\}$ form a Gröbner basis for I .

Proof. First we'll show that $(\tau_1, \dots, \tau_n) \subset (\tilde{\sigma}_1, \dots, \tilde{\sigma}_n)$. Since the latter ideal is a subset of I , it will follow that $(\tau_1, \dots, \tau_n) \subset I$. We'll accomplish this by inductively showing that $\tau_m = f_{m-1} \pm \tilde{\sigma}_m$ for some $f_{m-1} \in (\tau_1, \dots, \tau_{m-1})$. Congruence in the following is modulo $(\tau_1, \dots, \tau_{m-1})$.

$$\begin{aligned} \tau_m &= \sum_{i_1=m}^n x_{i_1} \tau_{m-1}(\vec{x}_{m-1}^{i_1}) - c_m \\ &= \sum_{i_1=m}^n x_{i_1} \left(\tau_{m-1} - \sum_{i_2=m-1}^{i_1-1} x_{i_2} \tau_{m-2}(\vec{x}_{m-2}^{i_2}) \right) - c_m \end{aligned}$$

$$\begin{aligned}
&\equiv - \sum_{i_1=m}^n \sum_{i_2=m-1}^{i_1-1} x_{i_1} x_{i_2} \tau_{m-2}(\vec{x}_{m-2}^{i_2}) - c_m \\
&\equiv \sum_{i_1=m}^n \sum_{i_2=m-1}^{i_1-1} \sum_{i_3=m-2}^{i_2-1} x_{i_1} x_{i_2} x_{i_3} \tau_{m-3}(\vec{x}_{m-3}^{i_3}) - c_m \\
&\vdots \\
&\equiv (-1)^m \sum_{i_1=m}^n \sum_{i_2=m-1}^{i_1-1} \cdots \sum_{i_{m-1}=2}^{i_{m-2}-1} x_{i_1} x_{i_2} \cdots x_{i_{m-1}} \tau_1(\vec{x}_1^{i_{m-1}}) - c_m \\
&\equiv (-1)^{m+1} \sum_{i_1=m}^n \sum_{i_2=m-1}^{i_1-1} \cdots \sum_{i_m=1}^{i_{m-1}-1} x_{i_1} x_{i_2} \cdots x_{i_m} - c_m \\
&= (-1)^{m+1} \tilde{\sigma}_m.
\end{aligned}$$

Now we must show that $(\text{init}(\tau_1), \dots, \text{init}(\tau_n)) = \text{init}(I) = (x_1, x_2^2, \dots, x_n^n)$. Remembering that τ_k is a function of x_k, x_{k+1}, \dots, x_n , we will achieve this by inductively proving the stronger fact that $\deg(\tau_m) = m$ and x_k^m is a monomial of τ_m for $m \leq k \leq n$. Suppose this condition holds for τ_k , where $1 \leq k < m$. As above,

$$\tau_m = \sum_{i=m}^n x_i \tau_{m-1} - \sum_{i=m}^n \sum_{j=m-1}^{i-1} x_i x_j \tau_{m-2}(\vec{x}_{m-2}^j) - c_m.$$

By the induction hypothesis, it's clear that $\deg(\tau_m) = m$. Furthermore, x_k^m ($m \leq k \leq n$) is a term of the first sum, but not of the second since $i \neq j$, completing the induction and the proof. \square

Corollary 3.1.4 $I = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_n)$.

Proof. Since $\{\tau_1, \tau_2, \dots, \tau_n\}$ form a Gröbner basis for I , we know that $(\tau_1, \tau_2, \dots, \tau_n) = I$. But we just showed that $(\tau_1, \tau_2, \dots, \tau_n) \subset (\tilde{\sigma}_1, \dots, \tilde{\sigma}_n) \subset I$. \square

3.2 The Hilbert function.

To prove our final result, we'll require the following lemma.

Lemma 3.2.1 *If $F(q) = \sum_{j \geq 0} f(j)q^j$ is a generating function for $f : \mathbf{N} \rightarrow \mathbf{Z}$, and*

$$g(k) = \sum_{i=0}^k f(i) - \sum_{i=0}^{k-a} f(i),$$

then

$$G(q) = \frac{(1 - q^a)F(q)}{(1 - q)}$$

is a generating function for g .

Proof. Observe that

$$\begin{aligned} \sum_{j \geq 0} \left(\sum_{i=0}^{j-a} f(i) \right) q^j &= f(0)q^a + (f(0) + f(1))q^{a+1} + \dots \\ &= q^a (f(0) + f(1)q + f(2)q^2 + \dots) (1 + q + q^2 + \dots) \\ &= q^a \frac{F(q)}{(1 - q)}. \end{aligned}$$

It follows that

$$\begin{aligned} G(q) &= \frac{F(q)}{1 - q} - q^a \frac{F(q)}{1 - q} \\ &= \sum_{j \geq 0} \left(\sum_{i=0}^j f(i) \right) q^j - \sum_{j \geq 0} \left(\sum_{i=0}^{j-a} f(i) \right) q^j \\ &= \sum_{j \geq 0} g(j)q^j, \end{aligned}$$

as claimed. □

Theorem 3.2.2 *The Hilbert function for the permutahedron in \mathbf{C}^n is given by the generating function*

$$\sum_{j \geq 0} H_{S/I}(j)q^j = \frac{\prod_{i=1}^n (1 - q^i)}{(1 - q)^{n+1}}.$$

Proof. By induction on n . We will require some notation. Let

$$\begin{aligned} S^n &:= \mathbf{C}[x_1, x_2, \dots, x_n], \\ J^n &:= (x_1, x_2^2, x_3^3, \dots, x_n^n), \\ \mathcal{H}_n(k) &:= H_{S^n/J^n}(k), \\ \mathcal{F}_n(k) &:= H_{S^{n+1}/J^n}(k), \\ \mathcal{B}_n(k) &:= H_{S^{n+1}/(J^n, x_{n+1}^n)}(k). \end{aligned}$$

The theorem holds for $n = 1$, so suppose it holds for a particular n . Consider the sequence

$$0 \longrightarrow \left(\frac{S^{n+1}}{J^n} \right)_{k-1} \xrightarrow{\cdot x_{n+1}} \left(\frac{S^{n+1}}{J^n} \right)_k \xrightarrow{\beta} \left(\frac{S^n}{J^n} \right)_k \longrightarrow 0,$$

where α is multiplication by x_{n+1} and β is the map setting x_{n+1} to zero. It is easy to see the sequence is exact by using the basic fact that a polynomial is in a given ideal generated by monomials iff each term of the polynomial is divisible by one of the generating monomials. For instance, $x_{n+1}f \in J^n$ iff each term of $x_{n+1}f$ is divisible by some x_i^i , where $1 \leq i \leq n$. But this condition holds iff f is divisible by some x_i^i , i.e. iff $f \in J^n$. This shows α is injective. By rank-nullity, we find that

$$\begin{aligned} \dim \left(\frac{S^{n+1}}{J^n} \right)_k &= \dim \left(\frac{S^n}{J^n} \right)_k + \dim \left(\frac{S^{n+1}}{J^n} \right)_{k-1} \\ &\Rightarrow \mathcal{F}_n(k) = \mathcal{H}_n(k) + \mathcal{F}_n(k-1). \end{aligned}$$

Noting that $\mathcal{F}_n(0) = \mathcal{H}_n(0) = 1$, we conclude that

$$\mathcal{F}_n(k) = \sum_{i=0}^k \mathcal{H}_n(i).$$

Consider next the following exact sequence:

$$0 \longrightarrow \left(\frac{S^{n+1}}{(J^n, x_{n+1}^n)} \right)_{k-1} \xrightarrow{\cdot x_{n+1}} \left(\frac{S^{n+1}}{J^{n+1}} \right)_k \xrightarrow{\beta} \left(\frac{S^n}{J^n} \right)_k \longrightarrow 0,$$

where α and β are as before. It follows that

$$\dim \left(\frac{S^{n+1}}{J^{n+1}} \right)_k = \dim \left(\frac{S^n}{J^n} \right)_k + \dim \left(\frac{S^{n+1}}{(J^n, x_{n+1}^n)} \right)_{k-1}$$

$$\Rightarrow \mathcal{H}_{n+1}(k) = \mathcal{H}_n(k) + \mathcal{B}_n(k-1).$$

We require one more exact sequence before we carry on with the induction:

$$0 \longrightarrow \left(\frac{S^{n+1}}{J^n} \right)_{k-n} \xrightarrow{\cdot x_{n+1}^n} \left(\frac{S^{n+1}}{J^n} \right)_k \longrightarrow \left(\frac{S^{n+1}}{(J^n, x_{n+1}^n)} \right)_k \longrightarrow 0,$$

where the penultimate map is the natural one to the quotient. We get that

$$\begin{aligned} \dim \left(\frac{S^{n+1}}{(J^n, x_{n+1}^n)} \right)_k &= \dim \left(\frac{S^{n+1}}{J^n} \right)_k - \dim \left(\frac{S^{n+1}}{J^n} \right)_{k-n} \\ &\Rightarrow \mathcal{B}_n(k) = \mathcal{F}_n(k) - \mathcal{F}_n(k-n). \end{aligned}$$

Combining the previous three results, we find that

$$\begin{aligned} \mathcal{H}_{n+1}(k) &= \mathcal{H}_n(k) + \mathcal{B}_n(k-1) \\ &= \mathcal{H}_n(k) + \mathcal{F}_n(k-1) - \mathcal{F}_n(k-n-1) \\ &= \mathcal{H}_n(k) + \sum_{i=0}^{k-1} \mathcal{H}_n(i) - \sum_{i=0}^{k-n-1} \mathcal{H}_n(i) \\ &= \sum_{i=0}^k \mathcal{H}_n(i) - \sum_{i=0}^{k-n-1} \mathcal{H}_n(i). \end{aligned}$$

We may now apply the lemma and the induction hypothesis to finish the proof:

$$\begin{aligned} \mathcal{H}_{n+1}(k) &= \frac{(1-q^{n+1}) \prod_{i=1}^n (1-q^i)}{(1-q)(1-q)^{n+1}} \\ &= \frac{\prod_{i=1}^{n+1} (1-q^i)}{(1-q)^{n+2}}. \end{aligned}$$

□

Corollary 3.2.3 *The Hilbert function of the permutahedron, $H_{S/I}(k)$, first reaches its full value at $k = \binom{n}{2}$ and the immediately preceding value is precisely one less, that is:*

$$H_{S/I} \left(\binom{n}{2} - 1 \right) = n! - 1 \quad \text{and} \quad H_{S/I} \left(\binom{n}{2} \right) = n!$$

Proof. Let $N := 1 + 2 + \cdots + (n - 1) = \binom{n}{2}$ and compute:

$$\begin{aligned} \sum_{j \geq 0} H_{S/I}(j)q^j &= \frac{\prod_{i=1}^n (1 - q^i)}{(1 - q)^{n+1}} \\ &= (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1})(1 + q + q^2 + \cdots) \\ &= \left(\sum_{j=0}^N a_j q^j \right) (1 + q + q^2 + \cdots) \end{aligned}$$

Where $\sum_{j=0}^N a_j = n!$, since there are $n!$ terms in the product. For $t \geq N$, the coefficient of q^t is $\sum_{j=0}^N a_j = n!$. The coefficient of q^{N-1} is $\sum_{j=0}^{N-1} a_j = n! - 1$, since $a_N = 1$. \square

Bibliography

- [1] Louis J. Billera and A. Sarangarajan, *The Combinatorics of Permutation Polytopes*, Formal Power Series in Discrete Mathematics and Theoretical Computer Science, vol. 24, American Mathematical Society, 1994, pp. 1–23.
- [2] David Eisenbud, *Commutative Algebra with a view to Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, 1994.
- [3] Oliver Gugenheim, *Inflections of parameterized surfaces*, Reed College Thesis, 1993.
- [4] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, 1977.
- [5] John Mulliken, *Lotsa Dots: Self-Dual Affine Monomial Varieties*, Reed College Thesis, 1995.
- [6] David Perkinson, *Inflections of toric varieties*, University of Oslo Preprint Series, 1994.
- [7] Judy Ridenour, *Combinatorics of Orbitopes*, Reed College Thesis, 1996.
- [8] Bernd Sturmfels, *Algorithms in Invariant Theory*, Texts and Monographs in Symbolic Computation, Springer-Verlag, 1993.