The Ranks of a Tropical Matrix

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Abstract

This thesis is a general investigation into the properties of the Kapranov rank and the tropical rank of a tropical matrix. Of note is the connection between the $\mathbb{C}$-representability of a matroid $\mathcal{M}$ and the equality of the Kapranov and tropical ranks of the matroid's cocircuit matrix, the lack of Gaussian elimination for the tropical rank, and the equivalence of the Kapranov and tropical ranks for 5×5 zero-one matrices.

Introduction

Tropical algebra, also know as min-plus or max-plus algebra, has recently become a hot topic in certain circles of mathematics due to its connections to various subfields of mathematics, for example algebraic geometry (14), synchronized systems (10) and computation biology (21). This algebra was given the name 'tropical' in honor of one of its pioneers, the Brazilian mathematician Imre Simon (19), by his French colleagues including Jean-Eric Pin (17).

This word 'ranks' in the title, refer to the fact that equivalent notions of the rank of a matrix over a field are no longer equivalent for a tropical matrix. One notion is that the rank of a matrix M is the smallest dimension linear space containing the columns of M, this will be known at the Kapranov rank. Another notion is that the rank of M is the largest nonsingular submatrix of M, and this will be known as the tropical rank of M. The aim of this thesis was to answer two of the questions posed at the end of (5), namely

(Q1) Is there an analogue to Gaussian elimination for computing the tropical rank?

(Q6) Is there a 5×5 matrix whose tropical rank not is not equal to its Kapranov rank?

Chapter one establishes some basic facts and examples about tropical algebra and tropical algebraic geometry, where the latter is used to define the Kapranov rank. The second chapter is devoted to the development of the Kapranov rank, the tropical rank, and properties of the tropical determinant. An improved proof of a combinatorial formula for the tropical rank of a zero-one matrix is given, Theorem 2.3.1. The third chapter is a development of matroid theory, which following (5), presents a proof of Theorem 3.4.12 that connects the $\mathbb{C}$-representability of a matroid $\mathcal{M}$ and the equality of the Kapranov and tropical rank of its cocircuit matrix. This theorem is then used to give an example of a matrix whose Kapranov rank exceeds its tropical rank. Chapter four is devoted to establishing the basics of NP-completeness and presenting an improved proof of Theorem 4.3.12, which first appeared in (11) and along with Theorem 2.3.1 implies that computing the tropical rank of a zero-one matrix is NP-complete. Which answers question (Q1) in the negative, at least until the P = NP question is settled. Finally, Appendix A presents computations that prove that every 5×5 zero-one matrix has equal tropical and Kapranov ranks, which partially answers question (Q6) above.

Chapter 1

Basics

1.1 Tropical Basics

Definition 1.1.1. Define the *tropical semiring* to be $(\mathbb{R}, \oplus, \odot)$ where

$$a \oplus b = \min\{a, b\} \quad \text{and} \quad a \odot b = a + b.$$

Examples of tropical addition and multiplication are

$$5 \oplus 11 = 5 \quad \text{and} \quad 5 \odot 11 = 16.$$

One can easily check that the additive structure is an abelian semigroup and the multiplicative structure is an abelian group with the multiplicative identity being 0. One can verify that tropical multiplication distributes over addition. If we wish, then we can also include the formal element ∞, to obtain an additive identity element where $a \oplus \infty = \min\{a, \infty\} = a$. However even with an additive identity, the additive structure is still not a group since there are not additive inverses. For example $2 \oplus x = 3$ has no solution since $2 \oplus x < 3$ for all x. The addition and multiplication tables are as follows:

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1
2	0	1	2	2	2	2	2
3	0	1	2	3	3	3	3
4	0	1	2	3	4	4	4
5	0	1	2	3	4	5	5
6	0	1	2	3	4	5	6

$\odot$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	8
3	3	4	5	6	7	8	9
4	4	5	6	7	8	9	10
5	5	6	7	8	9	10	11
6	6	7	8	9	10	11	12

These tropical operations carry over to $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ in the natural way. For example in $\mathbb{R}^2$,

$$4 \odot \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \odot 2 \\ 4 \odot 3 \\ 4 \odot 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 9 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \oplus 2 \\ 4 \oplus 3 \\ 4 \oplus 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}.$$

In $\mathbb{R}^{2\times 2}$, matrices tropically multiply in the natural way as well,

$$\begin{pmatrix}0 & 1\\ 3 & 4\end{pmatrix} \odot \begin{pmatrix}5 & 2\\ 0 & 3\end{pmatrix} = \begin{pmatrix}(0\odot 5)\oplus(1\odot 0) & (0\odot 2)\oplus(1\odot 3)\\ (3\odot 5)\oplus(4\odot 0) & (3\odot 2)\oplus(4\odot 3)\end{pmatrix}$$
$$= \begin{pmatrix}5\oplus 1 & 2\oplus 4\\ 8\oplus 4 & 5\oplus 7\end{pmatrix} = \begin{pmatrix}1 & 2\\ 4 & 5\end{pmatrix}.$$

How matrices multiply suggest the following modeling application, which (10) explores in depth. Suppose that one has a directed graph on vertices $V = \{1,\dots,n\}$ and weighted edges E. Then we can form the transition cost matrix $C \in \mathbb{R}^{n\times n}$ where C_{ij} is the weight given to the edge (i,j). By convention $C_{ii} = 0$ and if edge (i,j) does not exists then $C_{ij} = \infty$. If we interpret the weight of an edge (i,j) as the cost of moving from vertex i to vertex j, then $(C^{\odot m})_{ij}$ will be the minimum cost of moving from vertex i to vertex j in at most m steps.

Example 1.1.2. Suppose we have three cities $\{1,2,3\}$ and they are connected by the transition cost matrix

$$C = \begin{pmatrix}0 & 2 & 5\\ \infty & 0 & 2\\ 1 & 4 & 0\end{pmatrix}.$$

Then we have

$$C^{\odot 2} = C \odot C = \begin{pmatrix}0 & 2 & 4\\ 3 & 0 & 2\\ 1 & 3 & 0\end{pmatrix}.$$

In fact, by inspection we can see that $C^{\odot m} = C^{\odot 2}$ for $m \geq 2$, since any trip of size 3 or more steps contains a circuit.

Example 1.1.3. An interesting variation is if we say that one cannot spend the night in a city by saying that $C_{ii} = \infty$. Then the interpretation of $C^{\odot m}$ is the transition cost matrix for traveling from city i to city j in exactly m steps. In this case then

$$C = \begin{pmatrix}\infty & 2 & 5\\ \infty & \infty & 2\\ 1 & 4 & \infty\end{pmatrix}.$$

One can no longer see immediately what for example $C^{\odot 7}$ is, and one would need to write a special program to calculate $C^{\odot m}$ for an arbitrary m. However there is a way out of this, namely let

$$\widetilde{C} = \begin{pmatrix}0 & t^2 & t^5\\ 0 & 0 & t^2\\ t & t^4 & 0\end{pmatrix}.$$

For a polynomial $p(t)$ where positive and negative powers of t are allowed, let $\deg(p)$ be the number of zeros at $t = 0$. For example $\deg(t^2 - t^5) = 2$, $\deg(-5 + t + t^2) = 0$, $\deg(t^{-3} + 1 + t) = -3$, and by convention $\deg(0) = \infty$. The important idea here will be that $\deg(\widetilde{C}_{ij}) = C_{ij}$, because the map deg turns classical algebra into tropical algebra. Observe that

$$\deg(t^n + t^m) = \min\{\deg(t^n), \deg(t^m)\} = \deg(t^n) \oplus \deg(t^m)$$
$$\deg(t^n t^m) = \deg(t^n) + \deg(t^m) = \deg(t^n) \odot \deg(t^m).$$

In fact, more generally for $p, q \in \mathbb{R}_{\geq 0}[t, t^{-1}]$ we have

$$\deg(p + q) = \deg(p) \oplus \deg(q) \quad \text{and} \quad \deg(pq) = \deg(p) \odot \deg(q). \tag{1.1.1}$$

Therefore we get that $C^{\odot m} = \deg(\widetilde{C}^m)$. In particular, since

$$\widetilde{C}^7 = \begin{pmatrix} 8t^{17} & t^{12} + 12t^{20} & 5t^{15} + 8t^{23} \\ 4t^{14} & 8t^{17} & t^{12} + 8t^{20} \\ t^{11} + 8t^{19} & 5t^{14} + 8t^{22} & 12t^{17} \end{pmatrix}$$

we know that

$$C^{\odot 7} = \begin{pmatrix} 17 & 12 & 15 \\ 14 & 17 & 12 \\ 11 & 14 & 17 \end{pmatrix}.$$

This method gives us even more information, for $\widetilde{C}^m$ tells us how many different trips of size m there are for a given cost. In this case we can see that there are five trips of cost 15 and eight trips of cost 23 from city 1 to city 2 that take exactly 7 steps.

Tropical exponentiation in $\mathbb{R}$ has some interesting properties in its own right as well. For instance,

$$x^{\odot 2} = x \odot x = 2x$$

where the $2x$ is interpreted classically. Using the fact that $x^{\odot -1}$ should be the number such that $x^{\odot -1} \odot x = 0$, since 0 is the multiplicative identity, we get that $x^{\odot -1} = -x$. Therefore we have

$$x^{\odot a} = ax \text{ where } a \in \mathbb{Z}.$$

Similarly, we have that

$$\begin{aligned} (x \oplus y)^{\odot 2} &= (x \oplus y) \odot (x \oplus y) \\ &= (x \odot x) \oplus (x \odot y) \oplus (y \odot x) \oplus (y \odot y) \\ &= 2x \oplus (x \odot y) \oplus 2y. \end{aligned}$$

In fact, since classically we have that $\min\{2x, 2y\} \leq x + y$, the above becomes

$$(x \oplus y)^{\odot 2} = 2x \oplus 2y = x^{\odot 2} \oplus y^{\odot 2}.$$

This in fact generalizes and we get the following theorem in the tropical semiring:

Theorem 1.1.4 (The Freshman's Dream)**.** *In the tropical semiring, exponentiation distributes over addition, namely*

$$(x_1 \oplus \cdots \oplus x_n)^{\odot m} = x_1^{\odot m} \oplus \cdots \oplus x_n^{\odot m}.$$

1.2 Tropical Algebraic Geometry

Observe that in $\mathbb{R}[x_1, \ldots, x_n]$, tropical monomials

$$x_1^{\odot a_1} \odot \cdots \odot x_n^{\odot a_n} = a_1 x_1 + \cdots + a_n x_n$$

are just ordinary linear forms where $a_j \in \mathbb{Z}$. Tropical polynomials $F \in \mathbb{R}[x_1, \ldots, x_n]$ are of the form

$$F(x_1, \ldots, x_n) = \bigoplus_{a \in A} c_a \odot x_1^{\odot a_1} \odot \cdots \odot x_n^{\odot a_n} = \min\{c_a + a_1 x_1 + \cdots + a_n x_n \mid a \in A\}$$

with $A \subseteq \mathbb{Z}^n$ finite and $c_a \in \mathbb{R}$. In fact, tropical polynomials represent piecewise linear convex functions $F : \mathbb{R}^n \to \mathbb{R}$. To ease notation, we will write

$$F(\mathbf{x}) = \bigoplus_{\mathbf{a} \in A} c_{\mathbf{a}} \odot \mathbf{x}^{\odot \mathbf{a}} = \min\{c_{\mathbf{a}} + \mathbf{a} \cdot \mathbf{x} \mid \mathbf{a} \in A\}$$

where $\mathbf{x}^{\odot \mathbf{a}} = x_1^{\odot a_1} \odot \cdots \odot x_n^{\odot a_n}$ and $\mathbf{a} \cdot \mathbf{x} = a_1 x_1 + \cdots + a_n x_n$.

Example 1.2.1. Let F be the tropical polynomial in $\mathbb{R}[x_1, x_2]$ where

$$\begin{aligned} F(x_1, x_2) &= (3 \odot x_1^{\odot 2}) \oplus (1 \odot x_1 \odot x_2) \oplus (3 \odot x_1^{\odot 2}) \oplus (4 \odot x_1) \oplus (4 \odot x_2) \oplus 8 \\ &= \min\{3 + 2x_1,\ 1 + x_1 + x_2,\ 3 + 2x_2,\ 4 + x_1,\ 4 + x_2,\ 8\}. \end{aligned}$$

Then the graph of F is the following figure:

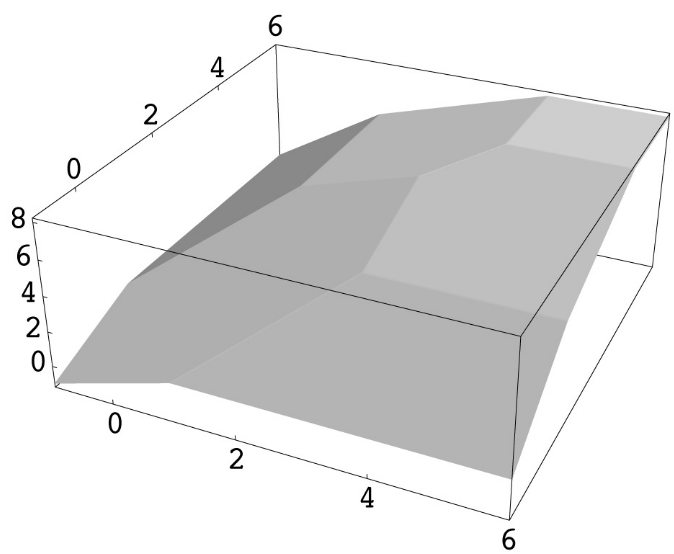

Figure 1.1: The tropical polynomial $F(x_1, x_2)$.

Definition 1.2.2. Let F be a tropical polynomial where $F(\mathbf{x}) = \bigoplus_{\mathbf{a} \in A} c_{\mathbf{a}} \odot \mathbf{x}^{\odot \mathbf{a}}$. Define the *tropical hypersurface* to be the set $\mathcal{T}(F)$ consisting of all $\mathbf{x} \in \mathbb{R}^n$ such that the minimum in $F(\mathbf{x}) = \min\{c_{\mathbf{a}} + \mathbf{a} \cdot \mathbf{x} \mid \mathbf{a} \in A\}$ is obtained by at least two $\mathbf{a} \in A$. Equivalently, $\mathcal{T}(F)$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ where F is not differentiable.

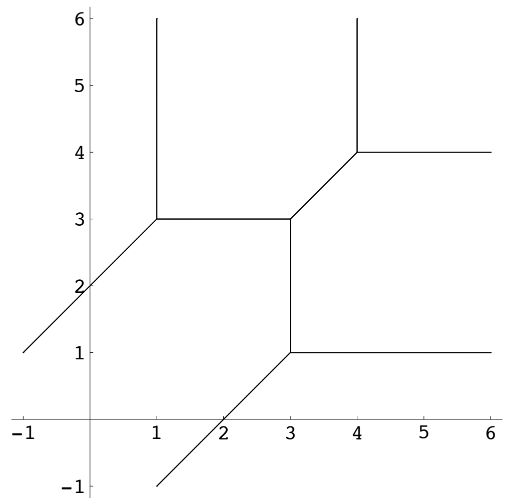

Figure 1.2: The tropical hypersurface $\mathcal{T}(F)$ for $F(x_1, x_2)$ in Example 1.2.1.

It turns out that we can study these tropical hypersurfaces in the language of algebraic geometry. However, in order to do so we need to cover some preliminaries to allow for the translation.

Definition 1.2.3. A *Puiseux series* with complex coefficients is a formal power series $p(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $c_j \in \mathbb{C}$ and $a_1 < a_2 < \cdots$ are rational numbers with a common denominator. These form a field denoted by $K = \mathbb{C}\{\{t\}\}$, which is the algebraic closure of field of Laurent series $\mathbb{C}(\!(t)\!)$ (Corollary 13.15 in (7)).

The field of Puiseux series has the valuation $\deg : K^* \to \mathbb{Q}$ where $\deg(f) = a_1$, its minimum exponent. This allows us to model tropical arithmetic as in Example 1.1.3, so given $p, q \in K$ we have that

$$\deg(pq) = \deg(p) + \deg(q) = \deg(p) \odot \deg(q) \tag{1.2.1}$$

and if the leading terms of p and q do not cancel, then

$$\deg(p + q) = \min\{\deg(p), \deg(q)\} = \deg(p) \oplus \deg(q). \tag{1.2.2}$$

It should be noted that this valuation also gives us a local ring of K and the ring's maximal ideal, which are

$$R_K = \{p \in K \mid \deg(p) \geq 0\} \text{ and } M_K = \{p \in K \mid \deg(p) > 0\}.$$

Observe that $\mathbb{C} = R_K / M_K$.

Let $f \in K[\mathbf{x}]$ where

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} p_{\mathbf{a}}(t)\mathbf{x}^{\mathbf{a}} \text{ with } A \subseteq \mathbb{Z}^n \text{ finite and } p_{\mathbf{a}}(t) \in K^*.$$

Then we can associate with f a tropical polynomial $\operatorname{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ where

$$\operatorname{trop}(f)(\mathbf{x}) = \bigoplus_{\mathbf{a} \in A} \deg(p_{\mathbf{a}}) \odot \mathbf{x}^{\odot \mathbf{a}}.$$

Definition 1.2.4. Let $I \subseteq K[\mathbf{x}]$ be an ideal. Then the *tropical variety* of ideal I is

$$\mathcal{T}(I) = \bigcap_{f \in I} \mathcal{T}(\mathrm{trop}(f)) \subseteq \mathbb{R}^n.$$

There turns out to be a strong connection, provided by the degree map, between $\mathcal{T}(I)$ and the variety $V(I) \subseteq (K^*)^n$. For an ideal $I \subseteq K[x_1, \ldots, x_n] = K[\mathbf{x}]$, let $V(I)$ be its variety in the algebraic torus $(K^*)^n$. So $V(I)$ is the set of all n-tuples of nonzero Puiseux series $(p_1, \ldots, p_n)$ such that $f(p_1, \ldots, p_n) = 0$ for all $f \in I \subseteq K[\mathbf{x}]$. Since what we tropically care about are the exponents of these power series, let us enlarge K so that the exponents can be real numbers.

Definition 1.2.5. Let $\tilde{K}$ be the field of *transfinite series* with complex coefficients and real exponents, so it consists of all power series of the form

$$p(t) = \sum_{s \in S} c_s t^s$$

where $S \subseteq \mathbb{R}$ is well ordered and $c_s \in \mathbb{C}$. The well ordered condition ensures that $\tilde{K}$ is closed under addition and multiplication.

The valuation for $\tilde{K}$ is $\deg : \tilde{K}^* \to \mathbb{R}$ where

$$\deg : \sum_{s \in S} c_s t^s \mapsto \min\{s \in S \mid c_s \neq 0\}.$$

Observe that $K \subsetneq \tilde{K}$ and their valuations agree on K.

Given an ideal $I \subseteq K[x_1, \ldots, x_n]$, let $\tilde{V}(I)$ be the variety in $(\tilde{K}^*)^d$ defined by I. We can then map $\tilde{V}(I)$ to $\mathbb{R}^n$ by extending the degree map coordinatewise, where

$$\deg(p_1, \ldots, p_n) = (\deg(p_1), \ldots, \deg(p_n)).$$

Hence $\deg(\tilde{V}(I)) \subseteq \mathbb{R}^n$ where,

$$\deg(\tilde{V}(I)) = \{(d_1, \ldots, d_n) \in \mathbb{R}^n \mid (p_1(t), \ldots, p_n(t)) \in \tilde{V}(I) \text{ and } \deg(p_j(t)) = d_j\}.$$

It turns out (Theorem 1.2.8) that the closure of $\deg(V(I))$ in $\mathbb{R}^n$ equals $\deg(\tilde{V}(I))$, which in turn equals $\mathcal{T}(I)$ for any ideal $I \subseteq K[\mathbf{x}]$.

Lastly we can also characterize $\mathcal{T}(I)$ using the theory of Gröbner bases, see (1) and (22) for an introduction to the theory.

Definition 1.2.6. For $\mathbf{w} \in \mathbb{R}^n$, let the $\mathbf{w}$-*weight* of the term $p_{\mathbf{a}}(t)\mathbf{x}^{\mathbf{a}}$ be

$$\mathrm{wgt}_{\mathbf{w}}(p_{\mathbf{a}}(t)\mathbf{x}^{\mathbf{a}}) = \deg(p_{\mathbf{a}}) + \mathbf{w} \cdot \mathbf{a},$$

where $p_{\mathbf{a}} \in K^*$. If v is be the smallest $\mathbf{w}$-weight of any term in $f \in K[\mathbf{x}]$ and $\tilde{f}(\mathbf{x}) = f(t^{w_1}x_1, \ldots, t^{w_n}x_n)$, then $t^{-v}\tilde{f} \in R_K[\mathbf{x}]$. Define the *initial form* $\mathrm{in}_{\mathbf{w}}(f)$ to be the residue of $t^{-v}\tilde{f}$ in $\mathbb{C}[\mathbf{x}] = R_K/M_K[\mathbf{x}]$.

Definition 1.2.7. Given an ideal $I \subseteq K[\mathbf{x}]$, define the *initial ideal of I with respect to $\mathbf{w}$* to be

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f) \mid f \in I \rangle \subseteq \mathbb{C}[\mathbf{x}].$$

Theorem 1.2.8 (Theorem 2.1 in (20)). *For an ideal $I \subseteq K[\mathbf{x}]$ the following subsets of $\mathbb{R}^n$ are equal:*

1. $\mathcal{T}(I)$;
2. $\{\mathbf{w} \in \mathbb{R}^n \mid \mathrm{in}_{\mathbf{w}}(I)$ *contains no monomial*$\}$;
3. *The closure of* $\deg(V(I))$ *in* $\mathbb{R}^n$;
4. $\deg(\tilde{V}(I))$.

Example 1.2.9. Let $\langle f \rangle \subseteq K[x_1, x_2]$, where

$$f(x_1, x_2) = t^3 x_1^2 + t^1 x_1 x_2 + t^3 x_2^2 + t^4 x_1 + t^4 x_2 + t^8$$

and therefore

$$\mathrm{trop}(f)(x_1, x_2) = (3 \odot x_1^{\odot 2}) \oplus (1 \odot x_1 \odot x_2) \oplus (3 \odot x_1^{\odot 2}) \oplus (4 \odot x_1) \oplus (4 \odot x_2) \oplus 8.$$

Observe that $\mathrm{trop}(f)(x_1, x_2) = F(x_1, x_2)$ from Example 1.2.1. By Theorem 1.2.8, for $\mathbf{w} \in \mathbb{R}^2$, we should expect that $\mathrm{in}_{\mathbf{w}}(f)$ is not a monomial only if $\mathbf{w} \in \mathcal{T}(F)$. If $\mathbf{w} = (3, 1) \in \mathbb{R}^2$, the $\mathbf{w}$-weights of the terms in f are

$$\begin{aligned} \mathrm{wgt}_{\mathbf{w}}(t^3 x_1^2) &= 3 + (3,1) \cdot (2,0) = 9 & \mathrm{wgt}_{\mathbf{w}}(t^1 x_1 x_2) &= 1 + (3,1) \cdot (1,1) = 5 \\ \mathrm{wgt}_{\mathbf{w}}(t^3 x_2^2) &= 3 + (3,1) \cdot (0,2) = 5 & \mathrm{wgt}_{\mathbf{w}}(t^4 x_1) &= 4 + (3,1) \cdot (1,0) = 7 \\ \mathrm{wgt}_{\mathbf{w}}(t^4 x_2) &= 4 + (3,1) \cdot (0,1) = 5 & \mathrm{wgt}_{\mathbf{w}}(t^8) &= 8 + (3,1) \cdot (0,0) = 8 \end{aligned}$$

and therefore the smallest $\mathbf{w}$-weight among the terms in f is $v = 5$. Now

$$\tilde{f}(x_1, x_2) = f(t^3 x_1, t^1 x_2) = t^9 x_1^2 + t^5 x_1 x_2 + t^5 x_2^2 + t^7 x_1 + t^5 x_2 + t^8,$$

so therefore

$$t^{-v} \tilde{f}(x_1, x_2) = t^{-5} \tilde{f}(x_1, x_2) = t^4 x_1^2 + x_1 x_2 + x_2^2 + t^2 x_1 + x_2 + t^3.$$

This mean that the residue of $t^{-v} \tilde{f} \in \mathbb{C}[x_1, x_2]$ is

$$\mathrm{in}_{\mathbf{w}}(f) = x_1 x_2 + x_2^2 + x_2,$$

so $\mathrm{in}_{\mathbf{w}}(f)$ is not a monomial. If we check Figure 1.2 we can verify that indeed $\mathbf{w} = (3, 1) \in \mathcal{T}(F)$, as predicted by Theorem 1.2.8. In fact, the minimum in $F(\mathbf{w})$ is obtained three times, by $1 \odot w_1 \odot w_2 = 3 \odot w_2^{\odot 2} = 4 \odot w_2 = 5$ and these terms correspond to the terms in $\mathrm{in}_{\mathbf{w}}(f)$.

Chapter 2

Two Ranks of a Tropical Matrix

2.1 The Kapranov Rank

One way of defining the classical rank of a matrix $M \in \mathbb{R}^{n \times m}$ is that $\text{rank}(M)$ is the smallest dimension of any linear space in $\mathbb{R}^n$ containing the columns of M. It is this sense of rank that the Kapranov rank models for tropical matrices.

Definition 2.1.1. A *tropical linear space* in $\mathbb{R}^n$ is any tropical variety $\mathcal{T}(I)$ where $I \subseteq \tilde{K}[\mathbf{x}]$ is an ideal generated by linear forms $p_1(t)x_1 + \cdots + p_n(t)x_n$ where $p_j(t) \in \tilde{K}$. The dimension of $\mathcal{T}(I)$ is given by n minus the minimal number of generators of I.

Definition 2.1.2. The *Kapranov rank* of $M \in \mathbb{R}^{n \times m}$, denoted $\text{k-rank}(M)$, is the smallest dimension of any tropical linear space in $\mathbb{R}^n$ containing the columns of M.

We will now present a different characterization of the Kapranov rank, which will allow us to compute the Kapranov rank with Gröbner bases. Let J_r be the ideal generated by all $(r+1) \times (r+1)$ subdeterminants of the $n \times m$ matrix of indeterminates (x_{ij}). J_r has the property that $F \in (\tilde{K}^*)^{n \times m}$ has classical rank at most r if and only if $F \in \tilde{V}(J_r)$, for being in $\tilde{V}(J_r)$ simply means that every $(r+1) \times (r+1)$ subdeterminant is zero.

Definition 2.1.3. Let $M \in \mathbb{R}^{n \times m}$ and $F \in (\tilde{K}^*)^{n \times m}$, then F is a *lift* of M if $\deg(F) = M$.

Theorem 2.1.4 (Theorem 3.3 in (5))**.** *Let $M = (m_{ij}) \in \mathbb{R}^{n \times m}$, then the following are equivalent:*

1. $\text{k-rank}(M) \leq r$
2. $M \in \mathcal{T}(J_r)$
3. *There exists a lift F of M with classical rank in $\tilde{K}^{n \times m}$ at most r.*

Proof. The equivalence (*2*) and (*3*) follows directly from the definition of the tropical variety and preceding discussion about the connection between rank and membership in $\tilde{V}(J_r)$.

Suppose (*3*) holds, so there is a lift F of M where the rank of F is at most r. Let V be an r-dimensional linear subspace of $\tilde{K}^n$, which contains the columns of F. Hence V is defined by an ideal I of $n-r$ linearly independent linear forms in $\tilde{K}[\mathbf{x}]$. This gives that $M \subseteq \mathcal{T}(I)$ where $\mathcal{T}(I)$ is a linear tropical space of dimension r. Therefore k-rank$(M) \leq r$.

Suppose (*1*) holds, so k-rank$(M) \leq r$. Let L be a tropical linear space of dimension r containing the columns of M. Let $I \subseteq \tilde{K}[\mathbf{x}]$ an ideal generated by linear form such that $L = \mathcal{T}(I)$. Then by the definition of $\mathcal{T}(I)$, for each column of M there exists a preimage, under the degree mapping, in $\tilde{V}(I) \subseteq (\tilde{K}^*)^n$. Let $F \in \tilde{K}^{n\times m}$ be the matrix whose columns are preimages in $\tilde{V}(I)$ of the columns of M. So F is a lift of M and since the columns of F are in the variety defined by I we have rank$(F) \leq r$. □

Corollary 2.1.5 (Corollary 3.4 in (5)). *The Kapranov rank of $M \in \mathbb{R}^{n\times m}$ is the smallest rank in $(\tilde{K}^*)^{n\times m}$ of any lift of M.*

Corollary 2.1.6. *The Kapranov rank of $M \in \mathbb{R}^{n\times m}$ is invariant under transposition, any permutation of its rows or columns, and tropically scaling any row or column.*

Corollary 2.1.7. *If $M \in \mathbb{R}^{n\times m}$ has two identical columns (rows), then the matrix $\widetilde{M}$ obtained by removing a duplicate column (row) is such that* k-rank$(\widetilde{M})$ = k-rank(M).

Using Theorem 1.2.8 and Theorem 2.1.4, we can give an algorithm using Gröber bases to compute the Kapranov rank, which is from (5).

Algorithm to Compute the Kapranov Rank: Let $M \in \mathbb{R}^{n\times m}$ and we want to know whether or not k-rank$(M) > r$.

Step 1: First fix a term order $\prec_M$ on $\mathbb{C}[\mathbf{x}]$, where $\mathbf{x}$ is the $n \times m$ matrix of indeterminates x_{ij}, which is a refinement of the partial ordering on monomial using the weight vector M.

Step 2: Compute the reduced Gröber basis $\mathcal{G}$ of J_r for the term order $\prec_M$. For each $g \in \mathcal{G}$, let $\text{in}_M(g)$ be its leading form with respect to the partial ordering due to the weight vector M.

Step 3: The ideal generated by the leading forms $\{\text{in}_M(g) \mid g \in \mathcal{G}\}$ is the initial ideal $\text{in}_M(J_r)$. Let $x^{\text{all}} = \prod_{ij} x_{ij}$, then compute the saturation

$$(\text{in}_M(J_r) : \langle x^{\text{all}} \rangle^{\infty}) = \{f \in \mathbb{C}[\mathbf{x}] \mid f \cdot (x^{\text{all}})^s \in \text{in}_M(J_r) \text{ for some } s \in \mathbb{N}\}, \quad (2.1.1)$$

which can be computed by various computational commutative algebra programs, such as `CoCoA`.

The reason this algorithm computes the Kapranov rank is because of the following theorem

Theorem 2.1.8 (Corollary 3.7 in (5)). *A matrix $M \in \mathbb{R}^{n\times m}$ has* k-rank$(M) > r$ *if and only if the ideal in* (2.1.1) *is the unit ideal* $\langle 1 \rangle$.

This will be the algorithm we use in the appendix to compute the Kapranov rank of various 5×5 zero-one matrices.

2.2 The Tropical Rank and Determinant

Another way of defining the classical rank of a matrix is to use the determinant, where the rank of M is the size of the largest square submatrix of M with nonzero determinant. In the tropical case this becomes the following:

Definition 2.2.1. $M = (m_{ij}) \in \mathbb{R}^{r\times r}$ is *tropically singular* if the minimum in

$$\det(M) = \bigoplus_{\sigma\in S_r} m_{1\sigma(1)} \odot \cdots \odot m_{r\sigma(r)} = \min\{m_{1\sigma(1)} + \cdots + m_{r\sigma(r)} \mid \sigma \in S_r\}$$

is attained at least twice, where S_r is the symmetric group on r elements.. The *tropical rank* of a matrix $M \in \mathbb{R}^{n\times m}$ is the largest integer r such that M has a nonsingular $r \times r$ minor. This will be denoted as t-rank$(M) = r$.

It will turn out that the tropical rank is not equivalent to the Kapranov rank (Theorem 3.5.3), but for now we have that the tropical rank is a lower bound for the Kapranov rank.

Theorem 2.2.2 (Proposition 4.1 in (5))**.** *For every $M \in \mathbb{R}^{n\times m}$ we have that*

$$\text{t-rank}(M) \leq \text{k-rank}(M).$$

Proof. Suppose M has a $r \times r$ minor M' that is tropically nonsingular. Then every lift F' of M' is classically nonsingular since the smallest exponent of t in its classical determinant occurs exactly once. Therefore by Theorem 2.1.4 it follows that $r \leq$ k-rank(M). □

Example 2.2.3. Let $M \in \mathbb{R}^{3\times 3}$ be the matrix

$$M = \begin{pmatrix} 2 & 7 & 7 \\ 1 & 3 & -2 \\ 1 & 3 & -2 \end{pmatrix}.$$

Observe that the upper-left 2×2 minor

$$M' = \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$$

is tropically nonsingular, because

$$m'_{11} \odot m'_{22} = 5 < 7 = m'_{12} \odot m'_{21}.$$

Therefore $2 \leq$ t-rank(M). However k-rank$(M) \leq 2$, for

$$F = \begin{pmatrix} t^2 + t^5 & t^7 & t^7 \\ t^1 & t^3 & t^{-2} + 1 \\ t^1 & t^3 & t^{-2} + 1 \end{pmatrix}$$

is a lift of M, since $\deg(F) = M$, and the classical rank of F in $\tilde{K}^{3\times 3}$ is 2. Hence by Corollary 2.1.5 we know that k-rank$(M) \leq 2$. By Theorem 2.2.2 we have that

$$2 \leq \text{t-rank}(M) \leq \text{k-rank}(M) \leq 2,$$

so therefore t-rank$(M) = 2 =$ k-rank(M).

In (5), the following two sufficient conditions for k-rank(M) = t-rank(M) are given:

Theorem 2.2.4 (Theorem 5.5 in (5))**.** *For a matrix $M \in \mathbb{R}^{n\times m}$,*

$$\text{k-rank}(M) = n \textit{ or } \text{k-rank}(M) = m \textit{ only if } \text{t-rank}(M) = \text{k-rank}(M).$$

Theorem 2.2.5. *For a matrix $M \in \mathbb{R}^{n\times m}$,*

$$\text{t-rank}(M) \leq 2 \textit{ only if } \text{t-rank}(M) = \text{k-rank}(M).$$

Proof. For the case where t-rank$(M) = 2$, see Theorem 6.5 in (5).

If t-rank$(M) = 1$, we know by definition that $M_{ij} + M_{kl} = M_{il} + M_{kj}$ for all i, j, k, and l. Let $F \in (\tilde{K}^*)^{n\times m}$ be given by $F_{ij} = t^{M_{ij}}$. Then $F_{ij}F_{kl} - F_{il} + F_{kj} = 0$, so rank$(F) < 2$. Hence k-rank$(M) = 1$.

Therefore if t-rank$(M) \leq 2$, then t-rank(M) = k-rank(M). □

It follows from these two theorems that if $M \in \mathbb{R}^{n\times m}$ and t-rank$(M) \neq$ k-rank(M), then n or m is at least 5. In the appendix we set up the computer computations that use Theorem 2.1.8 to prove the following:

Theorem 2.2.6. *If $M \in \{0,1\}^{n\times n}$ and* t-rank$(M) \neq$ k-rank(M)*, then $n \geq 6$.*

Many properties of the classical determinant and rank also hold for the tropical determinant and rank. We will now flush out those similarities. To ease notation, for $M \in \mathbb{R}^{r\times r}$ let $\Delta_M : S_r \to \mathbb{R}$ be defined by

$$\Delta_M(\sigma) = m_{1\sigma(1)} \odot \cdots \odot m_{r\sigma(r)} = \sum_{i=1}^{r} m_{i\sigma(i)}.$$

Hence we have the compact formula $\det(M) = \bigoplus_{\sigma\in S_r} \Delta_M(\sigma)$.

Proposition 2.2.7. *If $M \in \mathbb{R}^{r\times r}$, then* $\det(M) = \det(M^t)$.

Proof. Observe that $\Delta_M(\sigma) = \sum_{i=1}^{r} m_{i\sigma(i)} = \sum_{i=1}^{r} m_{\sigma^{-1}(i)i} = \Delta_{M^t}(\sigma^{-1})$. Hence

$$\det(M) = \bigoplus_{\sigma\in S_r} \Delta_M(\sigma) = \bigoplus_{\sigma\in S_r} \Delta_{M^t}(\sigma^{-1}) = \bigoplus_{\sigma\in S_r} \Delta_{M^t}(\sigma) = \det(M^t).$$

Therefore $\det(M) = \det(M^t)$. □

Corollary 2.2.8. *For $M \in \mathbb{R}^{n\times m}$,* t-rank$(M)$ = t-rank(M^t)*.*

Proof. Suppose t-rank$(M) = r$, so there is an $r \times r$ nonsingular minor A. Let A consist of the rows $\{x_1, \ldots, x_r\}$ and columns $\{y_1, \ldots y_r\}$. Now let $B \in \mathbb{R}^{r\times r}$ be the minor in M^t consisting of the rows $\{y_1, \ldots y_r\}$ and the columns $\{x_1, \ldots, x_r\}$. Then $B = A^t$.

In Proposition 2.2.7, we showed that $\Delta_A(\sigma) = \Delta_{A^t}(\sigma^{-1})$. Since $\Delta_A(\sigma)$ achieves its minimum exactly once, $\Delta_B(\sigma^{-1})$ achieves its minimum exactly once as well. Therefore B is an $r \times r$ nonsingular minor of M^t, so t-rank$(M) = r \leq$ t-rank(M^t).

Therefore by symmetry, t-rank(M) = t-rank(M^t). □

Proposition 2.2.9. *For $M \in \mathbb{R}^{r\times r}$, $\det(M)$ is invariant under any permutation of the rows and columns of M.*

Proof. By Proposition 2.2.7, it suffices to prove that if W is the matrix obtained by permuting the rows in M by $\tau \in S_r$, then $\det(M) = \det(W)$. Observe that $m_{ij} = w_{\tau(i)j}$, so we have that

$$\Delta_M(\sigma) = \sum_{i=1}^{r} m_{i\sigma(i)} = \sum_{i=1}^{r} w_{\tau(i)\,\sigma(i)} \sum_{i=1}^{r} w_{i\,\sigma\circ\tau^{-1}(i)} = \Delta_W(\sigma\circ\tau^{-1}).$$

This gives the following:

$$\det(M) = \bigoplus_{\sigma\in S_r} \Delta_M(\sigma) = \bigoplus_{\sigma\in S_r} \Delta_W(\sigma\circ\tau^{-1}) = \det(W).$$

Therefore $\det(M) = \det(W)$. □

Corollary 2.2.10. *For $M \in \mathbb{R}^{n\times m}$, $\text{t-rank}(M)$ is invariant under any permutation of its rows and columns.*

Proof. By Corollary 2.2.8, it suffices to prove that if W is the matrix obtained by permuting the rows in M by $\tau \in S_r$, then $\text{t-rank}(M) = \text{t-rank}(W)$.

Suppose that $\text{t-rank}(M) = r$, so there is an $r \times r$ nonsingular minor A. Let A consist of the rows $\{x_1, \dots, x_r\}$ and columns $\{y_1, \dots y_r\}$. Now let $B \in \mathbb{R}^{r\times r}$ be the minor in W consisting of the rows $\tau(\{x_1, \dots, x_r\})$ and columns $\{y_1, \dots y_r\}$. Then there exists $\psi \in S_r$ such that $B = A^\psi$ where A^ψ is the image of A after having its rows permuted by ψ.

Then by Proposition 2.2.9, we know that $\Delta_A(\sigma) = \Delta_B(\sigma\circ\psi^{-1})$. Since $\Delta_A(\sigma)$ obtains its minimum only we we that $\Delta_B(\sigma\circ\psi^{-1})$ obtains it minimum only once as well, so B is a nonsingular $r \times r$ minor of W. Hence $\text{t-rank}(M) = r \leq \text{t-rank}(W)$.

Therefore by symmetry $\text{t-rank}(M) = \text{t-rank}(W)$. □

Remark 2.2.11. Proposition 2.2.9 makes our life easy for within proofs we can assume without loss of generality that $\det(M) = \Delta_M(e)$ where $e \in S_r$ is the identity element.

Proposition 2.2.12. *The tropical determinant and $\Delta_{(\bullet)}(\sigma)$, for $\sigma \in S_r$, are both r-tropical linear functions mapping $\mathbb{R}^{r\times r}$ to $\mathbb{R}$.*

Proof. By Proposition 2.2.9 it suffices to prove tropically linearity in the first row.

Let $M, N \in \mathbb{R}^{r\times r}$ be such that $n_1 = c \odot m_1$ for some $c \in \mathbb{R}$ and $n_j = m_j$ for $j \geq 2$. We then have the following:

$$\Delta_N(\sigma) = \sum_{i=1}^{r} n_{i\sigma(i)} = (c + m_{j\sigma(j)}) + \sum_{i\neq j} m_{i\sigma(i)} = c + \sum_{i=1}^{r} m_{i\sigma(i)} = c \odot \Delta_M(\sigma).$$

Hence $\Delta_N(\sigma) = c \odot \Delta_M(\sigma)$ and $\det(N) = c \odot \det(M)$.

Now let $A, B \in \mathbb{R}^{r\times r}$ where $a_i = b_i = m_i$ for $i \geq 2$. So,

$$\begin{aligned}\Delta_{A\oplus B}(\sigma) &= (A\oplus B)_{1\sigma(1)} + \sum_{i=2}^{r}(A\oplus B)_{i\sigma(i)} = \min(a_{1\sigma(1)}, b_{1\sigma(1)}) + \sum_{i=2}^{r} m_{i\sigma(i)} \\ &= \Delta_A(\sigma) \oplus \Delta_B(\sigma) \quad \text{since } a_i = b_i = m_i \text{ for } i \geq 2.\end{aligned}$$

Hence $\Delta_{A\oplus B}(\sigma) = \Delta_A(\sigma) \oplus \Delta_B(\sigma)$ and $\det(A \oplus B) = \det(A) \oplus \det(B)$. Therefore the tropical determinant and $\Delta_{(\bullet)}(\sigma)$ are r-tropically linear functions. □

Proposition 2.2.13. *Let $M \in \mathbb{R}^{n\times m}$, and $N \in \mathbb{R}^{n\times m}$ where N is a matrix obtained by tropically scaling the rows and columns of M. Then* t-rank(M) = t-rank(N)*.*

Proof. Let t-rank$(M) = r$, A be a nonsingular $r \times r$ minor in M, and let B be the corresponding $r \times r$ minor in W. Since B can be obtained by scaling the rows and columns of A, by Proposition 2.2.12 we know that there is $c \in \mathbb{R}$ such that $\Delta_B(\sigma) = c + \Delta_A(\sigma)$. Since A is nonsingular, it follows that B is an $r \times r$ nonsingular minor of W and hence t-rank$(M) = r \leq$ t-rank(W). Therefore by symmetry, t-rank(M) = t-rank(W). □

From the above we see that we have two tropical rank preserving row-operations, permuting rows and scaling rows. However adding one row to another does not preserve the tropical rank of a matrix as illustrated by the following example:

Example 2.2.14. Let $L, M \in \mathbb{R}^{2\times 2}$ where

$$L = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then M can be obtained by tropically adding the second row of L to the first row. However we can see that t-rank$(L) = 2$ but t-rank$(M) = 1$, so unlike the classical case, adding one row to another does not preserve the tropical rank of a matrix.

Likewise in the classical case, adding one row of a matrix to another row leaves the determinant unchanged. However this does not happen in the tropical setting for $\det(L) = 1$ and $\det(M) = 0$. Without this third row-operation Gaussian elimination does not work for to compute the tropical rank of matrices. One can think of Gaussian elimination as a polynomial time algorithm for computing the classical rank of a matrix, so while Gaussian elimination does not work in the tropical setting perhaps it could have a tropical analogue. Unfortunately, in Chapter 4 we will prove that computing the tropical rank of a zero-one matrix is NP-complete, which will mean that is it highly unlikely that there is no analogue to Gaussian elimination for tropical matrices.

However we do have the Hungarian method (12), (15), which by only tropically scaling rows and columns turns a square matrix into a nonnegative matrix with tropical determinant equal to zero. Furthermore the Hungarian method is a polynomial time algorithm which can compute the tropical determinant, which is surprising since a brute force algorithm would have have to check $n!$ permutations for an $n \times n$ matrix.

The Hungarian Method: Take a matrix $M = \{m_1, \ldots, m_n\} \in \mathbb{R}^{n \times n}$.

Step 1: For each row, subtract its minimum value from each entry in the row. This corresponds to $m_i := -(\bigoplus_j m_{ij}) \odot m_i$ and the resulting matrix is nonnegative.

Step 2: For each column, subtract its minimum value from each entry in the column. This is just applying Step 1 to the transpose of M and then transposing back.

Step 3: Select rows and columns in a minimal way such that each 0 in the matrix is in one of the selected rows or columns. If the number of rows and columns chosen is n, then for the resulting matrix N, there exists $\sigma \in S_n$ such that $\Delta_N(\sigma) = 0$. Otherwise, continue to Step 4.

Step 4: Let m be the smallest entry in M that lies in none of the chosen rows and columns. Then subtract m from each element not in the chosen rows and column, and add m to each element that is in both a chosen row and column. This corresponds to tropically scaling each chosen row and column by m and then scaling every entry by $-m$. Go back to Step 3.

Example 2.2.15. Let $M \in \mathbb{R}^{4\times 4}$ be the matrix

$$M = \begin{pmatrix} 3 & 5 & 4 & 7 \\ -1 & 1 & 3 & -2 \\ 1 & 2 & 0 & -3 \\ 5 & 6 & 3 & 4 \end{pmatrix}.$$

We will use the Hungarian method to compute the tropical determinant of M. For Step 1, we subtract 3 from the 1st row, -2 from the 2nd row, -3 from the 3rd row, and 3 from the 4th row to obtain the matrix

$$\begin{pmatrix} 0 & 2 & 1 & 4 \\ 1 & 3 & 5 & 0 \\ 4 & 5 & 3 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}.$$

For Step 2, we subtract 2 from the 2nd column to obtain the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 4 \\ 1 & 1 & 5 & 0 \\ 4 & 3 & 3 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}.$$

For Step 3, we can cover all the zeros with three choices namely the 1st row the 4th row, and the 4th column

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{4} \\ 1 & 1 & 5 & \mathbf{0} \\ 4 & 3 & 3 & \mathbf{0} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Since we covered the zeros with three rows and columns, we move onto Step 4. We can see that $m = 1$ is the smallest entry outside of each chosen row and column, so we subtract 1 from every entry and add 1 to each chosen row and column to obtain the matrix

$$N = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 0 & 4 & 0 \\ 3 & 2 & 2 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}.$$

Going back to Step 3, there is no way to cover the zeros in N except by choosing 4 rows and columns, so there exists a $\sigma \in S_4$ such that $\Delta_N(\sigma) = 0$. In this case there are exactly two permutations namely, (34) and (12)(34). These same permutations give the determinant for M, so therefore

$$\det(M) = \Delta_M((34)) = 5 + -1 + -3 + 3 = 4 = 3 + 1 + -3 + 3 = \Delta_M((12)(34)).$$

Hence we see also that M is tropically singular with t-rank$(M) \leq 3$.

As a direct consequence of Proposition 2.2.12 and Proposition 2.2.13 we have the following:

Proposition 2.2.16. *Let $M \in \mathbb{R}^{n\times n}$ and H be the matrix resulting from the Hungarian method. Then for $\sigma \in S_n$, $\Delta_M(\sigma) = \det(M)$ iff $\Delta_H(\sigma) = 0$. Furthermore,* t-rank(M) = t-rank(H).

Therefore any question about the tropical rank or determinant of a square matrix can be reduced to a question about the tropical rank or determinant of a nonnegative square matrix with determinant equal to zero. We end this section with an alternative characterization of the tropical rank.

Definition 2.2.17. $V \subseteq \mathbb{R}^n$ is *tropically convex* if $(a \odot \mathbf{x}) \oplus (b \odot \mathbf{y}) \in V$ for all $\mathbf{x}, \mathbf{y} \in V$ and $a, b \in \mathbb{R}$.

Since a tropically convex set V is closed under scalar multiplication, this means that $\lambda \odot x = \lambda(1, \ldots, 1) + x \in V$ for all $\lambda \in \mathbb{R}$ and $x \in V$. Therefore it makes sense to identify V with its image in the $(n-1)$-dimensional *tropical projective space*:

$$\mathbb{TP}^{n-1} = \mathbb{R}^n/(1, \ldots, 1)\mathbb{R}.$$

The *tropical convex hull* of $S \subseteq \mathbb{R}^n$, denoted tconv(S), is the smallest convex set containing S. In fact,

$$\text{tconv}(S) = \left\{ \bigoplus_{j=1}^{m} a_j \odot \mathbf{x}^j \mid \mathbf{x}^j \in S, a_j \in \mathbb{R} \right\}.$$

Theorem 2.2.18 (Theorem 4.2 in (5))**.** *Let $M \in \mathbb{R}^{n\times m}$ be a matrix, then the tropical rank of M equals one plus the dimension of the tropical convex hull of the columns of M in $\mathbb{TP}^{n-1}$.*

2.3 The Tropical Rank of Zero-One Matrices

In (5), a combinatorial formula is given for the tropical rank of a zero-one matrix, or any matrix with two distinct entries. Given $\mathbf{x} \in \mathbb{R}^n$ where $\mathbf{x} = (x_1, \ldots, x_n)$ define the *zeros* of $\mathbf{x}$ to be

$$Z(\mathbf{x}) = \{j \mid x_j = 0\}. \tag{2.3.1}$$

Then define the *zero poset* of $V \subseteq \mathbb{R}^n$ to be

$$\text{poset}(V) = \left\{ \bigcup_{\mathbf{x} \in S} Z(\mathbf{x}) \mid S \subseteq V \right\}, \tag{2.3.2}$$

which has a partial ordering given by set inclusion. Finally define the *length* of $\text{poset}(V)$ to be the number of elements in a chain of maximum length in $\text{poset}(V)$, which will be denoted $\text{length}(V)$. For a matrix M, let $\text{length}(M) = \text{length}(V)$ where V is the set of the column vectors of M. Observe that $\text{length}(M)$ is invariant under a permutation of its rows and columns.

Theorem 2.3.1 (Proposition 4.3 in (5))**.** *Given a zero-one matrix $M \in \{0,1\}^{n\times m}$ with no column of all ones, then* $\text{t-rank}(M) = \text{length}(M)$ *where M is seen as a set of its column vectors.*

In (5), the proof that $\text{length}(M) \geq \text{t-rank}(M)$ claims that if $N \in \{0,1\}^{r\times r}$ is nonsingular then N can be transformed by row and column permutations into a matrix with 1's above the diagonal and 0's on and below the diagonal. However the following matrix provides a simple counterexample to this claim:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

However, one only needs to show that N can be transformed into a matrix with 0's on and 1's below the diagonal, which we will call being in standard form. We will give an constructive proof that this is possible, but first we need a few preliminaries.

Definition 2.3.2. A zero-one matrix $N \in \{0,1\}^{r\times r}$ is in *standard form* if $n_{ii} = 0$ for all i and $n_{ij} = 1$ if $i > j$.

Example 2.3.3. If $N \in \{0,1\}^{3\times 3}$ is in standard form, then

$$N = \begin{pmatrix} 0 & n_{12} & n_{13} \\ 1 & 0 & n_{23} \\ 1 & 1 & 0 \end{pmatrix}.$$

Observe that if N is in standard form, then N has full t-rank and full length. The main idea of the proof of Theorem 2.3.1 will be to find a standard form $r \times r$ minor N of the matrix M given that $\text{t-rank}(M) = r$.

Example 2.3.4. Let $M \in \{0,1\}^{4\times 4}$ be the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The top-left 3×3 minor of M is nonsingular, because $e \in S_3$ is only element such that $m_{1\sigma(1)} + m_{2\sigma(2)} + m_{3\sigma(3)} \leq 1$. Therefore t-rank$(M) \geq 3$. However

$$m_{11} + m_{22} + m_{34} + m_{43} = 1 = m_{14} + m_{22} + m_{33} + m_{41}$$

and $\det(M) \geq 1$, since $m_{1j} = 1$, so M is tropically singular. Therefore t-rank$(M) = 3$. Let $N \in \{0,1\}^{3\times 3}$ be the minor of M consisting of the fourth, third, and second rows of M and the first, third, and second columns of M,

$$N = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Hence N is a minor of M in standard form. Now let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4 \in \{0,1\}^4$ be the columns of M, so $Z(\mathbf{c}_1) = \{4\}$, $Z(\mathbf{c}_2) = \{2,3\}$, $Z(\mathbf{c}_3) = \{3,4\}$, and $Z(\mathbf{c}_4) = \{2,3\}$. Using the order of the columns in N we see that

$$Z(\mathbf{c}_1) \subsetneq Z(\mathbf{c}_1) \cup Z(\mathbf{c}_3) \subsetneq Z(\mathbf{c}_1) \cup Z(\mathbf{c}_3) \cup Z(\mathbf{c}_2)$$

is a chain of length 3 in poset(M), because

$$Z(\mathbf{c}_1) = \{3\},\ Z(\mathbf{c}_1) \cup Z(\mathbf{c}_3) = \{3,4\},\ \text{and } Z(\mathbf{c}_1) \cup Z(\mathbf{c}_3) \cup Z(\mathbf{c}_2) = \{2,3,4\}.$$

Therefore we have turned our knowledge of the fact that t-rank$(M) = 3$ into a chain of length 3 in poset(M).

Now any zero-one matrix $N \in \{0,1\}^{r\times r}$ defines a bipartite graph $G(N)$. The vertices are partitioned into $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$, and $\{a_i, b_j\}$ is an edge iff $n_{ij} = 0$.

In a bipartite graph $G = (V, E)$ with vertex partition $V = A \cup B$, subgraph $X = (V, E(X))$ is a *complete matching* if there exists a bijection $\sigma : A \to B$ such that $\{a, b\} \in E(X)$ iff $\sigma(a) = b$. The following lemma gives a necessary condition for a bipartite graph to have a unique complete matching.

Lemma 2.3.5 (Lemma on p. 89 in (13))**.** *Let $G = (V, E)$ be a bipartite graph with vertex partition $V = A \cup B$ and a unique complete matching X. Then there exists an edge $\{a, b\} \in E(X)$, where $a \in A$ and $b \in B$, such that*

$$\{a, b'\} \notin E(G) \textit{ for all } b' \in B \setminus \{b\}.$$

We can now present our proof of Theorem 2.3.1.

Proof of Theorem 2.3.1. Let $M \in \{0,1\}^{n \times m}$ with no column of all ones.

Suppose that length$(M) = r$. Observe that for zero-one vectors $\mathbf{x}$ and $\mathbf{y}$ that $Z(\mathbf{x} \oplus \mathbf{y}) = Z(\mathbf{x}) \cup Z(\mathbf{y})$. Hence without loss of generality we can suppose that each element of poset(M) corresponds to a column of M since adding a column to M that already is in the tropical span of its columns does not change the tropical rank of M by Theorem 2.2.18. Therefore, length$(M) = r$ implies that there are r columns in M with their zeros forming a chain of length r. From these r columns we can select an $r \times r$ minor of M such that, up to a permutation of rows and columns, there are zeros on the diagonal and ones below the diagonal. Hence this minor will be in standard form and tropically nonsingular. Therefore t-rank$(M) \geq r$.

Conversely, suppose that t-rank$(M) = r$ so there exists a nonsingular $r \times r$ minor N with minimal tropical determinant. We will prove that length$(N) = r$, which will imply that length$(M) \geq r$. The fact that N is nonsingular means that Δ_N obtains its minimum exactly once, and without loss of generality suppose $\Delta_N(e)$ is the minimum where $e \in S_r$ is the identity. Observe that $\Delta_N(e) \leq 1$ for if $n_{ii} = n_{jj} = 1$, then $n_{ij} + n_{ji} \leq n_{ii} + n_{jj}$ and hence $\Delta_N((ij)) \leq \Delta_N(e)$, which is a contradiction.

In fact $\Delta_N(e) = 0$. For suppose otherwise that $\Delta_N(e) = 1$, then we have an unique diagonal element $n_{jj} = 1$ and without loss of generality we may assume $n_{11} = 1$. Since N is nonsingular and $\det(N) = \Delta_N(e) = 1$, we know that $\Delta_N(\sigma) > 1$ for all $\sigma \in S_r \setminus e$. In particular if $j \neq 1$, then

$$1 < \Delta_N((1j)) = n_{1j} + n_{j1} + \sum_{i \neq 1,j} n_{ii}. \tag{2.3.3}$$

However, $n_{ii} = 0$ if $i \neq 1$, since n_{11} is the unique diagonal element equal to 1, so (2.3.3) becomes

$$1 < \Delta_N((1j)) = n_{1j} + n_{j1}. \tag{2.3.4}$$

Therefore it follows from (2.3.4) that $n_{1j} = n_{j1} = 1$ for all $j \neq 1$, and hence the first row and column in N is all ones. Since M does not have a column of all ones, it follows that we can choose K, an $r \times r$ minor of M where $k_{11} = 0$ and $k_{ij} = n_{ij}$ for $i \neq 1$. The idea here is that we can choose a row of M, which has a 0 in the column that corresponds to the first column in N. Observe that $\det(K) = 0$ since $\Delta_K(e) = 0$, and if $\Delta_K(\sigma) = 0$ then $\sigma(1) = 1$ for k_{11} is the only entry in the first column of K that is equal to 0. For $\sigma \in S_r$ such that $\sigma(1) = 1$, we have that $\Delta_N(\sigma) = 1 + \Delta_K(\sigma)$ since $n_{11} = 1, k_{11} = 0$ and $k_{ij} = n_{ij}$ for $i \neq 1$. Hence K is a nonsingular $r \times r$ minor of M with tropical determinant zero, which contradicts that N, with $\det(N) = 1$, has minimal tropical determinant among the nonsingular $r \times r$ minors. Therefore, in fact $\det(N) = \Delta_N(e) = 0$.

The fact that N is tropically nonsingular with tropical determinant $\Delta_N(e) = 0$, implies that $G = G(N)$ has a unique complete matching. First G has a complete matching X given by the identity permutation $e \in S_r$, since $\{a_i, b_i\} \in E(G)$ (i.e. $n_{ii} = 0$) for all i. Let X' be a complete matching, then we have a permutation $\sigma' \in S_r$ such that $\{a_i, b_{\sigma'(i)}\} \in E(G)$ and hence $n_{i\sigma'(i)} = 0$. However this means that $\Delta_N(\sigma') = \sum_{i=1}^{r} n_{i\sigma'(i)} = 0$, and since N is nonsingular $\sigma' = e$. Hence $X' = X$.

By the lemma, there exists a column in N with exactly one 0, and so a permutation of rows and columns transforms N into N' where $n'_{11} = 0$ and $n'_{i1} = 1$ for

$i \neq 1$. Proceeding to the nonsingular $(r-1) \times (r-1)$ lower-right minor of N', by induction on r it follows a permutation of rows and columns transforms N into a matrix N' that is in standard form. Since a matrix in standard form has full length, we know length$(N') = r$. Therefore length$(N) = r$. □

Observe that in our proof of 2.3.1, we proved the following two propositions.

Proposition 2.3.6. *Let $M \in \{0,1\}^{n\times m}$ be a matrix with no column of all ones and* t-rank$(M) = r$. *Then for all $a \leq r$ there exists an $a \times a$ nonsingular minor with tropical determinant zero.*

Proposition 2.3.7. *Let $N \in \{0,1\}^{r\times r}$ be tropically nonsingular, then* $\det(N) \leq 1$. *Furthermore,* $\det(N) = 0$ *iff N has no column (or row) of all ones.*

We can use this proposition to prove an upper bound on the sum of the tropical rank and tropical determinant of a zero-one matrix.

Proposition 2.3.8. *Let $M \in \{0,1\}^{n\times n}$ then* $\text{t-rank}(M) + \det(M) \leq 1 + n$.

Proof. Let $\text{t-rank}(M) = r$, so without loss of generality we may assume that N, the upper left $r \times r$ minor of M is tropically nonsingular. Furthermore, without loss of generality we may assume that $\Delta_N(e) = \det(N)$ where $e \in S_r$ is the identity. Finally by Proposition 2.3.7 we know $\det(N) \leq 1$. Hence for the identity $e \in S_n$,

$$\det(M) \leq \Delta_M(e) = \sum_{i=1}^{n} m_{ii} = \sum_{i=1}^{r} m_{ii} + \sum_{i=r+1}^{n} m_{ii} = \det(N) + \sum_{i=r+1}^{n} m_{ii} \leq 1 + (n-r).$$

Therefore $\text{t-rank}(M) + \det(M) \leq 1 + n$. □

One might hope that these results could be generalize to arbitrary matrices by means of the Hungarian method. To this end, say that a matrix $H \in \mathbb{R}^{n\times n}$ is in the *Hungarian form* if H is nonnegative and $\det(H) = 0$.

Lemma 2.3.9. *Let $\varphi : \mathbb{R}^{n\times n} \to \{0,1\}^{n\times n}$ where*

$$\varphi(M)_{ij} = \begin{cases} 0 & \text{if } m_{ij} = 0 \\ 1 & \text{if } m_{ij} \neq 0 \end{cases}.$$

If $H \in \mathbb{R}^{n\times n}$ is in the Hungarian form, then $\text{t-rank}(\varphi(H)) \leq \text{t-rank}(H)$.

Proof. Suppose that $\text{t-rank}(\varphi(H)) = r$. By Proposition 2.3.6, there exists a nonsingular $r \times r$ minor N of $\varphi(H)$ such that $\det(N) = 0$. Then for the corresponding minor N' of H, $\det(N') = 0$ and N' is nonsingular unless there are two $\sigma \in S_r$ such that $\Delta_N(\sigma) = \Delta_{N'}(\sigma) = 0$. Hence $r \leq \text{t-rank}(H)$ and therefore $\text{t-rank}(\varphi(H)) \leq \text{t-rank}(H)$. □

Unfortunately we do not have that t-rank$(\varphi(H))$ = t-rank(H) for all matrices in the Hungarian form as the following counterexample illustrates. Let $H \in \mathbb{R}^{4\times 4}$ be the matrix in Hungarian form where

$$H = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \varphi(H) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe t-rank$(H) = 3$, but by Theorem 2.3.1 we know that t-rank$(\varphi(H)) = 2$. Therefore in general all we can say for a matrix H in the Hungarian form is that t-rank$(\varphi(H)) \leq$ t-rank(H).

However we can use Proposition 2.2.16 and Lemma 2.3.9 to extend half of Theorem 2.3.1 to arbitrary square matrices.

Proposition 2.3.10. *Let $M \in \mathbb{R}^{n\times n}$, H be a matrix obtained by applying the Hungarian method to M, φ the map from Lemma 2.3.9. Then* length$(\varphi(H)) \leq$ t-rank(M).

Proof. By Theorem 2.3.1, Lemma 2.3.9, and Proposition 2.2.16 we know that

$$\text{length}(\varphi(H)) = \text{t-rank}(\varphi(H)) \leq \text{t-rank}(H) = \text{t-rank}(M).$$

Therefore length$(\varphi(H)) \leq$ t-rank(M). □

Chapter 3

Matroids

3.1 The Axioms and Basic Properties

There are numerous ways of axiomatizing the concept of a matroid. We refer the reader to the standard book (16), but perhaps the most intuitive notion of a matroid is as a combinatorial generalization of independent sets of vectors in a vector space. Suppose that you had a finite subset E of a vector space V and you formed the collection $\mathcal{I}$ of all subsets of E that are linearly independent sets in V. Then $\mathcal{I}$ would satisfy (I1), (I2), and (I3) from below, and we will make these the axioms of a matroid.

Definition 3.1.1. A *matroid* $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I} \subseteq 2^E$ such that:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $A \in \mathcal{I}$ and $A' \subseteq A$, then $A' \in \mathcal{I}$.

(I3) If $A_1, A_2 \in \mathcal{I}$ and $|A_1| < |A_2|$, then exists $e \in A_2 \setminus A_1$ such that $(A_1 \cup e) \in \mathcal{I}$.

E is called the *ground set* of $\mathcal{M}$ and sets in $\mathcal{I}$ are called *independent sets* of $\mathcal{M}$. A subset of E not in $\mathcal{I}$ is called *dependent*.

Example 3.1.2. Given a matrix $M \in \mathbb{C}^{n \times m}$, let E be the set of the m column vectors from M in $\mathbb{C}^n$ and let $\mathcal{I}$ be those subsets of E that are linearly independent. Then $(E, \mathcal{I})$ is a matroid and matroids that can be realized in this form are said to be $\mathbb{C}$-*representable*. By Theorem 3.4.12, it turns out that matroids that are not $\mathbb{C}$-representable lead to matrices with distinct Kapranov and tropical ranks

Now that we have independent sets, a natural choice for the name of a maximal independent set is a *basis*.

Definition 3.1.3. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. A set $I \in \mathcal{I}$ is a *basis* if I is maximal in $\mathcal{I}$ with respect to set inclusion. Denote the set of bases as $\mathcal{B}$.

Lemma 3.1.4. *In a matroid $\mathcal{M} = (E, \mathcal{I})$, if $B_1, B_2 \in \mathcal{B}$, then $|B_1| = |B_2|$. Furthermore if $I \in \mathcal{I}$ and $|I| = |B_1|$, then $I \in \mathcal{B}$.*

Proof. Let $B_1, B_2 \in \mathcal{B}$, and suppose that $|B_1| < |B_2|$. Then by (I3) we have that there exists $e \in B_2 \setminus B_1$ such that $(B_1 \cup e) \in \mathcal{I}$. However this contradicts that B_1 is maximal in $\mathcal{I}$. Therefore $|B_1| = |B_2|$.

For the second part, observe that if I is not a basis, then we have $I' \in \mathcal{I}$ with $|I'| > |I| = |B_1|$. By (I3) this means that there exists $e \in I' \setminus B_1$ such that $(B_1 \cup e) \in \mathcal{I}$, contradicting that B_1 is maximal in $\mathcal{I}$. Therefore $I \in \mathcal{B}$. □

Lemma 3.1.5. *The set of bases $\mathcal{B}$ in a matroid $\mathcal{M}$ satisfies the following:*

(B1) $\mathcal{B} \neq \emptyset$.

(B2) *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$.*

Proof. Since $\mathcal{I}$ is finite and nonempty, it follows that $\mathcal{B}$ is nonempty for maximal elements must exist in $\mathcal{I}$, proving (B1).

For (B2), we know that $B_1 \setminus x \in \mathcal{I}$ by (I2) and Lemma 3.1.4 tells us that $|B_1| = |B_2|$. Since $|B_1 \setminus x| < |B_2|$, by (I3) we have

$$y \in B_2 \setminus (B_1 \setminus x) \subseteq B_2 \setminus B_1 \tag{3.1.1}$$

such that $(B_1 \setminus x) \cup y \in \mathcal{I}$. By Lemma 3.1.4, we know that $(B_1 \setminus x) \cup y \in \mathcal{B}$ because $|(B_1 \setminus x) \cup y| = |B_1|$. Therefore there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$, proving (B2). □

It turns out that (B1) and (B2) provide an equivalent axiomization of matroids.

Theorem 3.1.6 (Theorem 1.2.3 in (16))**.** *Let E be a finite set and $\mathcal{B}$ a collection of subsets of E that satisfies (B1) and (B2). Let $\mathcal{I} = \{I \subseteq E \mid I \subseteq B \text{ for some } B \in \mathcal{B}\}$ be the collection of subsets of sets in $\mathcal{B}$. Then $\mathcal{M} = (E, \mathcal{I})$ is a matroid where $\mathcal{I}$ satisfies (I1), (I2), and (I3) and $\mathcal{M}$ has $\mathcal{B}$ as the set of its bases.*

Example 3.1.7. Let $G = (V, E)$ be the graph in Figure 3.1. Let $\mathcal{I} \subseteq 2^E$ be the collection of forests in G, so for example $\{1, 2, 3, 4\} \in \mathcal{I}$, but $\{1, 2, 3, 4, 5\} \notin \mathcal{I}$. Then $\mathcal{M}_G = (E, \mathcal{I})$ is a matroid. Its set of bases $\mathcal{B}$ is the collection of spanning forests in G, for example $\{1, 2, 3, 4, 6, 8\}$ is a basis in $\mathcal{M}_G$. The circuits in the graph G, $\{5, 6, 7\}$ for example, will correspond to the the minimal dependent sets in $\mathcal{M}_G$.

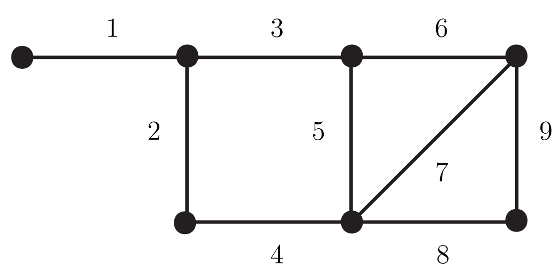

Figure 3.1: The graph $G = (V, E)$

We will generalize this example to an arbitrary matroid $\mathcal{M} = (E, \mathcal{I})$, by saying that $C \subseteq E$ is a *circuit* if C is a minimal dependent set. Denote the set of circuits of a matroid by $\mathcal{C}$.

Theorem 3.1.8 (Theorem 1.1.4 in (16))**.** *Let E be a finite set and $\mathcal{C}$ a collection of subsets of E. $\mathcal{C}$ is the collection of circuits of a matroid $\mathcal{M}$ on E if and only if $\mathcal{C}$ satisfies the following:*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) *If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*

(C3) *If $C_1, C_2 \in \mathcal{C}$, $e \in C_1 \cap C_2$, and $f \in C_1 \setminus C_2$, then there exists $C_3 \in \mathcal{C}$ such that $f \in C_3 \subseteq (C_1 \cup C_2) \setminus e$.*

Furthermore for $X \subseteq E$ in the matroid $\mathcal{M}$,

$$X \text{ is independent if and only if } C \nsubseteq X \text{ for all } C \in \mathcal{C}.$$

Proposition 3.1.9. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. If $I \subseteq E$ is independent and $(I \cup e) \subseteq E$ is dependent, then there is a unique circuit C in $\mathcal{M}$ such that*

$$e \in C \subseteq (I \cup e).$$

Proof. Since $(I \cup e)$ is dependent, there exists a circuit $C \subseteq (I \cup e)$ and any such circuit must contain e, for any subset of I is independent. Therefore if C_1 and C_2 are two such circuits, then by (C3) there exists a circuit $C_3 \subseteq (C_1 \cup C_2) \setminus e \subseteq I$, which is a contradiction. Therefore there is a unique circuit C such that $e \in C \subseteq (I \cup e)$. $\square$

Corollary 3.1.10. *Given a basis B and $e \in E \setminus B$, there is a unique circuit $C(e, B)$ such that $e \in C(e, B) \subseteq (B \cup e)$. We say that $C(e, B)$ is the* fundamental circuit *of e with respect to B.*

Lemma 3.1.11. *The set of bases $\mathcal{B}$ of a matroid $\mathcal{M} = (E, \mathcal{I})$ satisfies:*

(B2)* *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_2 \setminus B_1$, then there exists $y \in B_1 \setminus B_2$ such that $(B_1 \setminus y) \cup x \in \mathcal{B}$.*

Proof. By Corollary 3.1.10, $(B_1 \cup x)$ contains a unique circuit $C(x, B_1)$. Since $C(x, B_1) \nsubseteq B_2$, for $C(x, B_1)$ is dependent and B_2 is independent, there exists $y \in C(x, B_1) \setminus B_2$ and in fact $y \in B_1 \setminus B_2$. Furthermore, since $(B_1 \setminus y) \cup x$ is a subset of $B_1 \cup x$ and does not contain $C(x, B_1)$, we know that $(B_1 \setminus y) \cup x$ is independent. Since the cardinality of $(B_1 \setminus y) \cup x$ is the same B_1, if follows that in fact $(B_1 \setminus y) \cup x$ is a basis. $\square$

The next term from linear algebra we will bring in is the rank of a set. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, let $\rho : 2^E \to \mathbb{N}$ where

$$\rho(X) = \max\{|I| \mid I \in \mathcal{I} \text{ and } I \subseteq X\}.$$

So the *rank of* $X \subseteq E$ is the size of a largest independent set contained in X. The *rank of* $\mathcal{M}$ is $\rho(E)$ and will be denoted by $\rho(\mathcal{M})$. It turns out that one can axiomatize matroids in terms of a rank function.

Theorem 3.1.12 (Theorem 1.3.2 in (16))**.** *Let E be a finite set, $\rho : 2^E \to \mathbb{N}$ is the rank function of a matroid $\mathcal{M}$ on E if and only if ρ satisfies the following:*

(R1) *If $X \subseteq E$, then $\rho(X) \leq |X|$.*

(R2) *If $X \subseteq Y \subseteq E$, then $\rho(X) \leq \rho(Y)$.*

(R3) *If $X, Y \subseteq E$, then $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.*

Furthermore for $X \subseteq E$ in the matroid $\mathcal{M}$,

1. *X is independent if and only if $\rho(X) = |X|$;*
2. *X is a basis if and only if $\rho(\mathcal{M}) = \rho(X) = |X|$; and*
3. *X is a circuit if and only if $\rho(X \setminus x) = \rho(X) - 1 = |X|$ for all $x \in X$.*

Another characterization of matroids can be given in terms of the *closure* function, which is analogous to the span of a set of vectors. Given a matroid $\mathcal{M} = (E, \mathcal{I})$ with rank function ρ, let $\mathrm{cl} : 2^E \to 2^E$ by:

$$\mathrm{cl}(X) = \{e \in E \mid \rho(X) = \rho(X \cup e)\}.$$

Theorem 3.1.13 (Theorem 1.4.4 in (16))**.** *Let E be a finite set, $\mathrm{cl} : 2^E \to 2^E$ is the closure function of a matroid $\mathcal{M}$ on E if and only if cl satisfies:*

(CL1) *If $X \subseteq E$, then $X \subseteq \mathrm{cl}(X)$.*

(CL2) *If $X \subseteq Y \subseteq E$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.*

(CL3) *If $X \subseteq E$, then $\mathrm{cl}(X) = \mathrm{cl}(\mathrm{cl}(X))$.*

(CL4) *If $X \subseteq E$, $e \in E$, and $y \in \mathrm{cl}(X \cup e) \setminus \mathrm{cl}(X)$, then $e \in \mathrm{cl}(X \cup y)$.*

Furthermore for $X \subseteq E$ in the matroid $\mathcal{M}$,

$$X \text{ is independent if and only if } x \notin \mathrm{cl}(X \setminus x) \text{ for all } x \in X.$$

In linear algebra terms, this is expressing that a set of vectors is independent if and only if no vector lies in the span of the others. We say that a subset $X \subseteq E$ is *spanning* if $\mathrm{cl}(X) = E$.

Lemma 3.1.14. *Let $\mathcal{M}$ be a matroid on E with $X, Y \subseteq E$. If $\rho(X \cup y) = \rho(X)$ for all $y \in Y \setminus X$, then $\rho(X \cup Y) = \rho(X)$.*

Proof. By induction on $n = |Y \setminus X|$. If $n = 1$, then $X \cup y = X \cup Y$ so we are done by assumption. Now let $Y \setminus X = \{y_1, \ldots, y_{n+1}\}$ and by the induction hypothesis $\rho(X \cup \{y_1, \ldots, y_n\}) = \rho(X) = \rho(X \cup y_{n+1})$. Hence by the properties of the rank:

$$\begin{aligned}
\rho(X) + \rho(X) &= \rho(X \cup \{y_1, \ldots, y_n\}) + \rho(X \cup y_{n+1}) \\
&\geq \rho(X \cup Y) + \rho(X) && \text{by (R3)} \\
&\geq \rho(X) + \rho(X). && \text{by (R2)}
\end{aligned}$$

We invoke (R3) because

$$(X \cup \{y_1, \ldots, y_n\}) \cup (X \cup y_{n+1}) = X \cup Y$$
$$(X \cup \{y_1, \ldots, y_n\}) \cap (X \cup y_{n+1}) = X \cap Y.$$

So $\rho(X) + \rho(X) = \rho(X \cup Y) + \rho(X)$ and therefore $\rho(X \cup Y) = \rho(X)$. □

Corollary 3.1.15. *Let $\mathcal{M}$ be a matroid on E with $X \subseteq Y \subseteq E$, then*

1. $\rho(\mathrm{cl}(X)) = \rho(X)$;
2. $\rho(X) = \rho(Y)$ *only if* $Y \subseteq \mathrm{cl}(X)$; *and*
3. *If X is a flat and $\rho(X) = \rho(Y)$, then $X = Y$.*

Proof. (*1*) follows directly from the definition of $\mathrm{cl}(X)$ and Lemma 3.1.14. For (*2*), suppose that $y \in Y$ then $\rho(X) \leq \rho(X \cup y) \leq \rho(Y) = \rho(X)$. So $\rho(X) = \rho(X \cup y)$ and hence by definition $y \in \mathrm{cl}(X)$. Therefore $Y \subseteq \mathrm{cl}(X)$, which proves (*2*), and if X is a flat then $\mathrm{cl}(X) = X$ so $Y \subseteq \mathrm{cl}(X) = X \subseteq Y$, which proves (*3*). □

Therefore by Corollary 3.1.15 we have that $\mathrm{cl}(X)$ is the largest subset of E that contains X and has the same rank as X. Thinking in terms of a vector space, by analogy this is exactly what we would expect to happen in a matroid. Since in a vector space V the span of a set of vectors X, $\mathrm{span}(X)$, is the largest subset of V that contains X and has the same rank as X.

Definition 3.1.16. In a matroid $\mathcal{M}$, we say that $X \subseteq E$ is a *flat* or *closed set* of $\mathcal{M}$ if $\mathrm{cl}(X) = X$. A flat X with $\rho(X) = \rho(\mathcal{M}) - 1$ is called a *hyperplane*.

Proposition 3.1.17. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $X \subseteq E$, then*

1. *X is spanning if and only if $\rho(X) = \rho(\mathcal{M})$;*
2. *X is a basis if and only if X is a minimal spanning set; and*
3. *X is a hyperplane if and only if X is a maximal nonspanning set.*

Proof. For (*1*), if X is spanning then $\mathrm{cl}(X) = E$ and since Corollary 3.1.15 says that $\rho(X) = \rho(\mathrm{cl}(X))$, it follows that $\rho(X) = \rho(E) = \rho(\mathcal{M})$. For the converse, if $\rho(X) = \rho(\mathcal{M}) = \rho(E)$, then by Corollary 3.1.15 we have that $E \subseteq \mathrm{cl}(X)$ and therefore X is spanning since $\mathrm{cl}(X) = E$. Therefore we have proved that (*1*) holds, and it is clear that (*2*) follows from Theorem 3.1.12 and (*1*).

For (*3*), suppose that X is a hyperplane. By definition, $\rho(X) = \rho(\mathcal{M}) - 1$, so by (*1*) we know that X is not spanning. Let $e \notin X$, because X is a flat we know $e \notin \mathrm{cl}(X) = X$ and hence $\rho(X \cup e) = \rho(X) + 1 = \rho(\mathcal{M})$, which implies that $(X \cup e)$ is spanning. Therefore X is a maximal nonspanning set. Conversely, if X is a maximal nonspanning set then $\rho(X) < \rho(\mathcal{M})$ and $\rho(X \cup e) = \rho(\mathcal{M})$ for all $e \notin X$. Hence $\rho(X) = \rho(\mathcal{M}) - 1$ and since $\rho(X \cup e) = \rho(X)$ only if $e \in X$ we know $X = \mathrm{cl}(X)$. Therefore X is a flat with rank $\rho(\mathcal{M}) - 1$ and hence a hyperplane. □

Proposition 3.1.18. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $X \subset E$. Then*

1. *X is a circuit if and only if X is a minimal set with the property that*

$$x \in \operatorname{cl}(X \setminus x) \text{ for all } x \in X.$$

2. *$\operatorname{cl}(X) = X \cup \{e \in E \setminus X \mid \mathcal{M}$ has circuit C such that $e \in C \subseteq X \cup e\}$.*

Proof. Observe that by Theorem 3.1.13, (*1*) is just saying that a circuit is a minimal dependent set. For (*2*), let $e \in \operatorname{cl}(X) \setminus X$, so $\rho(X) = \rho(X \cup e)$. Let B be a basis for X, then $(B \cup e)$ is dependent so by Corollary 3.1.10 there exists a circuit C such that $e \in C \subseteq (B \cup e) \subseteq (X \cup e)$. Conversely, if $e \in E \setminus X$ and there exists a circuit C such that $e \in C \subseteq (X \cup e)$, then by (*1*) we know that $e \in \operatorname{cl}(C \setminus e)$ and by (CL2) we know $\operatorname{cl}(C \setminus e) \subseteq \operatorname{cl}(X)$. Therefore $e \in \operatorname{cl}(X)$. □

3.2 The Lattice of Flats

A *lattice* is a partially ordered set L such that for all $x, y \in L$, the least upper bound and greatest lower bound exist for the the pair x, y and they are denoted by $x \vee y$ and $x \wedge y$ respectively. Formally, a lattice is defined as follows:

Definition 3.2.1. A partially ordered set $(L, \leq)$ equipped with the functions

$$\vee : L \times L \to L \text{ and } \wedge : L \times L \to L$$

is a *lattice* if for every $x, y, z \in L$ the following are satisfied:

1. $x \leq x \vee y$ and $y \leq x \vee y$;
2. If $x \leq z$ and $y \leq z$, then $x \vee y \leq z$;
3. $x \wedge y \leq x$ and $x \wedge y \leq y$; and
4. If $z \leq x$ and $z \leq y$, then $z \leq x \wedge y$.

We say that $x \vee y$ is the *join* of x and y, and $x \wedge y$ is the *meet* of x and y.

Lemma 3.2.2. *Let $X_1, X_2 \subseteq E$ be flats in the matroid $\mathcal{M} = (E, \mathcal{I})$. Then $X_1 \cap X_2$ is a flat as well.*

Proof. Suppose that $X_1 \cap X_2$ was not a flat, so let $e \in \operatorname{cl}(X_1 \cap X_2) \setminus (X_1 \cap X_2)$. By Proposition 3.1.18 then, there exists a circuit C such that $e \in C \subseteq (X_1 \cap X_2) \cup e$. Since

$$(X_1 \cap X_2) \cup e \subseteq (X_i \cup e),$$

we have that $e \in C \subseteq (X_i \cup e)$. Therefore, again by Proposition 3.1.18, we have that $e \in \operatorname{cl}(X_i)$. Since X_i is a flat we know $\operatorname{cl}(X_i) = X_i$, so in fact $e \in X_i$. This means that $e \in X_1 \cap X_2$, which is a contradiction for we assumed that $e \in \operatorname{cl}(X_1 \cap X_2) \setminus (X_1 \cap X_2)$. Therefore $\operatorname{cl}(X \cap Y) = X \cap Y$. □

For a matroid $\mathcal{M}$, let $\mathcal{L}(\mathcal{M})$ be the collection of flats of $\mathcal{M}$ ordered under set inclusion. $\mathcal{L}(\mathcal{M})$ can be given a natural lattice structure, as the following proposition shows.

Proposition 3.2.3. *Given a matroid $\mathcal{M}$, $\mathcal{L}(\mathcal{M})$ is a lattice where for $X, Y \in \mathcal{L}(\mathcal{M})$,*

$$X \vee Y = \mathrm{cl}(X \cup Y) \text{ and } X \wedge Y = X \cap Y.$$

Proof. It follows from (CL2) and (CL3) that $\mathrm{cl}(X \cup Y)$ is the smallest flat containing X and Y. For $X \wedge Y$, $X \cap Y$ is the largest set contained in X and Y and by Lemma 3.2.2 we know $X \cap Y \in \mathcal{L}(\mathcal{M})$. □

Definition 3.2.4. Given flats $X \subseteq Y$ in $\mathcal{L}(\mathcal{M})$ we say that Y *covers* X if there does not exists a flat $F \in \mathcal{L}(\mathcal{M})$ such that $X \subsetneq F \subsetneq Y$.

Lemma 3.2.5. *If X and Y are flats of $\mathcal{M}$ and $X \subseteq Y$, then every maximal chain of flats from X to Y has length $\rho(Y) - \rho(X)$.*

Proof. It suffices to prove that Y covers X if and only if $\rho(X) + 1 = \rho(Y)$.

Suppose that Y covers X and let $y \in Y \setminus X$. Since $\mathrm{cl}(X \cup y)$ is a flat and $X \subsetneq \mathrm{cl}(X \cup y) \subseteq Y$, we know that $\mathrm{cl}(X \cup y) = Y$ because Y covers X. Now X is a flat and $y \notin X$, so we know that

$$\rho(X) + 1 = \rho(X \cup y) = \rho(\mathrm{cl}(X \cup y)) = \rho(Y).$$

Therefore $\rho(X) + 1 = \rho(Y)$. Conversely, suppose that $\rho(X) + 1 = \rho(Y)$. Let F be a flat such that $X \subseteq F \subseteq Y$. Now either $\rho(X) = \rho(F)$ or $\rho(F) = \rho(Y)$, then by Corollary 3.1.15 either $X = F$ or $F = Y$. Therefore Y covers X. □

Corollary 3.2.6. *If r is the rank of a matroid $\mathcal{M}$, then a maximal length chain of flats has length r. So r is the greatest number such that there exists flats F_j where*

$$\emptyset \neq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r = E. \tag{3.2.1}$$

Proposition 3.2.7 (Proposition 1.7.8 in (16))**.** *Every flat in a matroid $\mathcal{M}$ is the intersection of hyperplanes. In fact if X is a flat in $\mathcal{M}$ and $\rho(X) = \rho(\mathcal{M}) - k$ where $k \geq 1$, then there exists a set $\{H_1, \ldots, H_k\}$ of hyperplanes such that $X = \bigcap_j H_j$.*

Proof. We will induct on k. If $k = 1$, then X is a hyperplane so we are done. Now let X be a flat where $\rho(X) = \rho(\mathcal{M}) - k$. Since $\rho(X) < \rho(\mathcal{M})$ we know there exists $y \in E \setminus X$ and furthermore

$$\rho(\mathrm{cl}(X \cup y)) = \rho(X \cup y) = \rho(X) + 1 = \rho(\mathcal{M}) - (k - 1)$$

since X is a flat. By the induction hypothesis we know that there exists hyperplanes $H_1, \ldots, H_{k-1}$ such that $\mathrm{cl}(X \cup y) = \bigcap_{j=1}^{k-1} H_j$.

Let H_k be a maximal element in the set of flats F such that $X \subseteq F \subseteq E \setminus y$, this set is nonempty since X is such a flat. Then H_k is maximal nonspanning set and so is a hyperplane by Proposition 3.1.17. Furthermore

$$\mathrm{cl}(X \cup y) \supsetneq \mathrm{cl}(X \cup y) \cap H_k = \bigcap_{j=1}^{k} H_j \supset X. \tag{3.2.2}$$

However, as in the proof of Lemma 3.2.5 we have that $\mathrm{cl}(X \cup y)$ covers X and by Lemma 3.2.2, $\bigcap_{j=1}^{k} H_j$ is a flat. Therefore by (3.2.2), we have that $\bigcap_{j=1}^{k} H_j = X$. □

3.3 Matroid Duality

A common and often very powerful construction seen throughout mathematics is the construction of a dual object. For example given a vector space V over the field F, the dual space V^* is the F-vector space of all linear functionals $\phi : V \to F$. Another example is given a category $\mathbf{C}$ with morphisms $\mathbf{M}$ and objects $\mathbf{O}$, then the dual category $\mathbf{C}^*$ has morphisms $\mathbf{M}^*$ and the same objects $\mathbf{O}$. Here $\mathbf{M}^* = \{f^* \mid f \in \mathbf{M}\}$, where if $f : C \to D$ then $f^* : D \to C$ and $(f \circ g)^* = g^* \circ f^*$.

Proposition 3.3.1. *For a matroid $\mathcal{M}$ on ground set E with set of bases $\mathcal{B}(\mathcal{M})$, let*

$$\mathcal{B}^*(\mathcal{M}) = \{E \setminus B \mid B \in \mathcal{B}(\mathcal{M})\}.$$

Then $\mathcal{B}^*(\mathcal{M})$ is the set of bases for a matroid $\mathcal{M}^*$ on E.

Proof. We need to show that $\mathcal{B}^* = \mathcal{B}^*(\mathcal{M})$ satisfies (B1) and (B2) of Theorem 3.1.6. Since $\mathcal{B} \neq \emptyset$, we know that $\mathcal{B}^* \neq \emptyset$. Therefore (B1) is satisfied.

Let $B_1^* = E \setminus B_1$ and $B_2^* = E \setminus B_2$ be elements of $\mathcal{B}^*$, where $B_1, B_2 \in \mathcal{B}$, and let $x \in B_1^* \setminus B_2^*$. A moments thought shows that

$$B_1^* \setminus B_2^* = B_2 \setminus B_1,$$

so $x \in B_2 \setminus B_1$. By (B2)* of Lemma 3.1.11 we have there exists $y \in B_1 \setminus B_2 = B_2^* \setminus B_1^*$ such that $(B_1 \setminus y) \cup x \in \mathcal{B}$. Since $(B_1 \setminus y) \cup x \in \mathcal{B}$ and

$$E \setminus ((B_1 \setminus y) \cup x) = (B_1^* \setminus x) \cup y,$$

we have that $(B_1^* \setminus x) \cup y \in \mathcal{B}^*$ as desired. Therefore (B1) and (B2) hold for $\mathcal{B}^*$. □

Definition 3.3.2. Given a matroid $\mathcal{M}$ on ground set E with set of bases $\mathcal{B}(\mathcal{M})$, let the *dual matroid* $\mathcal{M}^*$ be the matroid on ground set E and with set of bases $\mathcal{B}(\mathcal{M}^*) = \mathcal{B}^*(\mathcal{M})$.

By Theorem 3.3.1, $\mathcal{M}^*$ is indeed a matroid and observe $(\mathcal{M}^*)^* = \mathcal{M}$. The bases of $\mathcal{M}^*$ are called the *cobases* of $\mathcal{M}$ and similar conventions hold for all named subsets of $E(\mathcal{M}^*)$. So in particular, the circuits, hyperplanes, independent sets, and spanning sets of $\mathcal{M}^*$ are called the *cocircuits*, *cohyperplanes*, *coindependent sets*, and *cospanning sets*. As expected, there are close relationships between these set as the following proposition shows.

Proposition 3.3.3. *Let $\mathcal{M}$ be a matroid on ground set E and $X \subseteq E$. Then*

1. *X is independent if and only if $E \setminus X$ is cospanning;*

2. *X is spanning if and only if $E \setminus X$ is coindependent;*

3. *X is a hyperplane if and only if $E \setminus X$ is a cocircuit; and*

4. *X is a circuit if and only if $E \setminus X$ is a cohyperplane.*

Proof. For (*2*), by Proposition 3.1.17 and the definitions we have the following series of equivalences:

$$\begin{aligned} X \text{ is spanning} &\iff \rho(X) = \rho(\mathcal{M}) \\ &\iff \text{Exists basis } B \text{ of } \mathcal{M} \text{ such that } B \subseteq X \\ &\iff E \setminus X \subseteq E \setminus B \text{ where } E \setminus B \text{ is a cobasis} \\ &\iff E \setminus X \text{ is coindependent} \end{aligned}$$

Therefore X is spanning if and only if $E \setminus X$ is coindependent.

For (*3*), by Proposition 3.1.17, (*2*), and the definitions we have the following series of equivalences:

$$\begin{aligned} X \text{ is a hyperplane} &\iff X \text{ is a maximal nonspanning set of } \mathcal{M} \\ &\iff E \setminus X \text{ is a minimal codependent set} \\ &\iff E \setminus X \text{ is a cocircuit} \end{aligned}$$

Therefore X is a hyperplane if and only if $E \setminus X$ is a cocircuit. (*1*) and (*4*) are the duals of the (*2*) and (*3*), respectively, so they are proved by applying (*2*) and (*3*) to the matroid $\mathcal{M}^*$. □

We can now characterize the rank of a matroid $\mathcal{M}$ in terms of its cocircuits.

Theorem 3.3.4. *Let $\mathcal{M}$ be a matroid on the ground set E with $\rho(\mathcal{M}) = r$. Let $\mathcal{C}^*$ be the set of cocircuits of $\mathcal{M}$ and let* $\operatorname{poset}(\mathcal{C}^*)$ *be the collection of all possible unions of cocircuits. Then* $\operatorname{length}(\operatorname{poset}(\mathcal{C}^*)) = \rho(\mathcal{M}) = r$.

Proof. Let $\mathcal{H}$ be the collection of the hyperplanes in $\mathcal{M}$ and let $\mathcal{P}(\mathcal{H})$ be the collection of all possible intersections of hyperplanes. By Proposition 3.2.7 we know that $\mathcal{P}(\mathcal{H}) = \mathcal{L}(\mathcal{M})$, and hence by Proposition 3.3.3

$$\operatorname{poset}(\mathcal{C}^*) = \{E \setminus X \mid X \in \mathcal{L}(\mathcal{M})\}.$$

We know that the longest length chain in $\mathcal{L}(\mathcal{M})$ is equal to $\rho(\mathcal{M}) = r$ by Corollary 3.2.6, so by taking the complement of (3.2.1) we get

$$E \supsetneq (E \setminus F_1) \supsetneq (E \setminus F_2) \supsetneq \cdots \supsetneq (E \setminus F_{r-1}) \supsetneq \emptyset$$

is a chain in $\operatorname{poset}(\mathcal{C}^*)$. Furthermore since $\operatorname{poset}(\mathcal{C}^*) = \{E \setminus X \mid X \in \mathcal{L}(\mathcal{M})\}$, this must be a maximal length chain. Therefore $\operatorname{length}(\operatorname{poset}(\mathcal{C}^*)) = \rho(\mathcal{M}) = r$. □

Definition 3.3.5. Given a matroid $\mathcal{M}$ on the ground set $E = \{v_1, \ldots, v_n\}$ and set of cocircuits $\mathcal{C}^* = \{C_1, \ldots, C_m\}$, then let $M \in \{0,1\}^{n \times m}$ be the matrix where $M_{ij} = 0$ if and only if $v_i \in C_j$. M is known as the *cocircuit matrix* of $\mathcal{M}$.

Corollary 3.3.6. *Let $M \in \{0,1\}^{n \times m}$ be the cocircuit matrix of matroid $\mathcal{M}$, then*

$$\text{t-rank}(M) = \rho(\mathcal{M}). \tag{3.3.1}$$

Proof. By Proposition 3.3.3 we know that no cocircuit is empty, since if X is a hyperplane then $X \subsetneq E$, and therefore no column of M is all ones. Therefore by Theorem 2.3.1, we know that t-rank(M) = length(poset(M)). Observe that poset(M) = poset$(\mathcal{C}^*)$, so length(poset(M)) = length(poset$(\mathcal{C}^*)$) $= \rho(\mathcal{M})$ by Theorem 3.3.4. Therefore t-rank$(M) = \rho(\mathcal{M})$. □

So we know what the circuits, hyperplanes, independent sets, and spanning sets of the dual matroid $\mathcal{M}^*$ are in terms of $\mathcal{M}$, but what about the rank function ρ^* on $\mathcal{M}^*$? To answer this question we first need the following lemma.

Lemma 3.3.7. *Given a matroid $\mathcal{M} = (E, \mathcal{I})$, let $I, I^* \subseteq E$ be disjoint where I is independent and I^* is coindependent. Then there is a basis B and a cobasis B^* such that $I \subseteq B$, $I^* \subseteq B^*$, and $B \cap B^* = \emptyset$.*

Proof. Let B be a maximal independent set such that $I \subseteq B \subseteq E \setminus I^*$, so therefore $\rho(E \setminus I^*) = \rho(B)$. However by Proposition 3.3.3, we know that $E \setminus I^*$ is spanning, so by Proposition 3.1.17 we know $\rho(E \setminus I^*) = \rho(\mathcal{M})$. Therefore $\rho(B) = \rho(\mathcal{M})$, so B is a basis. Now since $B \subseteq E \setminus I^*$, we know that $I^* \subseteq E \setminus B = B^*$. Therefore B and B^* are the required basis and cobasis. □

Proposition 3.3.8. *Given a matroid $\mathcal{M}$ on ground set E with rank function ρ, the rank function ρ^* for $\mathcal{M}^*$ is given by*

$$\rho^*(X) = |X| - \rho(\mathcal{M}) + \rho(E \setminus X).$$

Proof. Let $X \subseteq E$. Now let B^*_X be a maximal independent set in $\mathcal{M}^*$ contained in X, and let $B_{E\setminus X}$ be a maximal independent set in $\mathcal{M}$ contained in $E \setminus X$. Therefore

$$\rho^*(X) = |B^*_X| \quad \text{and} \quad \rho(E \setminus X) = \left|B_{E\setminus X}\right|. \tag{3.3.2}$$

Furthermore $B_{E\setminus X}$ is independent, B^*_X is coindependent, and they are disjoint. By Lemma 3.3.7, there is exists a basis B such that $B_{E\setminus X} \subseteq B$ and $B^*_X \subseteq (E \setminus B) = B^*$. Now $B_{E\setminus X} \subseteq B \cap (E \setminus X) \subseteq E \setminus X$, so by the maximally of $B_{E\setminus X}$ we know that

$$B_{E\setminus X} = B \cap (E \setminus X). \tag{3.3.3}$$

Similarly,

$$B^*_X = B^* \cap X. \tag{3.3.4}$$

From (3.3.3), it follows that $B = B_{E\setminus X} \uplus (B \cap X)$ and hence

$$|B \cap X| = |B| - \left|B_{E\setminus X}\right|. \tag{3.3.5}$$

Bringing this all together gives:

$$\begin{aligned} |X| &= |X \cap B| + |X \cap B^*| && \text{(by } B^* = E \setminus B) \\ &= (|B| - \left|B_{E\setminus X}\right|) + (|B^*_X|) && \text{(by (3.3.4) and (3.3.5))} \\ &= \rho(\mathcal{M}) - \rho(E \setminus X) + \rho^*(X) && \text{(by (3.3.2))}. \end{aligned}$$

Therefore $\rho^*(X) = |X| - \rho(\mathcal{M}) + \rho(E \setminus X)$. □

3.4 Matroid Representability

Definition 3.4.1. Given a field F, a matroid $\mathcal{M} = (E, \mathcal{I})$ is *F-representable* if there exists a vector space V over F and a mapping $\varphi : E \to V$ such that for $X \subseteq E$:

$$X \in \mathcal{I} \text{ if and only if } \varphi|_X \text{ is injective and } \varphi(X) \text{ is linearly independent in } V.$$

If a matroid $\mathcal{M} = (E, \mathcal{I})$ is F- representable, then we can think of a representation as a matrix in $M \in F^{n\times m}$ where $E = \{e_1, \dots, e_n\}$ and $m = \dim(V)$. Picking a basis for V, say $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, let $M \in F^{n\times m}$ be the matrix such that

$$\varphi(e_i) = M_{i1}\mathbf{v}_1 + \cdots + M_{im}\mathbf{v}_m.$$

Conversely, a matrix $M \in F^{n\times m}$ is said to be a *F-representation* if the mapping $e_i \mapsto (M_{i1}, \dots, M_{im})$ is a representation $E \to F^m$.

We now present a necessary and sufficient condition for a matroid $\mathcal{M}$ to be F-representable. Our presentation is based off of (24), but the theorem was first proved in (23).

Theorem 3.4.2. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid, then $\mathcal{M}$ is F-representable if and only if for each hyperplane H of $\mathcal{M}$ there exists a function $c_H : E \to F$ so that*

(H1) *For each hyperplane H, kernel$(c_H) = H$.*

(H2) *For hyperplanes H_1, H_2, H_3 of $\mathcal{M}$, if $\rho(H_1 \cap H_2 \cap H_3) \geq \rho(\mathcal{M}) - 2$, then there exists $\alpha_1, \alpha_2, \alpha_3 \in F^*$ such that $\alpha_1 c_{H_1} + \alpha_2 c_{H_2} + \alpha_3 c_{H_3} = 0$.*

To prove this theorem we need a few lemmas, which we will prove first.

Lemma 3.4.3. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid with basis $B = \{b_1, \dots, b_r\}$ where $r = \rho(\mathcal{M})$. Then for each b_j there exists an unique hyperplane H_j of $\mathcal{M}$ such that $B \setminus b_j \subseteq H_j$ and these hyperplanes are distinct.*

Proof. Let $H_j = \mathrm{cl}(B \setminus b_j)$, know that that H_j is a hyperplane since H_j is a flat and $\rho(H_j) = \rho(\mathcal{M}) - 1$. Observe that $b_j \notin H_j$, by Theorem 3.1.13 and the fact that B is independent. However, $b_i \in H_j$ if $i \neq j$, which proves that the H_j are distinct. □

Lemma 3.4.4. *Let F be a field, E be a set, and F^E be the vector space over F consisting of all functions $g : E \to F$. Let $f_1, \dots, f_r \in F^E$ be functions where there are distinct $b_1, \dots, b_r \in E$ such that $f_i(b_j) \neq 0$ if and only if $i = j$. Then $\{f_1, \dots, f_r\}$ is a linearly independent set in F^E.*

Proof. Suppose there exists $\alpha_j \in F$ such that $0 = \sum_{j=1}^{r} \alpha_j f_j$ in F^E. Then since $f_i(b_j) = 0$ if $i \neq j$, we have that

$$0 = \sum_{j=1}^{r} \alpha_j f_j(b_j) = \alpha_j f_j(b_j),$$

and since $f_j(b_j) \neq 0$, it follows that $\alpha_j = 0$. Therefore $\{f_1, \dots, f_r\}$ is a linearly independent set in F^E. □

Lemma 3.4.5. *Suppose that we have a matroid $\mathcal{M} = (E, \mathcal{I})$ and a set of functions $\{c_H : E \to F \mid H$ is a hyperplane in $\mathcal{M}\}$ that satisfy (H1) and (H2) from Theorem 3.4.2. Let $V = \operatorname{span}(\{c_H \mid H$ is a hyperplane in $\mathcal{M}\})$, which is a subspace of F^E. Let $B = \{b_1, \dots, b_r\}$ be a basis in $\mathcal{M}$ and let $H_j = \operatorname{cl}(B \setminus b_j)$ be the distinct hyperplanes from Lemma 3.4.3. Then $C_B = \{c_{H_1}, \dots, c_{H_r}\}$ is a basis for V.*

Proof. For $i, j \leq r$, we know that $b_j \notin H_j$ and $b_j \in H_i$ if $i \neq j$, so by (H1) we know that $c_{H_i}(b_j) \neq 0$ if and only if $i = j$. Hence by Lemma 3.4.4, we know that C_B is a linearly independent set in F^E and hence in V as well.

For a hyperplane H in $\mathcal{M}$, let $h = r - 1 - |H \cap B|$. We will prove that $c_H \in \operatorname{span}(C_B)$ by induction on h. If $h = 0$, then $|H \cap B| = r - 1$ and hence by Lemma 3.4.3 we know that $c_H \in C_B$. So if $h = 0$, then $c_H \in \operatorname{span}(C_B)$.

Let H be a hyperplane in $\mathcal{M}$ and suppose that $h = r - 1 - |H \cap B| \geq 1$. Then $H \cap B$ is independent, so by reindexing B and extending $H \cap B$ to B_H, a maximal independent subset of H, we have

$$H \cap B \subseteq \{b_1, \dots, b_l, a_{l+1}, a_{l+2}, \dots, a_{r-1}\} = B_H \subseteq H,$$

where $|H \cap B| = l = r - h - 1$. Then

$$L = \operatorname{cl}(\{b_1, \dots, b_l, a_{l+1}, a_{l+2}, \dots, a_{r-2}\})$$

is a rank $r-2$ flat contained in H. Let $b' \in B \setminus L$ and form hyperplane $H' = \operatorname{cl}(L \cup b')$. Similarly, let $b'' \in B \setminus H'$ and form the hyperplane $H'' = \operatorname{cl}(L \cup b'')$. These exist since $\rho(L) = r - 2$ and $\rho(H') = r - 1$, so neither can contain a basis so in particular cannot contain B.

H' and H'' are distinct from H since $|H' \cap B|, |H'' \cap B| \geq l+1$, and furthermore they are hyperplanes such that $r - 1 - |H' \cap B|, r - 1 - |H'' \cap B| \leq h - 1$. Therefore by the inductive hypothesis $c_{H'}, c_{H''} \in \operatorname{span}(C_B)$. Since $H \cap H' \cap H'' \supseteq L$, we know that $\rho(H \cap H' \cap H'') \geq \rho(L) = r - 2$ and therefore by (H2) we have that $c_H \in \operatorname{span}(c_{H'}, c_{H''}) \subseteq \operatorname{span}(C_B)$. Hence $c_H \in \operatorname{span}(C_B)$ and therefore by induction C_B spans $V = \operatorname{span}(\{c_H \mid H$ is a hyperplane in $\mathcal{M}\})$. Since we also proved that C_B is linearly independent, we therefore have proved that C_B is a basis for V. $\square$

Recall that for a vector space V over the field F, the *dual* space V^* is a vector space over F and consists of all linear functionals $\phi : V \to F$. Remember that if U is a subspace of V, then

$$W = \{\phi \in V^* \mid U \subseteq \ker(\phi)\}$$

is a subspace of V^* and $\dim(W) = \dim(V) - \dim(U)$.

Proof of Theorem 3.4.2. Suppose that (H1) and (H2) are satisfied by the set $\{c_H : E \to F \mid H$ is a hyperplane in $\mathcal{M}\}$, and form the vector space V over F where $V = \operatorname{span}(\{c_H \mid H$ is a hyperplane in $\mathcal{M}\}) \subseteq F^E$. Now for each $e \in E$, define the linear functional

$$L_e : V \to F \text{ where } L_e : f \mapsto f(e)$$

We claim that the mapping $\sigma : E \to V^*$ by $\sigma : e \mapsto L_e$ is a representation. To prove this it suffices to show that σ preserves independent and dependent sets, and to do this it suffices to shot that bases are mapped to independent sets and minimal dependent sets are mapped to dependent sets.

If $B = \{b_1, \ldots, b_r\} \subseteq E$ is a basis for $\mathcal{M}$, then by Lemma 3.4.5 we know that $C_B = \{c_{H_1}, \ldots, c_{H_r}\}$ is a basis for V and $c_{H_i}(b_j) \neq 0$ if and only if $i = j$. Hence $L_{b_j}(c_{H_i}) \neq 0$ if and only if $i = j$. Therefore by Lemma 3.4.4, we know that $L_{b_1}, \ldots, L_{b_r}$ are linearly independent. Therefore we have proved that σ maps bases of $\mathcal{M}$ to independent sets in V^*.

If $X = \{b_0, b_1, \ldots, b_k\}$ is a minimal dependent set where $k \leq r = \rho(\mathcal{M})$, then $X \setminus b_0$ is independent and may be extended to a basis $B = \{b_1, \ldots, b_r\}$ of $\mathcal{M}$. By Lemma 3.4.5 we have that a basis $C_B = \{c_{H_1}, \ldots, c_{H_r}\}$ for V with $L_{b_j}(c_{H_i}) \neq 0$ if and only if $i = j$. However $b_0 \in \text{cl}(\{b_1, \ldots, b_k\} \subseteq \text{cl}(B \setminus b_i) = H_i$ for $i > k$, so $L_{b_0}(c_{H_i}) = c_{H_i}(b_0) = 0$ for $i > k$ since $b_0 \in H_i$. Since $C_B = \{c_{H_1}, \ldots, c_{H_r}\}$ is a basis for V and $L_{b_0} : V \to F$ is a linear functional, we know that L_{b_0} is determined by is is values on C_B. In fact for $\alpha_j = L_{b_0}(c_{H_j})/L_{b_j}(c_{H_j}) = c_{H_j}(b_0)/c_{H_j}(b_j)$ we have

$$L_{b_0} = \sum_{j=1}^{r} \alpha_j L_{b_j} = \sum_{j=1}^{k} \alpha_j L_{b_j}$$

since $\alpha_j = c_{H_j}(b_0)/c_{H_j}(b_j) = 0$ for $j > k$. Therefore $\{L_{b_0}, \ldots, L_{b_k}\} = \sigma(X)$ is dependent. Hence $\sigma : E \to V^*$ by $\sigma(e) = L_e$ preserves independent and dependent sets, and therefore is a representation.

Now we want to prove that $\mathcal{M}$ having a set of functions that satisfy (H1) and (H2), is a necessary condition for $\mathcal{M}$ to be F-representable. Suppose that a matroid $\mathcal{M} = (E, \mathcal{I})$ has an F-representation $\varphi : E \to V$, where V is a vector space over F and without loss of generality we can assume that $\dim(V) = \rho(\mathcal{M})$. So for any hyperplane H in $\mathcal{M}$, $\text{span}(\varphi(H)) = U$ is a subspace of V of dimension $\rho(\mathcal{M}) - 1 = \dim(V) - 1$. Then there is a unique (up to a nonzero scalar multiple) linear functional $f_U : V \to F$ such that $\text{kernel}(f_U) = U$. Let $c_H = f_U \circ \varphi$. Observe that for $e \in E$,

$$c_H(e) = 0 \iff \varphi(e) \in U \iff e \in H,$$

so (H1) is satisfied. If H_1, H_2, and H_3 are hyperplanes of $\mathcal{M}$ such that there is a rank $r-2$ flat $L \subseteq H_1 \cap H_2 \cap H_3$, then the f_{U_i}, where $U_i = \text{span}(\varphi(H_i))$, are linear functionals whose kernels contain $\varphi(L)$. Since $\dim(\text{span}(\varphi(L))) = r - 2$, the subspace of linear functionals whose kernel contains $\varphi(L)$ has dimension 2. Therefore $c_{H_1}, c_{H_2}, c_{H_3}$ are linearly dependent and minimally as well since no two hyperplane functions are linearly dependent. Therefore (H2) holds for $c_H = f_{\text{span}(\varphi(H))} \circ \varphi$.

Therefore we have proved that conditions (H1) and (H2) are necessary and sufficient conditions for a matroid $\mathcal{M} = (E, \mathcal{I})$ to be F-representable. $\square$

It follows that if a matroid $\mathcal{M} = (E, \mathcal{I})$ is F-representable, then there exists the representation $\sigma : E \to V^*$ from the proof of Theorem 3.4.2. Let $\{H_1, \ldots, H_m\}$ be the hyperplanes in $\mathcal{M}$, and let $\{e_1, \ldots, e_n\}$ be the elements in E. Then we can think of σ as being a matrix in $F^{n\times m}$, where

$$\sigma_{ij} = L_{e_i}(c_{H_j}) = c_{H_j}(e_i).$$

So in fact the j^{th} column of σ is just c_{H_j}. Since the rows with zero in the j^{th} column correspond to the hyperplane H_j, it follows from Proposition 3.3.3 that the supports of the columns are precisely the cocircuits of $\mathcal{M}$. Where the *support* of $x \in F^n$ is $\text{support}(x) = \{j \mid x_j \neq 0\}$.

Corollary 3.4.6. *If a matroid $\mathcal{M}$ is F-representable, then there exists a matrix M, which is an F-representation in matrix form, such that the supports of the columns of M are exactly the cocircuits of M.*

We will use these results to prove the following theorem that relates matroid representability with the Kapranov rank of cocircuit matrices.

Theorem 3.4.7 (Theorem 7.3 in (5))**.** *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let M be is cocircuit matrix as in Definition 3.3.5. Then* $\text{k-rank}(M) = \rho(\mathcal{M})$ *if and only if $\mathcal{M}$ is $\mathbb{C}$-representable.*

In order to prove one direction of this theorem, we need a few preliminary lemmas, but first an example.

Example 3.4.8. Let k be the finite field $\mathbb{Z}/2\mathbb{Z}$. Then $p = x(x-1)$ is a polynomial in $k[x]$ and $p \neq 0$ as polynomials. However, for all $x \in k = \{0, 1\}$ we can see that $p(x) = 0$. The next lemma shows that this can only happen when k is a finite field, see (8) for a proof.

Lemma 3.4.9. *Let k be an infinite field and $p \in k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a polynomial, then $p = 0$ in $k[\mathbf{x}]$ if and only if $p(\mathbf{x}) = 0$ for all $\mathbf{x} \in k^n$.*

In a matroid $\mathcal{M} = (E, \mathcal{I})$, an element $e \in E$ is a *loop* if $e \notin I$ for all $I \in \mathcal{I}$. Observe that e being a loop is equivalent to e being in every flat, which is equivalent by Proposition 3.2.7 to e being in every hyperplane. By Proposition 3.3.3, e being in every hyperplane is equivalent to e not being in any cocircuit. Therefore in the cocircuit matrix M, a loop will correspond to a row of 1's.

Lemma 3.4.10. *Let M be a zero-one $n \times m$ matrix such that every column has at least one 0. Let $\tilde{M}$ be the matrix where a row of 1's is added on top of M. Then* $\text{k-rank}(M) = \text{k-rank}(\tilde{M})$*.*

Proof. We just need to prove that any lift of M gives rise to a lift for $\tilde{M}$ of the same rank. Suppose F is a lift for M, with columns $\mathbf{f}_j = (F_{1j}, \ldots, F_{nj}) \in (\tilde{K}^*)^n$ whose constant terms are $\mathbf{c}_j \in \mathbb{C}^n$. The condition that M has no column of 1's means that no $\mathbf{c}_j$ is all 0's. We want to find a vector $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{x} \cdot \mathbf{c}_j \neq 0$ for all $\mathbf{c}_j$. This would allow us to form the row in $(\tilde{K}^*)^m$:

$$\mathbf{r}_0 = (x_1 t)\mathbf{r}_1 + \cdots + (x_n t)\mathbf{r}_n$$

where $\mathbf{r}_i$ is the i^{th} row of F. Then the matrix with rows $\mathbf{r}_0, \ldots, \mathbf{r}_n$ would be a lift of $\tilde{M}$ with the same rank as M.

We can think of $\mathbf{x} \cdot \mathbf{c}_j$ as a nonzero linear polynomial $p_j \in \mathbb{C}[x_1, \ldots, x_n]$. If for every $\mathbf{x} \in \mathbb{C}^n$, there was some $p_j(\mathbf{x}) = 0$, then $p = \prod_{j=1}^m p_j$ is a polynomial in $\mathbb{C}[\mathbf{x}]$ such that $p(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{C}^n$. Since $\mathbb{C}$ is an infinite field, it follows by Lemma 3.4.9 that $p = 0$, so some $p_j = 0$, which is a contradiction. Hence there exists an $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{x} \cdot \mathbf{c}_j \neq 0$ for all $\mathbf{c}_j$. Therefore k-rank$(M) =$ k-rank$(\tilde{M})$. $\square$

Corollary 3.4.11. *If $\mathcal{M} = (E, \mathcal{I})$ is a matroid and $\tilde{\mathcal{M}} = (E \cup L, \mathcal{I})$ is a matroid with loops L added to $\mathcal{M}$, then $\rho(\mathcal{M}) = \rho(\tilde{\mathcal{M}})$ and for the respective cocircuit matrices we have* k-rank$(M) =$ k-rank$(\tilde{M})$.

Proof of Theorem 3.4.7. Let $\mathcal{M}$ be a matroid on $E = \{1, \ldots, n\}$ with cocircuits $\mathcal{C}^* = \{C_1, \ldots, C_m\}$. So the cocircuit matrix $M \in \{0,1\}^{n \times m}$ has $M_{ij} = 0$ if and only if $i \in C_j$.

Suppose that k-rank$(M) = \rho(\mathcal{M}) = r$. Then there is a lift $F \in (\tilde{K}^*)^{n \times m}$ such that rank$(F) = r$ and deg$(F) = M$. Let $\mathbf{v}_i = (v_{i1}, \ldots, v_{im}) \in \mathbb{C}^m$ be the vector of the constant terms from the entries in the i^{th} row of F and observe that

$$v_{ij} \neq 0 \iff M_{ij} = 0 \iff i \in C_j. \tag{3.4.1}$$

Let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{C}^m$, which we can think of as a matrix in $\mathbb{C}^{n \times m}$ where the i^{th} row is $\mathbf{v}_i$. It is the case that rank$(V) \leq$ rank$(F) = r$, since if a subdeterminant of F is zero then the corresponding subdeterminant in V is also zero. We claim that the mapping $E \to$ span(V) by $i \mapsto \mathbf{v}_i$ is a $\mathbb{C}$-representation of $\mathcal{M}$. It suffices to prove that $B = \{i_1, \ldots, i_r\}$ is a basis of $\mathcal{M}$ if and only if $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r}\}$ is a basis of V. Without loss of generality let $B = \{1, \ldots, r\}$.

If B is a basis, then we know $E \setminus B$ is a cobasis and therefore by Corollary 3.1.10 we have cocircuit $C(i, E \setminus B)$ for each $i \in B$. Again without loss of generality, let $C_i = C(i, E \setminus B)$ for $i \leq r$. Hence the upper left $r \times r$ minor of M is all 1's except for all 0's on the main diagonal. This means that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent by Lemma 3.4.4 since for $j \leq r$ we have $v_{ij} \neq 0$ if and only if $j = i$ by (3.4.1). Therefore since rank$(V) \leq r$, we have that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a basis and rank$(V) = r$.

If B is not a basis, then $\rho(B) < r$, so B can be extended to a hyperplane X where without loss of generality we have that $X = \{1, \ldots, r, r+1, \ldots, c\}$. Hence $E \setminus X$ is a cocircuit and without loss of generality let it be C_1, so $B \cap C_1 = \emptyset$. Therefore $M_{i1} = 1$ for $i \leq r$ and hence by (3.4.1) we have that $v_{i1} = 0$ for $i \leq r$. However since C_1 is not empty, some $v_{1j} \neq 0$. Therefore $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ does not span V and hence is not a basis.

Suppose that $\mathcal{M}$ is representable over $\mathbb{C}$. By Corollary 3.4.11, without loss of generality we can assume that $\mathcal{M}$ has no loops and therefore M has no row of 1's. By Corollary 3.4.6, we can have $\mathcal{M}$ be represented by a matrix $A \in \mathbb{C}^{n \times m}$ where the element $i \in E$ is mapped to row i in A, and the supports of the columns in A are exactly the cocircuits in $\mathcal{M}$. Observe that rank$(A) = \rho(\mathcal{M}) = r$. Without loss of generality let $\{1, \ldots, r\}$ be a basis for $\mathcal{M}$. If A' is the submatrix of the first r

rows of A, then there exists a matrix $C \in \mathbb{C}^{n-r\times r}$ such that

$$A = \begin{pmatrix} I_r \\ C \end{pmatrix} \cdot A'$$

where I_r is the $r \times r$ identity matrix. Because $\mathcal{M}$ does not have any loops, A and therefore C does not have a row of 0's (since the supports of the columns of A are the cocircuits). There exists a matrix $B' \in \mathbb{C}^{r\times m}$ such that every entry in

$$\begin{pmatrix} I_r \\ C \end{pmatrix} \cdot B'$$

is nonzero, because by Lemma 3.4.9 we can think of each entry in resulting matrix as being a polynomial over $\mathbb{C}$ in the entries in B'. Now define

$$F(t) = \begin{pmatrix} I_r \\ C \end{pmatrix} \cdot (A' + tB') \in (\tilde{K}^*)^{n\times m}.$$

Since F is the product of two matrices each with rank at most r, we know that $\text{rank}(F) \leq r$. However $\text{rank}(F(0)) \leq \text{rank}(F)$, $F(0) = A$, and $\text{rank}(A) = r$, so therefore $\text{rank}(F) = r$. We also have that $\deg(F) = M$, the cocircuit matrix, since $F(0) = A$ and the supports of the columns of A are the cocircuits of $\mathcal{M}$. Therefore $\text{k-rank}(M) = \rho(\mathcal{M})$. $\square$

Combining Corollary 3.3.6 and Theorem 3.4.7 we get the following theorem from (5) that gets at the heart of the difference between the Kapranov and tropical rank.

Theorem 3.4.12. *Let $\mathcal{M}$ be a matroid with cocircuit matrix $M \in \{0,1\}^{n\times m}$, then $\mathcal{M}$ is $\mathbb{C}$-representable if and only if*

$$\text{k-rank}(M) = \text{t-rank}(M).$$

3.5 The Fano Plane Matroid

Using Theorem 3.4.12, by presenting a matroid $\mathcal{M}$ that is not $\mathbb{C}$-representable, we can now finally prove that the Kapranov rank is not equivalent to the tropical rank. Such an example is provided by the Fano plane, the combinatorialists' coat of arms.

Definition 3.5.1. The *Fano plane* matroid $\mathcal{F} = (E, \mathcal{I})$ consists of the seven nonzero vectors in the vector space $(\mathbb{Z}/2\mathbb{Z})^3$ where independence is induced from the vector space.

Therefore in $\mathcal{F}$, the ground set is $E = \{\mathbf{x} = (x_1, x_2, x_3) \mid x_j \in \{0,1\} \text{ and } \mathbf{x} \neq 0\}$ and the seven hyperplanes are:

$$\begin{array}{llll} H_1 = V(x_1) & H_3 = V(x_3) & H_5 = V(x_2 + x_3) & H_7 = V(x_1 + x_2 + x_3) \\ H_2 = V(x_2) & H_4 = V(x_1 + x_2) & H_6 = V(x_3 + x_1) & \end{array}$$

We number the elements in E as

$$1 = (1,0,0) \qquad 3 = (0,0,1) \qquad 5 = (1,1,1) \qquad 7 = (0,1,1)$$
$$2 = (0,1,0) \qquad 4 = (1,0,1) \qquad 6 = (1,1,0)$$

So the hyperplanes in $\mathcal{F}$ are

$$H_1 = \{2,3,7\} \qquad H_3 = \{1,2,6\} \qquad H_5 = \{1,5,7\} \qquad H_7 = \{4,6,7\}$$
$$H_2 = \{1,3,4\} \qquad H_4 = \{3,5,6\} \qquad H_6 = \{2,4,5\}$$

Under this notation, we can represent the Fano plane matroid $\mathcal{F}$ graphically, where the hyperplanes are the lines passing through three points. The circle counts as a line and it is the hyperplane H_7. This picture also represents the bases of $\mathcal{F}$ for $B \subseteq E$ will be a basis if and only if B consists of three noncollinear points.

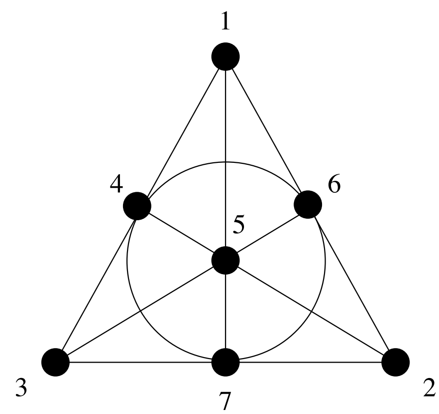

Figure 3.2: The Fano plane matroid and its hyperplanes.

It turns out that $\mathcal{F}$ is F-representable only if F has characteristic 2, which means in particular that $\mathcal{F}$ is not $\mathbb{C}$-representable. This was first proven in (25) and our proof will follow the one that appears in (18).

Proposition 3.5.2. *If the Fano plane $\mathcal{F} = (E, \mathcal{I})$ is representable then F has characteristic* 2.

Proof. Let $\varphi : E \to V$ be a representation where V is a vector space over F and $\varphi : (a,b,c) \mapsto \mathbf{v}_{abc}$. Since this is a representation and $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathcal{F}$, we know that $\{\mathbf{v}_{100}, \mathbf{v}_{010}, \mathbf{v}_{001}\}$ is a basis for $\operatorname{span}(\varphi(E))$. Since $\{\mathbf{v}_{111}, \mathbf{v}_{100}, \mathbf{v}_{010}, \mathbf{v}_{001}\}$ is a minimal dependent set, we know that

$$\mathbf{v}_{111} = a_1\mathbf{v}_{100} + a_2\mathbf{v}_{010} + a_3\mathbf{v}_{001} \quad \text{where } a_j \in F^*. \tag{3.5.1}$$

Similarly we know that $\{\mathbf{v}_{011}, \mathbf{v}_{010}, \mathbf{v}_{001}\}$ and $\{\mathbf{v}_{011}, \mathbf{v}_{111}, \mathbf{v}_{100}\}$ are also minimal dependent sets, so there exists $b_1, c_1, d_1, e_1 \in F^*$ such that

$$b_1\mathbf{v}_{010} + c_1\mathbf{v}_{001} = \mathbf{v}_{011} = d_1\mathbf{v}_{111} + e_1\mathbf{v}_{100}. \tag{3.5.2}$$

Substituting (3.5.1) into (3.5.2), gives

$$\mathbf{v}_{011} = d_1(a_1\mathbf{v}_{100} + a_2\mathbf{v}_{010} + a_3\mathbf{v}_{001}) + e_1\mathbf{v}_{100}. \tag{3.5.3}$$

Since $\mathbf{v}_{011}$ has an unique representation as the linear combination of $\mathbf{v}_{100}$, $\mathbf{v}_{010}$, $\mathbf{v}_{001}$, (3.5.2) and (3.5.3) tell us that $d_1a_1 + e_1 = 0$. Therefore we can rewrite (3.5.3) as

$$\mathbf{v}_{011} = d_1a_2\mathbf{v}_{010} + d_1a_3\mathbf{v}_{001}. \tag{3.5.4}$$

The same line of argument shows that since $\{\mathbf{v}_{101}, \mathbf{v}_{100}, \mathbf{v}_{001}\}$ and $\{\mathbf{v}_{101}, \mathbf{v}_{111}, \mathbf{v}_{010}\}$ are minimal dependent sets, we have $d_2 \in F^*$ such that

$$\mathbf{v}_{101} = d_2a_1\mathbf{v}_{100} + d_2a_3\mathbf{v}_{001}. \tag{3.5.5}$$

Likewise since $\{\mathbf{v}_{110}, \mathbf{v}_{100}, \mathbf{v}_{010}\}$ and $\{\mathbf{v}_{110}, \mathbf{v}_{111}, \mathbf{v}_{001}\}$ are minimal dependent sets, we have $d_3 \in F^*$ such that

$$\mathbf{v}_{110} = d_3a_1\mathbf{v}_{100} + d_3a_2\mathbf{v}_{001}. \tag{3.5.6}$$

Finally we have that since $\{\mathbf{v}_{011}, \mathbf{v}_{101}, \mathbf{v}_{110}\}$ is a minimal dependent set, we have $f_j \in F^*$ such that

$$f_1\mathbf{v}_{011} + f_2\mathbf{v}_{101} + f_3\mathbf{v}_{110} = 0. \tag{3.5.7}$$

Substituting (3.5.4), (3.5.5), and (3.5.6) into (3.5.7) gives

$$\begin{aligned} 0 &= f_1(d_1a_2\mathbf{v}_{010} + d_1a_3\mathbf{v}_{001}) + f_2(d_2a_1\mathbf{v}_{100} + d_2a_3\mathbf{v}_{001}) + f_3(d_3a_1\mathbf{v}_{100} + d_3a_2\mathbf{v}_{001}) \\ &= (f_2d_2a_1 + f_3d_3a_1)\mathbf{v}_{100} + (f_3d_3a_2 + f_1d_1a_2)\mathbf{v}_{010} + (f_1d_1a_3 + f_2d_2a_3)\mathbf{v}_{001}. \end{aligned}$$

However since $\{\mathbf{v}_{100}, \mathbf{v}_{010}, \mathbf{v}_{001}\}$ is a basis, this gives

$$f_2d_2a_1 = -f_3d_3a_1 \tag{3.5.8}$$
$$f_3d_3a_2 = -f_1d_1a_2 \tag{3.5.9}$$
$$f_1d_1a_3 = -f_2d_2a_3. \tag{3.5.10}$$

Forming $a = (f_2d_2a_1)(f_3d_3a_2)(f_1d_1a_3) = \prod_j a_jd_jf_j$, it follows from (3.5.8), (3.5.9), (3.5.10), and $a_j, d_j, f_j \neq 0$, that

$$a = -a \neq 0.$$

Therefore the characteristic of F is 2. □

It follows from Proposition 3.5.2 that $\mathcal{F}$ is not $\mathbb{C}$-representable, and therefore by Theorem 3.4.12 we get that for the cocircuit matrix M of the Fano plane $\mathcal{F}$ where

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

that t-rank(M) < k-rank(M). In fact since the rank of the Fano plane matroid is 3, we know by Corollary 3.3.6 the t-rank$(M) = 3$. Using Theorem 2.1.8, one can compute that k-rank$(M) = 4$. Since we can join the Fano matroid with other matroids and have the resulting matroid remain not $\mathbb{C}$-representable, we have the following:

Theorem 3.5.3. *For all $n \geq 7$ there exists a matrix $M \in \mathbb{R}^{n\times n}$ such that*

$$\text{t-rank}(M) < \text{k-rank}(M).$$

Chapter 4

NP-completeness and the Tropical Rank

In this chapter will present a proof, based on the one that appears in (11), that computing the tropical rank of a zero-one matrix is NP-complete, answering question (Q1) at the end of (5). This proof will appear in Section 4.3, and the first two sections of this chapter will serve as a brief primer on the the theory of NP-completeness based on the standard book (9).

4.1 The Basics of P and NP

The study of NP-completeness deals with decision problems, which are problems with a definitive yes/no answer.

Definition 4.1.1. A *decision problem* Π is an order pair $\Pi = (D_\Pi, Y_\Pi)$, where $Y_\Pi \subset D_\Pi$. The elements of $D = D_\Pi$ are called *instances* and the elements in $Y = Y_\Pi$ are the *yes-instances*.

To describe decision problems we use the INSTANCE/QUESTION format. For example consider the problem:

VERTEX COVER
INSTANCE: A graph $G = (V, E)$ and $K \in \mathbb{N}$.
QUESTION: Is there a set $S \subseteq V$ with $|S| \leq K$ such that for every $\{u, v\} \in E$ either $u \in S$ or $v \in S$?

Then for the vertex cover decision problem VC, $D_{VC} = \mathcal{G} \times \mathbb{N}$ where $\mathcal{G}$ is the collection of (finite) graphs and $(G, K) \in Y_{VC}$ if and only if the answer to the question in the case of (G, K) is 'yes.' Now suppose that we had a program that could compute the answer for any instance of VC and we wanted to know the size of the smallest vertex cover for a given graph $G = (V, E)$. Then we could compute (G, K) for each $K \in \mathbb{N}$ with $K \leq |E|$ and pick the smallest K such that $(G, K) \in Y_{VC}$, which exists since $(G, |E|) \in Y_{VC}$. Using this idea, many optimization problems can be reduced to decision problems.

Definition 4.1.2. For a finite set Σ, denote by Σ^* the set of all finite strings of symbols from Σ, including the empty string ϵ. A subset $L \subseteq \Sigma^*$ is called a *language over the alphabet* Σ.

Example 4.1.3. If $\Sigma = \{0, 1\}$, then ϵ, 0, 1, 00, 01, 10, 11, and 000 are examples of elements from $\{0, 1\}^*$. Likewise $\{\epsilon, 1, 001, 100011\}$, the set of all finite strings with exactly three 1's, and $\{0, 1\}^*$ are examples of languages over $\{0, 1\}$.

We relate decision problems with languages with the notion of an encoding scheme. Where an *encoding scheme for* Π is an injective function $e : D_\Pi \to \Sigma^*$ where Σ is some alphabet. Such an encoding of a decision problem, results in a partition of Σ^*:

$$\Sigma^* = e(Y_\Pi) \uplus e(D_\Pi \setminus Y_\Pi) \uplus (\Sigma^* \setminus e(D_\Pi)).$$

It is the first set in this partition that we are interested in. It is the language over Σ we will associate with Π and e where:

$$L[\Pi, e] = e(Y_\Pi).$$

The idea going forward will be that if a result holds for the language $L[\Pi, e]$, then it will hold for the decision problem Π under encoding e.

Now given a problem Π what kind of encoding schemes should we allow so that a notion of the computational complexity of Π is preserved? This turns out to be a difficult question to answer. However, we know that such encoding schemes should be 'concise' so as not to make an easy problem computationally hard by adding irrelevant data, and it should be 'decodable' meaning that it is easy to compute the inverse of e. Furthermore, if e and e' are allowable encoding schemes then we would want a result about the computability of $L[\Pi, e]$ to hold if and only if the same result about $L[\Pi, e']$ holds. The utility behind this approach is that it allows us to talk about results involving a decision problem independent of any particular encoding scheme. Let's informally say that such an encoding is 'reasonable' and we refer the reader to (9) for a longer discussion about what 'reasonable' encoding schemes might be.

For convenience let's suppose that every decision problem has an *input-length* function Length : $D_\Pi \to \mathbb{N}$ which is polynomially related to reasonable encoding schemes. This mean that for any reasonable encoding scheme e there exists polynomials $p, q \in \mathbb{Z}[x]$ such that for all instances $I \in D_\Pi$:

$$\text{Length}(I) \leq p(|e(I)|) \text{ and } |e(I)| \leq q(\text{Length}(I)),$$

where $|e(I)|$ is the length of the string $e(I)$ in Σ^*. For example in the VERTEX COVER problem, we could take

$$\text{Length}(G, K) = |V| \text{ where } G = (V, E).$$

A *deterministic Turing machine* (DTM) consists of a two-way infinite sequence of *tape squares* labeled by the integers, a *read-write head* to write symbols on the tape squares, and a *finite state control* to shift the tape following a finite set of rules.

Definition 4.1.4. A *program* for a DTM consists of the following:

1. A finite set Γ of *tape symbols* with a subset $\Sigma \subseteq \Gamma$ of *input symbols* and a *blank symbol* $b \in \Gamma \setminus \Sigma$;
2. A finite set Q of *states* with a *start-state* q_0, and *halt-states* q_Y and q_N; and
3. A *transition function* $\delta : (Q \setminus \{q_N, q_Y\}) \times \Gamma \to Q \times \Gamma \times \{-1, 1\}$.

The program then runs as follows. The *input* is any string $x \in \Sigma^*$, and the string x is placed in the tape squares 1 through $|x|$, one symbol per square, and every other tape square contains the blank symbol b. The program then starts with the read-write head at square 1 in state q_0 and proceeds in accordance to the transition function σ. If at any time the state q is q_N or q_Y then the program halts and returns the answer 'no' or 'yes' respectively. Otherwise if $q \in Q \setminus \{q_N, q_Y\}$, then the read-write head reads the symbol $s \in \Gamma$ at the head's present location z and computes $\delta(q, s) = (q', s', \Delta)$. The read-write head erases s and writes s' at position z and then moves to position $z + \Delta$ and switches to state q' and the process now repeats.

We say that a DTM program M with input alphabet Σ *accepts* $x \in \Sigma^*$ if M halts in state q_Y when given x as an input. The language L_M *recognized* by the program M is given by

$$L_M = \{x \in \Sigma^* \mid M \text{ accepts } x\}.$$

A DTM program M is an *algorithm* if it halts for all inputs $x \in \Sigma^*$. Hence an algorithm is a function from $\Sigma^* \to \{q_N, q_Y\}$.

Definition 4.1.5. We say that a DTM algorithm M *solves* the decision problem Π under encoding scheme e if and only if and $L_M = L[\Pi, e]$.

For $x \in \Sigma^*$, the *time* a DTM algorithm M takes to compute $M(x)$ is just the number of steps occurring in the computation until the program halts.

Definition 4.1.6. Given a DTM algorithm M, its *time complexity function* is given by $T_M : \mathbb{N} \to \mathbb{N}$ where

$$T_M(n) = \max\{t \mid x \in \Sigma^*, |x| = n, \text{ and computing } M(x) \text{ takes time } t\}.$$

We can now define a polynomial time algorithm and the class P.

Definition 4.1.7. A DTM algorithm M is a *polynomial time algorithm* if there exists a polynomial $p \in \mathbb{Z}[x]$ such that $T_M(n) \leq p(n)$ for all $n \in \mathbb{N}$. P is then the collection of languages defined as

$$\mathrm{P} = \{L \mid \text{there exists polynomial time algorithm } M \text{ such that } L = L_M\}.$$

We say that a decision problem Π *belongs* to P if there exists an encoding scheme e such that $L[\Pi, e] \in \mathrm{P}$, and restricting ourselves to reasonable encoding schemes we will just say that Π belongs to P.

Returning to the VERTEX COVERING problem, there is no known polynomial time algorithm solution, however given an arbitrary instance (G, K) and arbitrary subset of vertices C it is easy to check if $|C| \leq K$ and if C is a vertex covering. It is problems like this, where it easy to check if a proposed object is a solution to an existence question, that we will want to be NP problems.

The counterpart of a DTM here will be a *nondeterministic Turing machine* (NDTM), which is just like a DTM with the addition of a *guessing module*. Programs for NDTM are just the same as for DTM, except in running them there is first a guessing stage. So a tape with an input $x \in \Sigma^*$ is fed into the NDTM, and then the guessing module guesses a string $g \in \Gamma^*$, having the read-write head write the guess on tape squares 0 to $-|g| + 1$. The guessing stage then ends and the program starts to run at square $-|g| + 1$.

For a given guess $g \in \Gamma^*$, if computation halts at q_Y then the guess g is said to be an *accepting computation*, and if the computation halts at q_N or never halts, then the guess g is said to be a *nonaccepting computation*. We say that a NDTM program M *accepts* an input $x \in \Sigma^*$ if there is at least one guess that is an accepting computation. The language *recognized* by M is

$$L_M = \{x \in \Sigma^* \mid M \text{ accepts } x\}.$$

The *time* required by an NDTM program M to accept a string $x \in L_M$ is the minimum number of steps required to get the program to halt at q_Y (counting both the guessing and computing stage) over all accepting computations. Then the *time complexity function* $T_M : \mathbb{N} \to \mathbb{N}$ for M is

$$T_M(n) = \max\{t \mid x \in L_M, |x| = n, \text{ and the time for } M \text{ to accept } x \text{ is } t\},$$

and by convention if the set is empty, then $T_M(n) = 0$. As expected, a NDTM program M is a *polynomial time NDTM program* if there exists a polynomial $p \in \mathbb{Z}[x]$ such that $T_M(n) \leq p(n)$ for all $n \in \mathbb{N}$. So the class NP of languages is defined as follows:

$$\text{NP} = \{L \mid \text{there exists polynomial time NDTM program } M \text{ such that } L = L_M\}.$$

Just as in the case of P, we say that a decision problem Π *belongs* to NP if there exists an encoding scheme e such that $L[\Pi, e] \in \text{NP}$. However we can be informal and say that Π belongs to NP if it is clear that it leads to a language in NP under some reasonable encoding.

Observe that $\text{P} \subseteq \text{NP}$, since given a polynomial time algorithm M such that $L = L_M$, then M with the addition of a guessing module is a NDTM program and for each input in $x \in \Sigma^*$ the program halts for the empty guess. Further discussions about the relationship between P and NP are far beyond the scope of this thesis (it one of the Clay Foundation's Millennium Problems to determine if $\text{P} = \text{NP}$), however we will end by quoting the following result.

Theorem 4.1.8 (Theorem 2.1 on p.33 in (9))**.** *If a decision problem Π is in NP, then there exists a polynomial $p \in \mathbb{Z}[x]$ such that Π can be solved by a DTM algorithm having time complexity $O(2^{p(n)})$.*

4.2 NP-completeness and Cook's Theorem

Since the question of whether P $=$ NP is currently intractable, there is no known way of showing that a problem Π belongs to NP $\setminus$ P. So the question of NP-completeness deals with the weaker statement: if P $\neq$ NP, then $\Pi \in$ NP $\setminus$ P. The way of addressing this is the notion of a polynomial transformation.

Definition 4.2.1. A *polynomial transformation* from language $L_1 \subseteq \Sigma_1^*$ to language $L_2 \subseteq \Sigma_2^*$ is a function $T : \Sigma_1^* \to \Sigma_2^*$ such that

1. There is a polynomial time DTM program that computes T; and
2. $T^{-1}(L_2) = L_1$ or equivalently for $x \in \Sigma_1^*$, $x \in L_1$ if and only if $T(x) \in L_2$.

If there exists a polynomial transformation form L_1 to L_2, then we say that L_1 *transforms* to L_2 and denote this by $L_1 \propto L_2$.

The importance of polynomial transformations in the theory of NP-completeness lies in the following two propositions:

Proposition 4.2.2. *If $L_1 \propto L_2$ and $L_2 \in$ P, then $L_1 \in$ P.*

Proof. Let Σ_1 and Σ_2 be the alphabets of L_1 and L_2 respectively and let $T : \Sigma_1^* \to \Sigma_2^*$ be the polynomial transformation from L_1 to L_2. Now let M_T and M_2 be the polynomial time DTM algorithms that compute T and recognize L_2 respectively.

Then we claim that the composition $M_2 \circ M_T = M_1$, is a polynomial time DTM algorithm that recognizes L_1. M_1 is the program that takes $x \in \Sigma_1^*$ and first computes $T(x)$ and then computes $M_2(T(x))$. Observe that, $x \in L_1$ if and only if $T(x) \in L_2$ if and only if $M_2(T(x))$ halts at q_Y, so it is a DTM algorithm such that $L_{M_1} = L_1$. Furthermore, if the time complexity function for T and M_2 are p_T and p_2 respectively, then $T_{M_1} \leq p_T + p_2(p_T)$ which is a polynomial. Therefore $M_1 = M_2 \circ M_T$ is a polynomial time DTM algorithm such that $L_{M_1} = L_1$ and hence $L_1 \in$ P. $\square$

In practice we will use the contrapositive, namely that if $L_1 \propto L_2$ and $L_1 \notin$ P, then $L_2 \notin$ P. At the decision problem level, we say that $\Pi_1 \propto \Pi_2$ if there exists encoding schemes e_1 and e_2 such that $L[\Pi_1, e_1] \propto L[\Pi_2, e_2]$. Restricting ourselves to the use of reasonable encoding schemes, we can think of a polynomial transformation from Π_1 to Π_2 as a function $T : D_{\Pi_1} \to D_{\Pi_2}$ such that:

1. T is computable by a polynomial time algorithm; and
2. $T^{-1}(Y_{\Pi_2}) = Y_{\Pi_1}$.

The other important property of polynomial transformations is that the $\propto$ relation is transitive among languages.

Proposition 4.2.3. *If L_1, L_2, and L_3 are languages, $L_1 \propto L_2$, and $L_2 \propto L_3$, then $L_1 \propto L_3$.*

Proof. Let Σ_1, Σ_2, and Σ_3 be the alphabets of L_1, L_2, and L_3 respectively. Let $T_1 : \Sigma_1^* \to \Sigma_2^*$ be a polynomial transformation from L_1 to L_2 and let $T_2 : \Sigma_2^* \to \Sigma_3^*$ be the polynomial transformation from L_2 to L_3. Finally let p_1 and p_2 be polynomials which bound the time complexity functions for T_1 and T_2, respectively.

Then $T = T_2 \circ T_1 : \Sigma_1^* \to \Sigma_3^*$ is a transformation such that

$$T^{-1}(L_3) = T_1^{-1}(T_2^{-1}(L_3)) = T_1^{-1}(L_2) = L_1. \tag{4.2.1}$$

Furthermore the time complexity function for T is bounded by the polynomial $p_1 + p_2(p_1)$, since T first computes $T_1(x)$ and then $T_2(T_1(x))$. Hence by the above, which shows that T runs in polynomial time, and (4.2.1) we have that T is a polynomial transformation from L_1 to L_3. Therefore $L_1 \propto L_3$. □

We can define two languages L_1 and L_2 to be *polynomial equivalent* if $L_1 \propto L_2$ and $L_2 \propto L_1$, and this is an equivalence relation by Proposition 4.2.3. We can now define the class of NP-complete languages.

Definition 4.2.4. A language L is NP-*complete* if $L \in \mathrm{NP}$ and $L' \propto L$ for all $L' \in \mathrm{NP}$.

Corollary 4.2.5. *If L_1 is NP-complete, $L_2 \in \mathrm{NP}$, and $L_1 \propto L_2$, then L_2 is NP-complete.*

Proof. Let $L \in \mathrm{NP}$. Since L_1 is NP-complete, we know that $L \propto L_1$, and hence by Proposition 4.2.3 we have that $L \propto L_2$. Therefore L_2 is NP-complete. □

As expected we informally say that a problem Π is NP-complete if and only $\Pi \in \mathrm{NP}$ and $\Pi' \propto \Pi$ for all $\Pi' \in \mathrm{NP}$. For problems Π_1 and Π_2, we can think of $\Pi_1 \propto \Pi_2$ as saying that Π_2 is harder that Π_1. So by Proposition 4.2.2 we see that NP-complete problems are, in this sense, the hardest problems in NP. For if an NP-complete problem Π could be solved in polynomial time, then all NP problems could be as well. Therefore an NP-complete problem Π has the property mentioned at the beginning of this section, namely that if $\mathrm{P} \neq \mathrm{NP}$ then $\Pi \in \mathrm{NP} \setminus \mathrm{P}$.

Now while Corollary 4.2.5 is nice, for it gives us a method of proving that a problem is NP-complete from a known NP-complete problem, we need a known NP-complete problem first. We are saved by the celebrated theorem of Cook (3) which supplies us with an NP-complete problem. The problem is known as SATISFIABILITY (SAT) and is as follows:

SATISFIABILITY
INSTANCE: A set U of variables and a collection C of clauses over U.
QUESTION: Is there a truth value assignment $\varphi : U \to \{T, F\}$ such that every clause c in C is satisfied by φ?

Example 4.2.6. Let $U = \{u_1, u_2, u_3\}$ and $C = \{\{u_1, \neg u_2\}, \{u_2, u_3\}, \{\neg u_1, u_3\}\}$. We interpret the clause $\{u_1, \neg u_2\}$ as saying 'u_1 or not-u_2.' Then C is satisfied by the function $\varphi : \{u_1, u_2, u_3\} \to \{T, F\}$, where $\varphi(u_1) = T, \varphi(u_2) = T$, and $\varphi(u_3) = T$.

Theorem 4.2.7. *SATISFIABILITY is NP-complete.*

In (9) numerous proofs of NP-completeness are given, using Corollary 4.2.5 by giving a polynomial transformation from a known NP-complete problem to another NP problem. In particular they prove in Theorem 3.3 on page 54 the following:

Theorem 4.2.8. *VERTEX COVER is NP-complete.*

4.3 Computing the Tropical Rank is Hard

Let $A_1, \ldots, A_n \subseteq A$, where A is a finite set. Recall that

$$\text{poset}(\{A_1, \ldots, A_n\}) = \left\{ \bigcup_{j \in S} A_j \mid S \subseteq \{1, \ldots, n\} \right\}.$$

For $C_1, \ldots, C_m \in \text{poset}(\{A_1, \ldots, A_n\})$, we say that

$$\mathcal{C} = C_1, \ldots, C_m \text{ is a } \textit{chain} \text{ if } \emptyset \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_{m-1} \subsetneq C_m$$

and we say that m is the *length* of the chain $\mathcal{C}$. A chain $\mathcal{C}$ is said to have *maximal length* if no other chain in the poset has a greater length, and the *length* of a poset is the length of a maximal length chain.

Proposition 4.3.1. *Let $A_1, \ldots, A_n \subseteq A$ and $W = \text{poset}(\{A_1, \ldots, A_n\})$. For a maximal length chain $\mathcal{C} = C_1, \ldots, C_m$ in W it is the case that for each $j \leq m$ there is a j_i such that $C_{j-1} \cup A_{j_i} = C_j$.*

Proof. Without loss of generality let $C_{j-1} = A_1 \cup \cdots \cup A_x$ and $C_j = A_{j_1} \cup \cdots \cup A_{j_y}$. For each j_i, we have that $C_{j-1} \subseteq C_{j-1} \cup A_{j_i} \subseteq C_j$. Since $\mathcal{C}$ is a maximal chain, it follows that either $C_{j-1} \cup A_{j_i} = C_j$, in which case we are done, or $C_{j-1} = C_{j-1} \cup A_{j_i}$.

If $C_{j-1} = C_{j-1} \cup A_{j_i}$ for all j_i, then $A_{j_i} \subseteq C_{j-1}$ for all j_i. This would imply that $C_j = A_{j_1} \cup \cdots \cup A_{j_y} \subseteq C_{j-1}$, which is false. Therefore for some j_i, we have that $C_{j-1} \cup A_{j_i} = C_j$, which proves the proposition. □

This proposition shows that by renumbering the A_i in $\text{poset}(\{A_1, \ldots, A_n\})$, we may assume that a maximal length chain $\mathcal{C} = C_1, \ldots, C_m$ is such that

$$C_j = \bigcup_{i=1}^{j} A_j.$$

This section will be devoted to proving that the problem of computing the tropical rank of a zero-one matrix is NP-complete following (11). Our method of attack will be to show that finding the length of $\text{poset}(\{A_1, \ldots, A_n\})$ is NP-complete. So we need to prove that MAXIMAL LENGTH (MAX) is NP-complete where:

MAXIMAL LENGTH
INSTANCE: $A_1, \ldots, A_n \subseteq S$ with $\bigcup_{j=1}^{n} A_j = S$, $K \in \mathbb{N}$.
QUESTION: Does there exist $L \geq K$ such that for some $A_{j_1}, \ldots, A_{j_L}$,

$$A_{j_1} \subsetneq A_{j_1} \cup A_{j_2} \subsetneq \cdots \subsetneq \bigcup_{i=1}^{L} A_{j_i} = S?$$

In order to prove that the problem of computing the tropical rank of a matrix in $\{0,1\}^{n\times m}$ is NP-complete, it suffices to prove that MAX is NP-complete. This is because Theorem 2.3.1 and Proposition 4.3.1 tell us an instance of MAX has length less then or equal to K if and only if the the corresponding $\{0,1\}$ matrix has tropical rank less than or equal to K.

Before diving into the proof that MAX is NP-complete, we will first prove that the problem MINIMAL LENGTH (MIN), which looks very similar to MAX, is NP-complete.

MINIMAL LENGTH
INSTANCE: $A_1,\dots,A_n \subseteq S$ with $\bigcup_{j=1}^n A_j = S$, $K \in \mathbb{N}$.
QUESTION: Does there exist $L \leq K$ such that for some $A_{j_1},\dots,A_{j_L}$,

$$A_{j_1} \subsetneq A_{j_1} \cup A_{j_2} \subsetneq \cdots \subsetneq \bigcup_{i=1}^{L} A_{j_i} = S?$$

Proposition 4.3.2. *MINIMAL LENGTH is NP-complete.*

Proof. First off, MIN is in NP since a nondeterministic algorithm need only guess a subset of $\{A_1,\dots,A_n\}$ and an ordering, then check to see that the subset is of the proper size and that the ordering gives a proper chain with final element being equal to S.

We will transform VERTEX COVER into MINIMAL LENGTH. Given a graph $G=(V,E)$, label the vertices and edges so $V=\{v_1,\dots,v_n\}$ and $E=\{e_1,\dots e_m\}$. For each vertex, v_j form the set $A_j=\{i \mid v_j \in e_i\}$. We claim that there is a vertex cover $S \subseteq V$ with $|S| \leq K$ if and only if there is a $L \leq K$ such that for some $A_{j_1},\dots,A_{j_L}$,

$$A_{j_1} \subsetneq A_{j_1} \cup A_{j_2} \subsetneq \cdots \subsetneq \bigcup_{i=1}^{L} A_{j_i} = E.$$

Suppose that $S=\{v_{j_1},\dots,v_{j_L}\}$ is a vertex cover, $|S|=L\leq K$, and without loss of generality suppose that no proper subset of S is a vertex cover. Then choose the sets $A_{j_1},\dots,A_{j_L}$. Since each edge has one of the v_{j_i} as an end point, we know that $\bigcup_{i=1}^{L} A_{j_i}=E$. Furthermore, if for some k we had

$$A_{j_1} \cup \cdots \cup A_{j_{k-1}} = A_{j_1} \cup \cdots \cup A_{j_{k-1}} \cup A_{j_k},$$

then $S \setminus v_{j_k}$ would be a vertex cover as well, which cannot be the case. Hence we have

$$A_{j_1} \subsetneq A_{j_1} \cup A_{j_2} \subsetneq \cdots \subsetneq \bigcup_{i=1}^{L} A_{j_i} = E,$$

as desired. Conversely suppose that we had a minimal chain given by $A_{j_1},\dots,A_{j_L}$ with $L \leq K$. Then in particular $\bigcup_{i=1}^{L} A_{j_i}=E$, so $S=\{v_{j_1},\dots,v_{j_L}\}$ is a vertex cover with $|S|\leq K$.

Hence VERTEX COVER can be mapped to MINIMAL LENGTH by a polynomial transformation. Therefore MINIMAL LENGTH is NP-complete. □

Now we will present a couple of propositions that will allow us to prove that MAXIMAL LENGTH is NP-complete. This series of propositions and proofs is based off of Theorem 13 in (11), where it is proven that computing the tropical rank of a zero-one matrix is NP-complete. We will be working with colored cycles, *c-cycles*, defined as follows:

Definition 4.3.3. Let C be a graph that is a circuit, so $C = (V, E)$ where

$$V = \{v_0, v_1, \dots, v_{2n}\} \text{ and } E = \{(v_0, v_1), (v_1, v_2), \dots, (v_{2n-1}, v_{2n}), (v_{2n}, v_0)\}.$$

C is a *c-cycle* if each edge is assigned a color from the distinct $\{c_0, \dots, c_n\}$, where c_0 is assigned to (v_{2n}, v_0) and for $j > 0$, c_j is assigned to (v_{2j-2}, v_{2j-1}) and (v_{2j-1}, v_{2j}). In this case c_0 will be called the cycle's *color* and (v_{2n}, v_0) the cycle's *final edge*.

Example 4.3.4. Figure 4.1 is an example of a c-cycle for $n = 2$. In this case c_1 is 'black', c_2 is 'dashed black', and c_0, the cycle's color, is 'gray'.

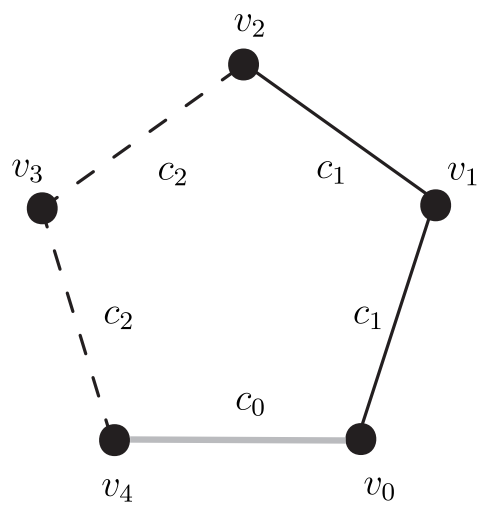

Figure 4.1: A c-cycle for $n = 2$.

For a general colored graph G, where $\mathcal{C} = e_1, \dots, e_m$ is an ordering of a subset of its edges, we say that $\mathcal{C}$ is a *chain* iff every e_j has a color or vertex that does not appear in any e_i for all $i < j$. The chain $\mathcal{C}$ is said to have length m.

Proposition 4.3.5. *Let C be a c-cycle, labeled and colored as is in the definition. For $j \leq 2n$ let $e_j = (v_{j-1}, v_j)$ and let $e_{2n+1} = (v_{2n}, v_0)$. Then $\mathcal{C} = e_1, e_2, \dots, e_{2n}, e_{2n+1}$ is a chain of length $2n + 1$.*

Proof. For $j \leq 2n$, e_j contains the vertex v_j that does not appear in any e_i for $i < j$. Observe that e_{2n+1} is the cycle's final edge and therefore no other edge contains its color, c_0. Therefore $\mathcal{C}$ is a chain of length $2n + 1$. $\square$

Definition 4.3.6. For a c-cycle C, let $\mathcal{C}_C = \mathcal{C}$ where $\mathcal{C}$ is from Proposition 4.3.5. Define $\mathcal{C}_C$ to be C's *natural chain* and let $\mathcal{C}^*_C = e_1, e_2, \dots, e_{2n}$ be C's natural chain without its final edge.

Proposition 4.3.7. *Suppose $\mathcal{C} = e_1, e_2, \ldots, e_{2n+1}$ is a chain consisting of all the edges of a c-cycle C. Then e_{2n+1} is the final edge and every other e_j has a vertex that does not appear in any e_i for $i < j$.*

Proof. Each of the $2n+1$ vertices of C appear in any collection of $2n$ edges of C. Hence e_{2n+1} must have a color that no other edge in C has, and therefore e_{2n+1} is the final edge. Now for some $j \leq 2n$, suppose that the vertices of e_j appear in e_a or e_b where $a, b < j$. Then e_a and e_b are the adjacent edges of e_j in C, and therefore without loss of generality we know that e_j and e_a have the same color. This contradicts the fact that e_j has a color or vertex that does not appear in any e_i for $i < j$. Therefore e_j must contain a vertex that does not appear in any e_i for any $i < j$. $\square$

Definition 4.3.8. Given chains $\mathcal{C}_1 = e_1, e_2, \ldots, e_n$ and $\mathcal{C}_2 = f_1, f_2, \ldots, f_m$, their *concatenation* is

$$\mathcal{C}_1, \mathcal{C}_2 = e_1, \ldots, e_n, f_1, \ldots, f_m.$$

Proposition 4.3.9. *Suppose that G is a colored graph consisting of n disjoint c-cycles $C_1, \ldots, C_n$, where each cycle's color is distinct and let L be the length of the longest chain in G. Then under a renumbering of the cycles, there is a chain $\mathcal{C}$ of length L where*

$$\mathcal{C} = \mathcal{C}_1, \ldots, \mathcal{C}_m, \mathcal{C}^*_{m+1}, \ldots, \mathcal{C}^*_n.$$

Here $\mathcal{C}_j$ is the natural chain for the c-cycle C_j and $\mathcal{C}^*_j$ is the natural chain without its final edge for the c-cycle C_j.

Remark 4.3.10. Observe while the collection of vertices for each c-cycle are pairwise disjoint and each c-cycle's color is distinct, the collection of colors appearing in the c-cycles are not necessarily pairwise disjoint. For example C_1, C_2, C_3 could be as follows where C_1's color is 'black', C_2's color is 'gray', and C_3's color is 'dashed black'.

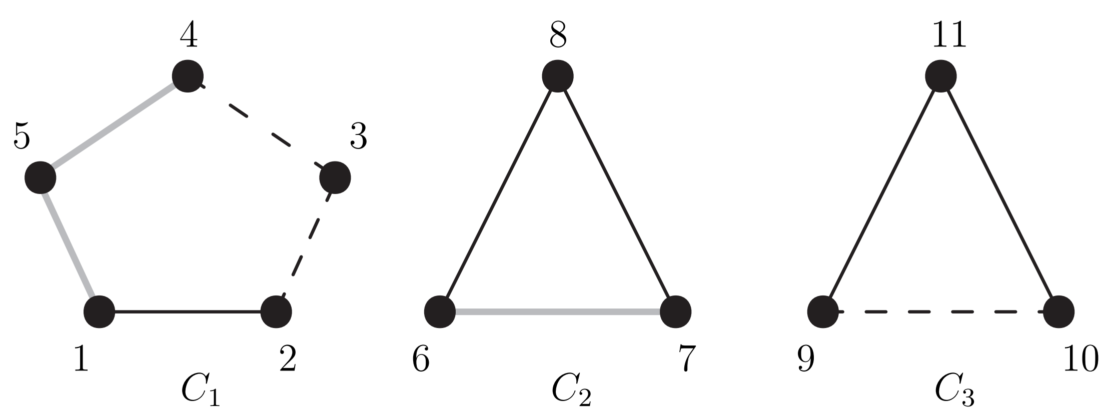

Figure 4.2: Three disjoint c-cycles with distinct colors.

Example 4.3.11. Before beginning the proof of Proposition 4.3.9, we will do an example first. Observe that

$$\mathcal{L} = (7,8), (10,11), (11,9), (8,6), (6,7), (2,3), (3,4), (4,5), (5,1), (9,10)$$

is a chain in Figure 4.2, where (i, j) is the edge between vertex i and vertex j. In $\mathcal{L}$ cycles C_2 and C_3 are completed, so let

$$\mathcal{L}' = (7,8), (10,11), (11,9), (8,6), (6,7), (9,10)$$

be the chain obtained by removing the edges in $\mathcal{L}$ that are from cycle C_1. In $\mathcal{L}'$, we can pull edges from cycle C_2 in front of edges from C_3 since C_2 is the first cycle in $\mathcal{L}'$ that is completed. Pulling these edges into the front gives the chain

$$\mathcal{L}'' = (7,8), (8,6), (6,7), (10,11), (11,9), (9,10) = \mathcal{C}_2, \mathcal{C}_3.$$

Let $\mathcal{N} = \mathcal{C}_1^*$, then we have that

$$\mathcal{C} = \mathcal{L}'', \mathcal{N} = \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_1^*$$

is a chain of length 9, which was the length of $\mathcal{L}$, and $\mathcal{C}$ is of the desired form for Proposition 4.3.9.

Proof of Proposition 4.3.9. Let e be the total number of edges in G. Observe that

$$\mathcal{C} = \mathcal{C}_1, \mathcal{C}_2^*, \ldots, \mathcal{C}_n^*$$

is a chain, for $\mathcal{C}_1$ is a chain and in $\mathcal{C}_j^*$ each edge is adding a vertex. The length of $\mathcal{C}$ is $e - (n-1)$ because it contains every edge except for $n-1$ final edges. Therefore $L \geq e - n + 1$ and by the pigeon-hole principle, any maximal length chain contains at least one completed cycle.

Let $\mathcal{L} = e_1, e_2, \ldots, e_L$ be a maximal length chain. From $\mathcal{L}$ remove every edge that does not belong to a completed cycle in $\mathcal{L}$, and call the remaining chain

$$\mathcal{L}' = e_{i_1}, \ldots, e_{i_a}.$$

Let e_{i_b} be the first edge in $\mathcal{L}'$ that completes a cycle and without loss of generality let it be C_1. Let $j \leq b$ with $e_{i_j} \in C_1$ and $e_{i_{j-1}} \notin C_1$. We claim that then,

$$e_{i_1}, \ldots, e_{i_{j-2}}, e_{i_j}, e_{i_{j-1}}$$

is a valid chain. But this follows from Proposition 4.3.7 for $e_{i_{j-1}}$ contains a vertex that does not appear in any e_{i_c} for $c < j-1$ and that vertex cannot appear in e_{i_j} for e_{i_j} is not in the same cycle as $e_{i_{j-1}}$. So the above list is in fact a valid chain and therefore we can permute all the C_1 edges to the beginning of $\mathcal{L}$.

It follows from induction on the number of completed cycles in $\mathcal{L}'$, that by numbering the completed cycles in $\mathcal{L}'$ in the order that they are completed, the edges in $\mathcal{L}'$ can be permuted into the following chain:

$$\mathcal{L}'' = \mathcal{C}_1, \ldots, \mathcal{C}_m$$

We can then concatenate $\mathcal{L}''$ with the chain

$$\mathcal{N} = \mathcal{C}^*_{m+1}, \ldots, \mathcal{C}^*_n,$$

because each edge in $\mathcal{N}$ is adding a new vertex from a chain that does not appear in $\mathcal{L}''$.

The resulting chain, $\mathcal{C} = \mathcal{L}'', \mathcal{N}$ is at least as long as $\mathcal{L}$. Therefore $\mathcal{C}$ has length L and is of desired form. □

Theorem 4.3.12. *MAXIMAL LENGTH is NP-complete.*

Proof. MAX is in NP for the same reason MIN is in NP, since a nondeterministic algorithm need only to guess a subset of $\{A_1, \ldots, A_n\}$ and an ordering, then check to see that the subset is of the proper size and that the ordering gives a proper chain with final element being equal to S.

We will transform VC into MAX. The idea will be to turn a graph G into a disjoint collection of c-cycles, where each cycle's color is distinct, so that we can apply Proposition 4.3.9.

Let $G = (V, E)$ be a graph with N vertices and assign each vertex $v_j \in V$, a unique color c_j. Without loss of generality we may assume that there are no isolated vertices in G, because no isolated vertex will be part of a minimal vertex cover. Now for each vertex v_j create a c-cycle C_j as follows:

> C_j's color will be c_j and the other colors in C_j will be exactly those colors c_i such that $(v_i, v_j) \in E$ is an edge in G.

Then the collection, Γ, of these c-cycles will be a colored graph consisting of n disjoint c-cycles $C_1, \ldots, C_N$ where each cycle's color is distinct. Therefore we will be able to apply Proposition 4.3.9 to Γ.

For example the graph G in Figure 4.3 transforms into the set of the three c-cycles from Figure 4.2. This is done by assigning v_1 to the color 'black', v_2 to the color 'gray', and v_3 to the color 'dashed black'.

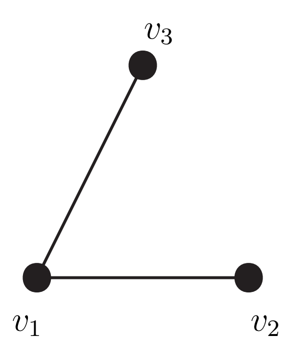

Figure 4.3: The graph G transforms into Figure 4.2.

Now consider the graph Γ consisting of all c-cycles C_j created for G. For each edge e in Γ, identify e with the set $A_e = \{c_i, v_a, v_b\}$ where c_i is the color of edge e and v_a, v_b are the two vertices of e. We can see that finding the longest chain in poset($\{A_e \mid e \in \Gamma\}$) is the same as finding the longest chain of edges in Γ. Let E be the number of edges in Γ. We claim that there is a vertex cover of G of size K if and only if the length of the longest chain of edges in Γ is at least $E - K$.

Suppose that we have a vertex cover of G of size K. We will show that there is a chain of edges in Γ of length $E-K$. Renumber the vertices of G so that $\{v_1, \ldots, v_K\}$ is a vertex cover. This means that for $i, j > K$ we know that (v_i, v_j) is not an edge in G and hence C_i and C_j do not contain the other's color. Then in the notation of Proposition 4.3.9, the following is a chain in Γ:

$$\kappa = \mathcal{C}_{K+1}, \ldots, \mathcal{C}_N, \mathcal{C}_1^*, \ldots, \mathcal{C}_K^*.$$

Observe that the length of κ is $E - K$.

Suppose now that we know that the length of the longest chain of edges in Γ is at least $E - K$, where $K < N$ by Proposition 4.3.9. Let the length of the longest chain be $E - K'$ where $K' \leq K$. It follows from Proposition 4.3.9, that after a renumbering of the cycles and their edges we have a maximal length chain of edges of the form:

$$\mathcal{C} = \mathcal{C}_{K'+1}, \ldots, \mathcal{C}_N, \mathcal{C}_1^*, \ldots, \mathcal{C}_{K'}^*.$$

This means if $i, j > K' + 1$ and $i \neq j$, then C_i and C_j do not have the other's color. Therefore in G there are no edges of the form (v_i, v_j) if $i, j > K'$. Hence the set of vertices $\{v_1, \ldots, v_{K'}\}$ form a vertex cover for the graph G. Since $K' \leq K$, we know that $\{v_1, \ldots, v_K\}$ form a vertex cover of size K for the graph G.

Therefore we have proved that there is a vertex cover of G of size K if and only if the length of the longest chain of edges in Γ is at least $E - K$. Furthermore in the worst case scenario where $G = K_N$, Γ has $2N^2 - N$ vertices and $2N^2 - N$ edges, so Γ can be constructed from G in polynomial time. Therefore since we have a polynomial transformation from VC to MAX, which proves that MAXIMAL LENGTH is NP-complete. $\square$

Appendix A

5×5 Zero-One Matrices

A.1 The Setup

If $M \in \{0,1\}^{5\times 5}$ and t-rank$(M) \neq$ k-rank(M), then by Theorem 2.2.4 and Theorem 2.2.5:

$$\text{t-rank}(M) = 3 \text{ and } \text{k-rank}(M) = 4.$$

In this appendix we will present the computations that prove Theorem 2.2.6, which states that there are no matrices $M \in \{0,1\}^{5\times 5}$ such that t-rank$(M) \neq$ k-rank(M).

Let $M \in \{0,1\}^{5\times 5}$ be such that t-rank$(M) = 3$, then it has a chain of length three and does not have a chain of length four. We can think of a chain of length three as being an ordered partition of $\{1,2,3,4,5\}$ into three nonempty sets and up to a permutation of $\{1,2,3,4,5\}$ there are exactly six of these. The six cases are:

5.1 :$\{1\},\{2\},\{3,4,5\}$	**5.2** :$\{1\},\{2,3\},\{4,5\}$	**5.3** :$\{1\},\{2,3,4\},\{5\}$
5.4 :$\{1,2\},\{3\},\{4,5\}$	**5.5** :$\{1,2\},\{3,4\},\{5\}$	**5.6** :$\{1,2,3\},\{4\},\{5\}$

So in our chain of length three, the first column adds the first set, the second column adds the second set, and the third column adds the third set. So M can look like one of the following matrices (where a star means the entry can be 0 or 1):

$$C_1 = \begin{pmatrix} 0 & * & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \\ 1 & 1 & 0 & * & * \\ 1 & 1 & 0 & * & * \end{pmatrix} C_2 = \begin{pmatrix} 0 & * & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \\ 1 & 1 & 0 & * & * \end{pmatrix} C_3 = \begin{pmatrix} 0 & * & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \end{pmatrix}$$

$$C_4 = \begin{pmatrix} 0 & * & * & * & * \\ 0 & * & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \\ 1 & 1 & 0 & * & * \end{pmatrix} C_5 = \begin{pmatrix} 0 & * & * & * & * \\ 0 & * & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \end{pmatrix} C_6 = \begin{pmatrix} 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \end{pmatrix}$$

Thinking of each cases as an ordered partition of the number 5 into three parts, we have that

$$\mathbf{5.1} = (1,1,3) \qquad \mathbf{5.2} = (1,2,2) \qquad \mathbf{5.3} = (1,3,1)$$
$$\mathbf{5.4} = (2,1,2) \qquad \mathbf{5.5} = (2,2,1) \qquad \mathbf{5.6} = (3,1,1)$$

We say that a matrix $M \in \{0,1\}^{5\times5}$ with $\text{t-rank}(M) = 3$ is in case $\mathbf{5.j}$ if it can be permuted to look like C_j, has no length 3 chain $(a,b,c) < \mathbf{5.j}$ under the lexicographic ordering, and has no column of 1's.

Proposition A.1.1. *If $M \in \{0,1\}^{5\times5}$, $\text{t-rank}(M) = 3$, and M is in case* $\mathbf{5.1}$, $\mathbf{5.2}$, *or* $\mathbf{5.4}$, *then* $\text{t-rank}(M) = \text{k-rank}(M)$.

Proof. Observe that $M_{44} = M_{54}$ and $M_{45} = M_{55}$. For otherwise c_1, c_2, c_4, c_3 would be a chain of length four, where c_j is the j^{th} column of M, which contradicts that $\text{t-rank}(M) = 3$. Therefore the last two rows of M and let $\tilde{M} \in \{0,1\}^{4\times5}$ be M with the last column removed. Hence,

$$\text{t-rank}(M) = \text{t-rank}(\tilde{M}) \text{ and } \text{k-rank}(\tilde{M}) = \text{k-rank}(M). \tag{A.1.1}$$

Since $\tilde{M} \in \{0,1\}^{4\times5}$ it follows that

$$\text{t-rank}(\tilde{M}) = \text{k-rank}(\tilde{M}). \tag{A.1.2}$$

Therefore $\text{t-rank}(M) = \text{k-rank}(M)$. □

The goal going forward will be to find all the representative matrices in the cases $\mathbf{5.3}$, $\mathbf{5.5}$, and $\mathbf{5.6}$ and then compute their Kapranov rank using Theorem 2.1.8. In order to do this, there are a few computational issues that must be dealt with first in order to make this feasible. For instance, there are $2^{15} = 32768$ matrices that are of the form C_3 and checking which of these have tropical rank 3 and computing their Kapranov rank would take a considerable amount of time on a standard computer. So the following three sections will be devoted to getting a better upper estimate of the matrices in the cases $\mathbf{5.3}$, $\mathbf{5.5}$, and $\mathbf{5.6}$. For instance, we will reduce case $\mathbf{5.3}$ to checking $2^3 \cdot 45 = 360$ matrices.

With these better upper bounds, we generated all of the representative matrices in each case. A naive `GAP` algorithm was used to check to see if each matrix had a length 4 chain, and those that did were thrown out. The remaining matrices were then run through a `CoCoA` algorithm based off of Theorem 2.1.8, to check if they had Kapranov rank greater than 3 (see Section A.5 for the actual code). It turned out that no matrix that had Tropical rank 3 had Kapranov rank greater than 3. This fact along with Proposition A.1.1 proves that there is no matrix $M \in \{0,1\}^{5\times5}$ such that $\text{t-rank}(M) = 3 < \text{k-rank}(M)$. By the remarks at the beginning of this section, this means we have proved Theorem 2.2.6,

$$\text{t-rank}(M) = \text{k-rank}(M) \text{ if } M \in \{0,1\}^{n\times n} \text{ and } n \leq 5.$$

A.2 Case 5.3

We have that M with t-rank$(M) = 3$ is of the following form:

$$M = \begin{pmatrix} 0 & * & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \end{pmatrix}.$$

Without loss of generality we can assume that no column is all 1's and the columns are distinct. Furthermore, we can assume that $(1,3,1)$ is the smallest length three chain in M under the lex order.

If $M_{54} = 1$ and $M_{i4} \neq M_{j4}$ for $i,j \in \{2,3,4\}$, then c_1, c_2, c_4, c_3 would be a chain and a similar result holds for c_5. Therefore the last two columns of M can look like either

$$f_1 = \begin{pmatrix} a \\ b \\ b \\ b \\ 1 \end{pmatrix} \text{ or } f_2 = \begin{pmatrix} c \\ d \\ e \\ f \\ 0 \end{pmatrix} \text{ where } a,b,c,d,e,f \in \{0,1\}. \tag{A.2.1}$$

If $c_4 = f_1$ and $b = 1$, then $a = 0$, since there is no column of 1's, and hence $c_1 = c_4$ which cannot happen. Therefore the last two columns of M can look like either

$$f_1 = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ or } f_2 = \begin{pmatrix} c \\ d \\ e \\ f \\ 0 \end{pmatrix}. \tag{A.2.2}$$

Now in the third column of M, at least two of M_{23}, M_{33}, M_{43} must be equal to 0, for otherwise c_1, c_3, c_2 would be a chain of the form $(1,2,2)$ or $(1,1,3)$ both of which are smaller that $(1,3,1)$. Hence by permuting the rows of M we know that the first three columns look like

$$M = \begin{pmatrix} 0 & * & * & & \\ 1 & 0 & * & & \\ 1 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ 1 & 1 & 0 & & \end{pmatrix}. \tag{A.2.3}$$

A similar statement holds for the fourth or fifth columns if they are of form f_2, namely that at least two of d, e, f are equal to 0. Therefore the last two columns must be drawn from

$$f_1 = \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, f_2 = \begin{pmatrix} * \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, f_3 = \begin{pmatrix} * \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, f_4 = \begin{pmatrix} * \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, f_5 = \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{A.2.4}$$

A.3 Case 5.5

We have that M with t-rank$(M) = 3$ is of the following form:

$$M = \begin{pmatrix} 0 & * & * & * & * \\ 0 & * & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \end{pmatrix}.$$

Without loss of generality we can assume that no column is all 1's and the columns are distinct. Furthermore, we can assume that $(2, 2, 1)$ is the smallest length three chain in M under the lex order.

By an argument similar to that for (A.2.1) and (A.2.2), we have that the last two columns look like either

$$f_1 = \begin{pmatrix} a \\ b \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ or } f_2 = \begin{pmatrix} c \\ d \\ e \\ f \\ 0 \end{pmatrix} \text{ where } a, b, c, d, e, f \in \{0, 1\}. \tag{A.3.1}$$

Now neither c_3, c_4, nor c_5 can be of the form

$$c_n = \begin{pmatrix} * \\ * \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

for then c_1, c_n, c_2 would be a chain of the form $(2, 1, 2)$ which is less than $(2, 2, 1)$. Permuting gives that M is of the form

$$M = \begin{pmatrix} 0 & * & * & & \\ 0 & * & * & & \\ 1 & 0 & * & & \\ 1 & 0 & 0 & & \\ 1 & 1 & 0 & & \end{pmatrix}, \tag{A.3.2}$$

where c_4 and c_5 are drawn from

$$f_1 = \begin{pmatrix} * \\ * \\ 0 \\ 0 \\ 1 \end{pmatrix}, f_2 = \begin{pmatrix} * \\ * \\ 0 \\ 1 \\ 0 \end{pmatrix}, f_3 = \begin{pmatrix} * \\ * \\ * \\ 0 \\ 0 \end{pmatrix}. \tag{A.3.3}$$

A.4 Case 5.6

We have that M with t-rank$(M) = 3$ is of the following form:

$$M = \begin{pmatrix} 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 1 & 0 & * & * & * \\ 1 & 1 & 0 & * & * \end{pmatrix}.$$

Without loss of generality we can assume that no column is all 1's and the columns are distinct. Furthermore, we can assume that $(3, 1, 1)$ is the smallest length three chain in M under the lex order.

Now if $M_{44} = M_{54} = 1$, then $M_{14} = M_{24} = M_{34}$, for otherwise c_4, c_1, c_2, c_3 would be a chain. However, then if $M_{44} = M_{54} = 1$, then c_4 is either all 1's or equal to c_1, both contradictions. These statements hold just as well for the fifth column. Therefore the last two columns of M are drawn from

$$f_1 = \begin{pmatrix} a \\ b \\ c \\ 0 \\ 1 \end{pmatrix} \text{ or } f_2 = \begin{pmatrix} d \\ e \\ f \\ g \\ 0 \end{pmatrix} \text{ where } a, b, c, d, e, f, g \in \{0, 1\}. \tag{A.4.1}$$

Observe that neither c_2, c_4, nor c_5 can be of the form

$$c_n = \begin{pmatrix} a \\ b \\ c \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad c_m = \begin{pmatrix} a \\ b \\ c \\ 1 \\ 0 \end{pmatrix}$$

with at most one of a, b, and c being equal to 0, for then c_n, c_1, c_3 or c_m, c_1, c_2 would be a chain with c_n or c_m adding less than 3 and hence the chain would be of a form less than $(3, 1, 1)$.

Bringing this altogether gives that M can be permuted to be the following

$$M = \begin{pmatrix} 0 & * & * & & \\ 0 & 0 & * & & \\ 0 & 0 & * & & \\ 1 & 0 & 0 & & \\ 1 & 1 & 0 & & \end{pmatrix} \text{ or } \begin{pmatrix} 0 & * & * & & \\ 0 & 0 & * & & \\ 0 & 0 & 0 & & \\ 1 & 0 & 1 & & \\ 1 & 1 & 0 & & \end{pmatrix}, \tag{A.4.2}$$

where c_4 and c_5 are drawn from

$$f_1 = \begin{pmatrix} a \\ b \\ c \\ 0 \\ 1 \end{pmatrix}, f_2 = \begin{pmatrix} d \\ e \\ f \\ 1 \\ 0 \end{pmatrix}, f_3 = \begin{pmatrix} * \\ * \\ * \\ 0 \\ 0 \end{pmatrix}, \tag{A.4.3}$$

with $a, b, c, d, e, f \in \{0, 1\}$, $a + b + c \leq 1$, and $d + e + f \leq 1$.

A.5 CoCoA Code

The following is the explicit CoCoA code used to compute the Kapranov rank of a matrix M using Theorem 2.1.8:

```
// Determine whether the Kapranov rank of the matrix M has rank
// greater than T.  The answer is "True" if the output is Ideal(1).
Define IsKRank(M,T)
  NumRows:=Len(M);
  NumCols:=Len(M[1]);
  FM:=Flatten(List(M));
  L:=Len(FM);
  W1:=(Max(FM)+1)*NewList(L,1)-FM;
  W:=Mat(Concat([W1],Submat(Identity(L),2..L,1..L)));

  NewRingName := NewId();
  Var(NewRingName)::= Q[x[1..NumRows,1..NumCols]],Ord(W);
  Using Var(NewRingName) Do
    D:=Mat([[x[I,J]|J In 1..NumCols]|I In 1..NumRows]);
    J:=Ideal(Minors(T+1,D));
    G:=ReducedGBasis(J);
    IG:=[InitialForm(X,FM)|X In G];
    S:= Saturation(Ideal(IG),Ideal(Product(Indets())));
    If S=Ideal(1) Then
      Return True;
    Else
      Return False;
    EndIf;
  EndUsing;
EndDefine;

// Given a polynomial F and a weight vector W for the variables,
// grab the initial form of F.
Define InitialForm(F,W)
  M:=Monomials(F);
  B:=Min([ScalarProduct(Log(X),W)|X In M]);
  Return Sum([X|X In M And ScalarProduct(Log(X),W)=B]);
EndDefine;
```

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