The h-Vector Conjecture for Matroids with Cycle Systems

A Thesis
Presented to
The Division of Mathematical and Natural Sciences
Reed College

In Partial Fulfillment
of the Requirements for the Degree
Bachelor of Arts

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December 2023

Approved for the Division
(Mathematics)

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Acknowledgements

This thesis would not be possible without my advisor, David Perkinson. Thank you for your patience and your passion for sharing the wonder of math with your students. Thank you for bringing students into the exploration of this area of research.

Thank you to the many math teachers in my life that have fostered my interest and helped me learn, in particular, Jamie Pommersheim. Jamie, thank you for your infectious enthusiasm. I must also thank my peers, who shared my curiosity and helped me when I fell behind.

To my parents, who brought me into this world: thank you for your unconditional love and support. Thank you for sending me to college.

Thank you to Dr. Jan Chozen Bays, Roshi.

The theorems proved here rely on the results of many mathematicians, known and unknown. Thank you for your curiosity and energy.

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Abstract

We provide an introduction to matroids, the Tutte polynomial, and h-vectors. We then prove Stanley's h-vector conjecture for a subset of linear matroids using deletion-contraction and what we call "cycle systems."

Introduction

The h-vector conjecture has remained unproven since its proposal in 1977 by Richard Stanley. Many mathematicians have worked on the problem and produced proofs for different cases of the conjecture. Here, we provide the necessary background information and prove another case of the h-vector conjecture in the area of linear matroids (Theorem 2.1.1). This thesis is the first exposition of this theorem and proof. The key ideas for the proof were found during a summer 2023 research project by Lily Factora, Sanay Sehgal, and Lixing Yi, advised by David Perkinson. That project was based on earlier work by Perkinson, Scott Corry, and Anton Dochtermann.

Chapter 1 introduces matroids, their Tutte polynomials, and their h-vectors. We primarily focus on linear and graphic matroids. In Chapter 2, we define cycle systems and coparking functions. We then prove the h-vector conjecture for linear matroids with cycle systems. We also show how the deletion-contraction tree of a matroid partitions the matroid's bases and coparking functions; we use this to find a bijection between the two.

Chapter 1

Matroids

1.1 Matroids

Definition 1.1.1 (Matroid)**.** A *matroid* is a pair of finite sets (E, I), where E is called the *ground set* and $I \subseteq 2^E$ satisfies the following properties:

- $\emptyset \in I$.
- For any $A \in I$, any subset $A' \subseteq A$ also has $A' \in I$.
- Exchange axiom: For any $A, B \in I$ with $|A| > |B|$, there exists $a \in A \setminus B$ such that $B \cup \{a\} \in I$.

Elements of I are called *independent sets*. An independent set that is maximal under inclusion is called a *basis* of the matroid. The *rank* of a matroid is the size of any basis (well-defined by the exchange axiom). A *circuit* of a matroid is a minimal dependent set.

Example 1.1.1 (Vector Matroid)**.** Take any field F and an $m \times n$ matrix A with entries in F. The associated matroid's ground set is the set of column indices, that is, $E = \{1, \ldots, n\}$. The independent sets are the sets of indices whose columns are linearly independent. In linear matroids, there may be repeated columns.

For example, take the matrix

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 3 & 3 & 1 \end{pmatrix}.$$

The independent sets are $\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}, \{3,4\}$.

Example 1.1.2 (Cycle Matroid)**.** Let $G = (V, E)$ be a finite multigraph. This means that G has a finite set V of vertices and a finite set E of edges. There may be multiple edges between a given pair of vertices. The *cycle matroid* given by G, denoted $M(G)$, has ground set E. Its independent sets are the collections of edges that do not contain a cycle. In graph theory, these sets of edges are called *forests*.

Two matroids (E, I) and (E', I') are *isomorphic* if there is a bijection $f\colon E \to E'$ such that

$$A \in I \iff f(A) \in I'.$$

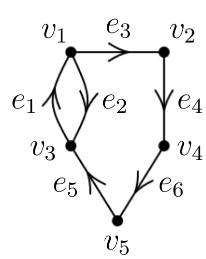
A matroid is *linear* if it is isomorphic to a vector matroid. In this paper, whenever a linear matroid is mentioned, we will assume an identification has been made with a vector matroid.

A matroid is *graphic* if it is isomorphic to the cycle matroid of some graph.

Every graphic matroid is isomorphic to a linear matroid (Oxley, 2011, Proposition 5.1.2), as indicated by the following example.

Example 1.1.3 (Graphic to Linear Matroid)**.** If $M = M(G)$ for some graph G, number each vertex and pick an arbitrary orientation for each edge. For each edge e_i, the entries of the *signed incidence vector* d_i correspond to the vertices of G. If e_i is oriented out of v_j, then the j-th entry of d_i is -1. If e_i is oriented into v_j, then the j-th entry of d_i is 1. If e_i does not have an endpoint at v_j, then the j-th entry of d_i is 0. The matrix of the vector matroid uses d_i's as its columns.

For example, take the arbitrarily labeled graph



Following the rules above, the associated matrix for this graph is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}
\begin{array}{c} \begin{array}{cccccc} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \end{array} \\
\begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array}.$$

It is easier to list the dependent sets than the independent sets of the associated vector matroid. They are any sets containing the following subsets:

$$\{1, 2\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}.$$

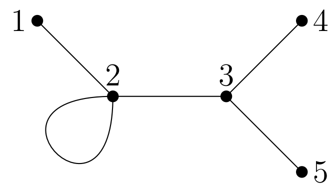
Notably, these minimal dependent subsets correspond exactly with the circuits of the graphic matroid, which has circuits

$$\{e_1, e_2\}, \{e_1, e_3, e_4, e_5, e_6\}, \{e_2, e_3, e_4, e_5, e_6\}.$$

Therefore, the independent sets correspond as well.

Definition 1.1.2 (Bridge, Loop)**.** For a matroid (E, I), any $e \in E$ is called a *bridge* if it is in every basis for the matroid. Any $e \in E$ is called a *loop* if it is not in any basis.

Example 1.1.4. Consider the cycle matroid (as defined above) on the following graph:



The edges from 1 to 2 and from 2 to 3 are both bridges. The edge from 2 to itself is a loop.

1.2 The Tutte Polynomial

In this section, we will give three equivalent definitions of the Tutte polynomial. For their equivalence, see White (1992), which shows that the first two definitions are equivalent in Theorem 6.2.2 and Equation 6.13 and that the last definition is equivalent in Equation 7.11 and Theorem 7.3.7.

Let $M = (E, I)$ be a matroid.

TP I. Tutte Polynomial by Deletion and Contraction. If there exists a non-bridge $e \in E$, the *deletion* of M by e is the matroid

$$M \setminus e := (E \setminus \{e\}, I'),$$

where

$$I' = \{A \mid A \in I \text{ and } e \notin A\}.$$

Now take a non-loop $e \in E$. The *contraction* of M by e is the matroid

$$M/e := (E \setminus \{e\}, I''),$$

where

$$I'' = \{A \mid A \cup \{e\} \in I\}.$$

The Tutte polynomial $T_M(x, y)$ can be defined iteratively using deletion and contraction. For a non-bridge, non-loop $e \in E$,

$$T_M(x, y) = T_{M \setminus e}(x, y) + T_{M/e}(x, y).$$

If all edges are bridges or loops, and there are B bridges and L loops, then

$$T_M(x, y) = x^B y^L.$$

In particular, if $M = \emptyset$, then $T_M(x, y) = 1$.

Example 1.2.1. We will now calculate the Tutte polynomial for the cycle matroid M on the following graph, known as the *diamond graph*:

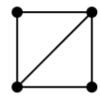
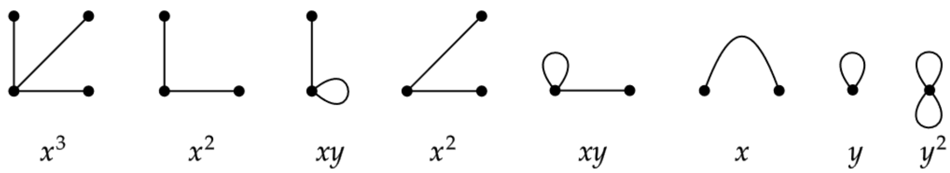


Figure 1.1 demonstrates the deletion/contraction process for this graph. The dashed edges represent e in the above definition.

The resulting deleted and contracted graphs consist only of bridges and loops. Arranged neatly and labeled by their Tutte polynomials, they are:



So the Tutte polynomial of M is their sum:

$$\begin{aligned} T_M(x,y) &= x^3 + x^2 + xy + x^2 + xy + x + y + y^2 \\ &= x^3 + 2x^2 + x + 2xy + y + y^2. \end{aligned}$$

TP II. Tutte Polynomial by Rank. The *rank* of a subset $S \subseteq E$ is the cardinality of a maximal independent subset of S.

The Tutte polynomial of M is

$$T_M(x,y) = \sum_{\text{subsets } S \subseteq E} (x-1)^{r(E)-r(S)} (y-1)^{|S|-r(S)},$$

where r is the rank function.

Example 1.2.2. For the same matroid as that of Example 1.2.1, we will now calculate the Tutte polynomial using TP II. We will look at every subset of E and find its rank. We can do this quickly by looking at all subsets $S \subseteq E$ with no elements, then one element, and so on.

$\|S\|$	rank	how many	polynomial term
0	0	1	$(x-1)^3$
1	1	5	$(x-1)^2$
2	2	10	$(x-1)$
3	2	2	$(x-1)(y-1)$
3	3	8	1
4	3	5	$(y-1)$
5	3	1	$(y-1)^2$

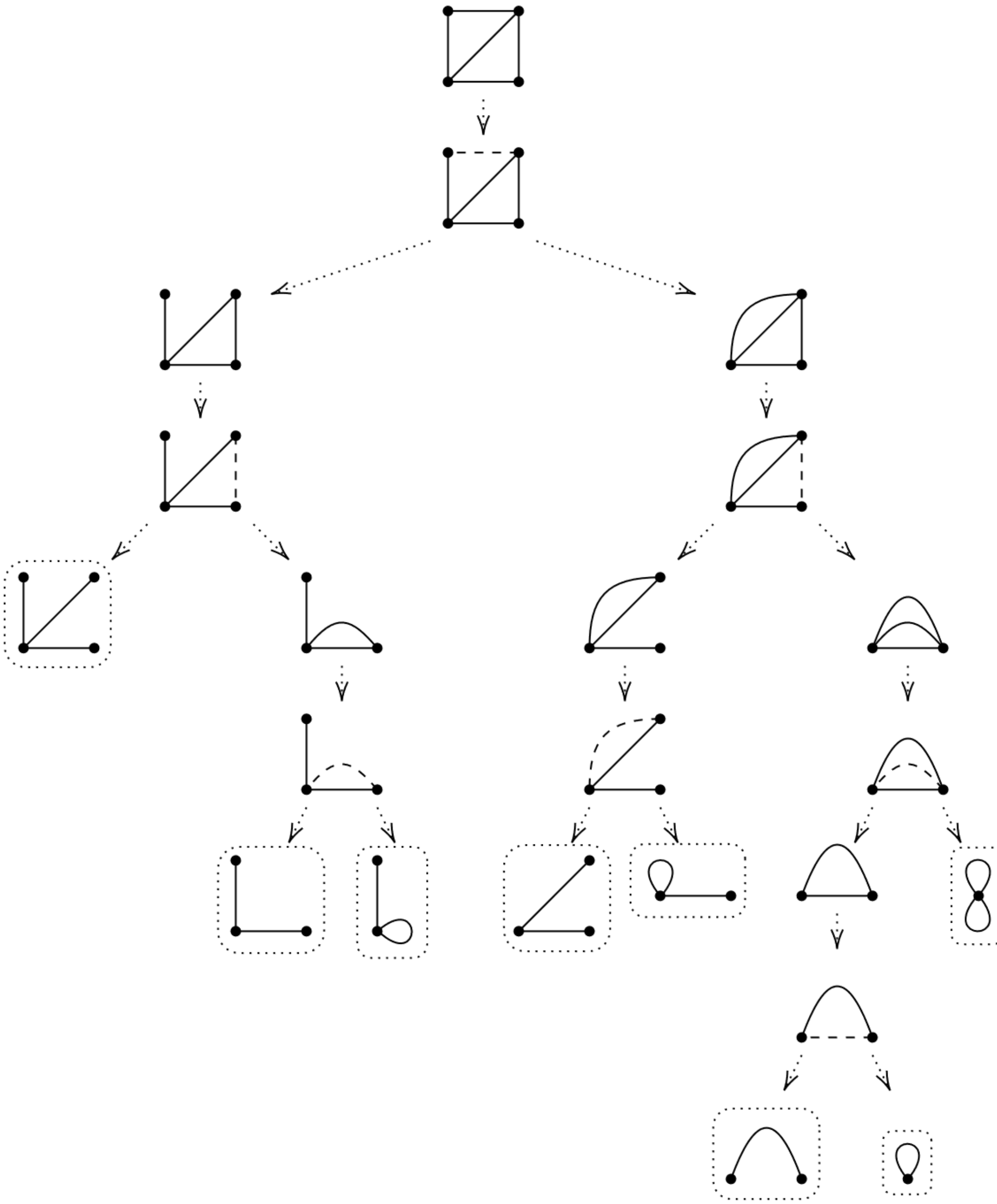


Figure 1.1: Deletion and contraction visualized for Example 1.2.1.

Having found the terms, we can now expand and simplify the Tutte polynomial.

$$1(x-1)^3 + 5(x-1)^2 + 10(x-1) + 2(x-1)(y-1) + 8(1) + 5(y-1) + 1(y-1)^2$$

$$= x^3 + 2x^2 + x + 2xy + y + y^2.$$

This is exactly what we found in Example 1.2.1 when we calculated the Tutte polynomial for the same matroid using deletion and contraction.

TP III. Tutte Polynomial by Activity. Let B be a basis for M.

Take $e \in E \setminus B$. The *fundamental circuit* for (B, e) is the unique circuit in $B \cup \{e\}$. This circuit is unique by Oxley (2011), Proposition 1.1.6.

A *cut*[1] is a minimal subset of E that is contained in every basis of M. If $e \in B$, the *fundamental cut* for (B, e) is the unique cut of M that is disjoint from $B \setminus \{e\}$. This cut is unique by Oxley (2011), Chapter 2.1, Exercise 10.

Both the fundamental circuit and fundamental cut contain their defining e.

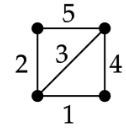
Choose a total ordering on E. Keeping B fixed, an element $e \in E \setminus B$ is *externally active* if it is the smallest element in its fundamental circuit with B. An element $e \in B$ is *internally active* if it is the smallest element in its fundamental cut with B.

The *external activity* of B, denoted here by $\varepsilon(B)$, is the number of externally active elements in $E \setminus B$. The *internal activity* of B, denoted here by $i(B)$, is the number of internally active elements in B. The Tutte polynomial can be defined as follows:

$$T_M(x, y) = \sum_{\text{bases } B} x^{i(B)} y^{\varepsilon(B)}.$$

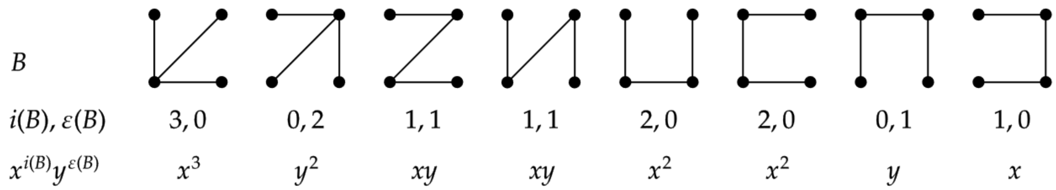
An element $e \in E \setminus B$ is *externally passive* if it is not externally active. An element $e \in B$ *internally passive* if it is not internally active. The *external passivity* of a basis B is the number of externally passive elements in $E \setminus B$. The *internal passivity* of B is the number of internally passive elements in B.

Example 1.2.3. We will use TP III to calculate the Tutte polynomial for the same graphic matroid as in Examples 1.2.1 and 1.2.2. Consider the diamond graph's matroid with labeled edges



We can look at each of its bases B and find their term $x^{i(B)} y^{\varepsilon(B)}$ in the Tutte polynomial:

[1]Some literature refers to *cuts* as *cocircuits*.



Just as in Examples 1.2.1 and 1.2.2, the Tutte polynomial for this matroid is

$$x^3 + 2x^2 + x + 2xy + y + y^2.$$

1.3 h-Vectors

As with the Tutte polynomial, we will give two equivalent definitions of the h-vector. Let $M = (E, I)$ be a matroid.

HV I. h-Vector by Tutte. Evaluate the Tutte polynomial of M at $(x, 1)$:

$$T_M(x, 1) = h_0 x^r + h_1 x^{r-1} + \cdots + h_{r-1} x + h_r = \sum_{i=0}^{r} h_{r-i} x^i,$$

where $h_i \in \mathbb{N}$ and $r = \text{rank}(E)$.[2] Then, the *h-vector* for M is $(h_0, \ldots, h_r)$.

HV II. h-Vector by Rank. All matroids can be thought of as simplicial complexes. The following definition of h-vector is motivated by the idea that M is a simplicial complex and f_i is the number of faces of dimension i.

For a matroid $M = (E, I)$ with rank r, define an f-vector $f = (f_0, \ldots, f_{r-1})$ by

$$f_i = \#\{A \in I : |A| = i + 1\}.$$

Let $f_{-1} = 1$ unless $E = \emptyset$. Then, define

$$f(x) = \sum_{i=0}^{r} f_{i-1} x^{r-i} = \sum_{i=0}^{r} f_{r-i-1} x^i.$$

Now define $h(x) = f(x - 1)$. The result is

$$h(x) = f(x-1) = \sum_{i=0}^{r} h_{r-i} x^i,$$

and the *h-vector* is defined $(h_0, \ldots, h_r)$, the coefficients of the polynomial.

Theorem 1.3.1. *The definitions HV I and HV II of h-vector are equivalent.*

[2] In cycle matroids, we have the equivalent $r = \#\{\text{vertices}\} - \#\{\text{connected components}\}$.

Proof. Using r as the rank function, the TP II definition of Tutte polynomial states

$$T_M(x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

Plugging this into the HV I definition for h-vector, we get

$$T_M(x,1) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(0)^{|A|-r(A)},$$

whose terms are only nonzero when $|A| = r(A)$, i.e. A is independent. So,

$$\begin{aligned} T_M(x,1) &= \sum_{A \subseteq I} (x-1)^{r(E)-r(A)} \\ &= \sum_{A \subseteq I} (x-1)^{r(E)-|A|}. \end{aligned}$$

In HV II we defined $h_i = \#\{A \in I : |A| = i\}$ with $h_0 = 1$ unless $E = \emptyset$. Using this, we can combine like terms in our sum:

$$\begin{aligned} T_M(x,1) &= \sum_{A \subseteq I} (x-1)^{r(E)-|A|} \\ &= \sum_{i=0}^{r(E)} h_i (x-1)^{r(E)-i} \\ &= \sum_{i=0}^{r(E)} h_{r(E)-i} (x-1)^{r(E)-(r(E)-i)} \\ &= \sum_{i=0}^{r(E)} h_{r(E)-i} (x-1)^{i}. \end{aligned}$$

We have reached the polynomial from HV II. So, the polynomials in HV I and HV II are the same, and thus the h-vectors they define are the same. □

Definition 1.3.1 (Multicomplex)**.** Take $\mathbb{N}^r = \mathbb{Z}^r_{\geq 0} = \{(a_1, \ldots, a_r) : a_i \in \mathbb{N}\}$ and establish a partial order by, for all $a, b \in \mathbb{N}^r$,

$$a \leq b \iff a_i \leq b_i \text{ for } i = 1, \ldots, r.$$

A *multicomplex* is a finite subset $D \subset \mathbb{N}^r$ such that if $a \in D$ and $b < a$, then $b \in D$. The *degree* of an element $a \in D$ is the sum of its values, that is,

$$\deg(a) = \sum_{i=1}^{r} a_i.$$

If $r = 0$, then the only nonempty multicomplex is $\{()\}$, whose element has degree 0.

A multicomplex D is considered *pure* if all of its maximal elements have the same degree.

The *degree vector* for a multicomplex D is

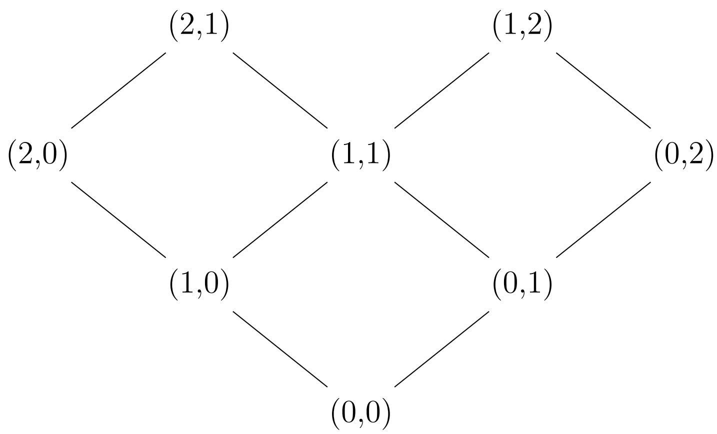
$$\deg(D) = (d_0, d_1, \ldots, d_\ell)$$

where

$$d_i = \#\{a \in D : \deg(a) = i\}.$$

and ℓ is the maximal degree among the elements of D.

Example 1.3.1. Take the multicomplex D generated by $\{(2,1),(1,2)\}$. It is $D = \{(2,1),(1,2),(2,0),(1,1),(0,2),(1,0),(0,1),(0,0)\}$. We can use a diagram to show the structure of the partial order:



The degree vector of D is $(1,2,3,2)$. The maximal elements of D are $(2,1)$ and $(1,2)$, which both have degree 3. Therefore, D is a pure multicomplex.

Conjecture 1.3.1 (Stanley's h-Vector Conjecture)**.** The h-vector of any matroid is the degree vector of some pure multicomplex.

Note: To be precise, the h-vector conjecture is that the *non-zero* entries of an h-vector form the degree vector of a pure multicomplex. We know that any zero entries in the h-vector occur at the end of the vector, and the number of zero entries is the number of bridges in the matroid.

Example 1.3.2. In Examples 1.2.1, 1.2.2, and 1.2.3, we saw that the Tutte polynomial of the graphic matroid M associated to the diamond graph was $x^3 + 2x^2 + x + 2xy + y + y^2$. Using HV I, we can find the matroid's h-vector.

$$\begin{aligned} T_M(x,1) &= x^3 + 2x^2 + x + 2x + 1 + 1 \\ &= x^3 + 2x^2 + 3x + 2, \end{aligned}$$

and hence the h-vector for M is $(1,2,3,2)$. In Example 1.3.1, we found a pure multicomplex with degree vector $(1,2,3,2)$, as conjectured.

Definition 1.3.2 (Dual Matroid)**.** For a matroid $M = (E,I)$, the *dual matroid* is $M^* = (E, I^*)$ where

$$I^* = \{X \subseteq E \mid \exists \text{ a basis } B \text{ for } M \text{ such that } X \cap B = \emptyset\}.$$

Its bases are the complements of bases for M.

Stanley's h-vector conjecture is known for certain types of matroids. One of these types is cographic matroids, that is, matroids whose duals are graphic. This was proved using chip-firing theory by Merino López (1997), and an exposition can be found in Corry & Perkinson (2018). Chapter 2 is an attempt to "dualize" the cographic proof so that it works for graphic matroids. As it turns out, the proof works for a certain class of linear matroids.

Chapter 2

Coparking Functions

2.1 Cycle Systems

Definition 2.1.1 (Cycle System)**.** Take a linear matroid $M = (E, I)$ for a matrix A with n columns (as seen in Example 1.1.1). Then, $E = \{1, \ldots, n\} = [n]$. The *cycle space* of M, denoted $\mathrm{Cyc}(M)$, is the defined as the kernel of A, and its elements are called *cycles*[1].

For any $i \in E$, the i-th column of A is called e_i. Any $C \in \mathrm{Cyc}(M)$ has form $C = (c_1, \ldots, c_n) \in K^n$ with

$$\sum_{i\in\{1,\ldots,n\}} c_i e_i = 0.$$

We define the *support* of a cycle C as

$$\mathring{C} = \{i \ : \ c_i \neq 0\}.$$

Let $\mathrm{Cyc}(M) = \mathrm{Span}(\{C_1, \ldots, C_g\})$. For a subset $S \subseteq [g]$, we define the *unique intersection set* $\mathring{C}_S$ by

$$\mathring{C}_S = \{i \in E : \text{for all } j \in S,\ i \text{ appears in exactly one } \mathring{C}_j\}.$$

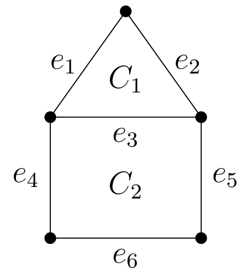
A *cycle system* for M is a basis $\mathcal{C} = \{C_1, \ldots, C_g\}$ of $\mathrm{Cyc}(M)$ such that, for all nonempty subsets $S \subseteq [g]$, the set $\{e_i : i \in \mathring{C}_S\}$ is linearly dependent. We call this the *unique intersection property.*

Definition 2.1.2 (Coparking Function)**.** Take a linear matroid M with a cycle system $\mathcal{C} = \{C_1, \ldots, C_g\}$. An element $a = (a_1, \ldots, a_g) \in \mathbb{Z}^g_{\geq 0}$ is a *coparking function* if, for all nonempty $S \subseteq [g]$, there exists an $i \in S$ such that

$$a_i < |\mathring{C}_S \cap \mathring{C}_i|.$$

Example 2.1.1 (Finding Coparking Functions)**.** Take the labeled graphic matroid with cycle system $\mathcal{C} = \{C_1, C_2\}$ below.

[1]Unfortunately, graph theorists use the word "circuit" to refer to what matroid theorists call a "cycle". Graph theorists also use the word cycle, but its meaning differs from that of matroid cycles. In this paper, we use the standard matroid terminology.



Specifically, $\mathring{C}_1 = \{1,2,3\}$ and $\mathring{C}_2 = \{3,4,5,6\}$. The reader may check that $\mathcal{C}$ is a cycle system. We will now find the associated coparking functions.

S	$\mathring{C}_S$	$\mathring{C}_S \cap \mathring{C}_1$	a_1	$\mathring{C}_S \cap \mathring{C}_2$	a_2
1	$1,2,3$	$1,2,3$	$a_1 < 3$	—	—
2	$3,4,5,6$	—	—	$3,4,5,6$	$a_2 < 4$
$1,2$	$1,2,4,5,6$	$1,2$	$a_1 < 2$	$4,5,6$	$a_2 < 3$

For a vector $a = (a_1, a_2)$ to be considered a coparking function, for each S, it must satisfy *either* the condition on a_1 or the condition on a_2. The resulting conditions for a are:

$$a_1 < 3$$
$$a_2 < 4$$
$$a_1 < 2 \quad \text{or} \quad a_2 < 3.$$

So, the coparking functions for this graphic matroid are

$$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(2,1),(1,2),(0,3),(2,2),(1,3).$$

Note that these coparking functions form a pure multicomplex.

Lemma 2.1.1. *The coparking functions of a linear matroid with a cycle system form a pure multicomplex.*

Proof. Let M be a linear matroid with a cycle system $\mathcal{C} = \{C_1, \dots, C_g\}$, and let P^* denote the set of coparking functions of M.

First, we will show that P^* is a multicomplex. Take $a = (a_1, \dots, a_g) \in P^*$, and let $b \in \mathbb{N}^g$ with $b \le a$ (using the partial ordering defined in Definition 1.3.1). Because $a \in P^*$, there exists i such that

$$a_i < |\mathring{C}_S \cap \mathring{C}_i|.$$

It follows that

$$b_i \le a_i < |\mathring{C}_S \cap \mathring{C}_i|,$$

so $b \in P^*$.

To show that P^* is pure is much more involved. We begin by producing an ordered list of indices of a coparking function a. Let $S = [g]$. Find an $i \in S$ such that $a_i < |\mathring{C}_S \cap \mathring{C}_i|$. This i exists because a is a coparking function. Then, replace S

with $S \setminus \{i\}$ and repeat until $S = \emptyset$. Use the order that we removed i's from $[g]$ to define a list $i_1, \ldots, i_g$.

For $m \in [g]$, define

$$U_m = \mathring{C}_{i_m} \cap \mathring{C}_{\{i_m,\ldots,i_g\}}$$

and see that

$$a_{i_m} < |U_m|.$$

Letting R be the set of bridges in M, we want to show that

$$\bigsqcup_{m=1}^{g} U_m = E \setminus R.$$

Induction on integer j shows that, for all j, the edge $\ell \notin \cup_{m=1}^{j} U_m$ if and only if $\ell \in \mathring{C}_i \cup R$ for some $i \in \{i_{j+1}, \ldots, i_g\}$. (When $j = g$, the latter condition is just $\ell \in R$). Therefore,

$$\begin{aligned} \deg(a) = \sum_{m=1}^{g} a_{i_k} &\leq \sum_{m=1}^{g} (|U_m| - 1) \\ &= |E \setminus R| - g \\ &= |E| - |R| - g. \end{aligned}$$

We see that $|E| - |R| - g$ does not depend on a, and so it is the maximal degree of any coparking function. Now let $c = (c_1, \ldots, c_g)$ where $c_{i_m} = |U_m| - 1$ for each index i_m. This c is a coparking function with maximal degree. Also, $a \leq c$. So, a coparking function is maximal if and only if it has maximal degree. Therefore, P^* is a pure multicomplex with degree $|E| - |R| - g$. □

Having proved Lemma 2.1.1, the multicomplex side of the h-vector conjecture is ready for proving. Now, we have to prepare the h-vector side, specifically with regards to the Tutte polynomial and deletion-contraction.

The following lemma shows that cycle systems are preserved through deletion and contraction. Its proof will use properties of commutative diagrams, short exact sequences, and the snake lemma. For more information about these techniques, see Appendix A.

Lemma 2.1.2. *Consider a linear matroid M with cycle system $\mathcal{C} = \{C_1, \ldots, C_g\}$ and let $k \in \mathring{C}_{[g]} \cap \mathring{C}_g$.*

1. *For $i = 1, \ldots, g-1$ the k-th entry of C_i is 0. Delete this componenet to define the cycle C'_i in $E \setminus \{k\}$. Then $\mathcal{C}' = \{C'_1, \ldots, C'_{g-1}\}$ is a cycle system on $M' = M \setminus k$.*
2. *For non-loop k, let $M'' = M/k$. For $i = 1, \ldots, g$, let C''_i be formed by removing the k-th component from C_i. Then $\mathcal{C}'' = \{C''_1, \ldots, C''_g\}$ is a cycle system for M''.*

Proof. Part (1): Because $\mathcal{C}' \subseteq \mathcal{C}$, it is already linearly independent and has the unique intersection property. It remains to show that $\dim(\mathrm{Cyc}(M')) = g - 1$.

Below is a commutative diagram with exact sequences as rows and columns. We use ι to represent the inclusion map. We use

$$V = \operatorname{Span}(\{e_i : k \neq i \in E\}) = \operatorname{Span}(\{e_i : i \in E\}).$$

These spans are equal because e_k is in a cycle.

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Cyc}(M') & \longrightarrow & K^{n-1} & \xrightarrow{A'} & V & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Cyc}(M) & \longrightarrow & K^{n} & \xrightarrow{A} & V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{cok}(\iota) & \longrightarrow & K & \xrightarrow{A'} & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

By the snake lemma, there is an exact sequence

$$0 \longrightarrow 0 \longrightarrow \operatorname{cok}(\iota) \longrightarrow K \longrightarrow 0,$$

so $\dim(\operatorname{cok}(\iota)) = \dim(K) = 1$. Now we can look at the short exact sequence

$$0 \longrightarrow \dim(\operatorname{Cyc}(M')) \longrightarrow \dim(\operatorname{Cyc}(M)) \longrightarrow \operatorname{cok}(\iota) \longrightarrow 0,$$

which reveals that

$$\begin{aligned}
g &= \dim(\operatorname{Cyc}(M)) \\
&= \dim(\operatorname{Cyc}(M')) + \dim(\operatorname{cok}(\iota)) \\
&= \dim(\operatorname{Cyc}(M')) + 1.
\end{aligned}$$

Hence, $g - 1 = \dim(\operatorname{Cyc}(M'))$. We have shown that $\mathcal{C}'$ has the unique intersection property, is linearly independent, and has cardinality equal to the dimension of $\operatorname{Cyc}(M')$. Therefore, $\mathcal{C}'$ is a cycle system for M'.

Part (2): To show that $\mathcal{C}''$ is a basis, we will show that $g = |\mathcal{C}''| = \dim(\operatorname{Cyc}(M''))$ and that $\mathcal{C}''$ inherits linear independence from $\mathcal{C}$. Then, we will show that the unique intersection property is maintained, and so the basis $\mathcal{C}''$ is a cycle system.

We will once again use a commutative diagram with exact rows and columns. We use π_M for the projection from $\operatorname{Cyc}(M)$ onto $\operatorname{Cyc}(M'')$, which is described in the statement of the lemma. We use π_K for the projection from K^n onto K^{n-1} by removing the vector's k-th entry. Finally, A'' is A with the k-th column removed.

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & K & \longrightarrow & Ke_k & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Cyc}(M) & \longrightarrow & K^n & \xrightarrow{A} & V & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_K} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Cyc}(M'') & \longrightarrow & K^{n-1} & \xrightarrow{A''} & V/Ke_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{cok}(\pi_M) & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

By the snake lemma, the sequence

$$0 \longrightarrow K \longrightarrow Ke_k \longrightarrow \mathrm{cok}(\pi_M) \longrightarrow 0$$

is exact. The vector spaces K and Ke_k is are isomorphic since k is a non-loop, i.e., $e_k \neq 0$. So, by dimension count, $\mathrm{cok}(\pi_M) = 0$. Now, we can look at the exact sequence

$$0 \longrightarrow \mathrm{Cyc}(M) \xrightarrow{\pi_M} \mathrm{Cyc}(M'') \longrightarrow \mathrm{cok}(\pi_M) \longrightarrow 0$$

and conclude by exactness that $\dim(\mathrm{Cyc}(M)) = \dim(\mathrm{Cyc}(M''))$. Therefore, π_M is an isomorphism and

$$|\mathcal{C}''| = g = \dim(\mathrm{Cyc}(M)) = \dim(\mathrm{Cyc}(M'')).$$

Further, because π_M is an isomorphism, $\mathcal{C}''$ inherits linear independence from $\mathcal{C}$. Thus, by dimension argument and linear independence, $\mathcal{C}''$ is a basis for $\mathrm{Cyc}(M'')$.

We will now show that $\mathcal{C}''$ has the unique intersection property. Let $S'' \subseteq \mathcal{C}''$ be nonempty. Now let $S \subseteq \mathcal{C}$ be the corresponding set under π_M. Then, $\mathring{C}''_{S''} = \mathring{C}_S \setminus \{k\}$. Because $\mathcal{C}$ has the unique intersection property, there exists a non-trivial solution to the equation $0 = \sum_{i \in \mathring{C}_S} a_i e_i$ where $a_i \in K$. Because k is not a loop and so $e_k \neq 0$, there exists $j \neq k$ with $a_j \neq 0$. Applying π_M to both sides, we get

$$0 = \sum_{i \in \mathring{C}_S} a_i \pi_M(e_i) = \sum_{i \in \mathring{C}_S \setminus \{k\}} a_i \pi_M(e_i).$$

This is non-trivial relation because of the existence of a_j. Thus, the set $\{e_i : i \in \mathring{C}''_{S''}\}$ is dependent for all $S'' \subseteq \mathcal{C}''$, and so $\mathcal{C}''$ has the unique intersection property. We have shown that $\mathcal{C}''$ is a cycle system for M''. □

The next lemma partitions the coparking functions of a matroid (linear, with a cycle system) into coparking functions that come from its deletion by an edge and coparking functions that come from from its contraction by the same edge.

Lemma 2.1.3. *Let $\mathcal{C} = \{C_1, \ldots, C_g\}$ be a cycle system for a linear matroid M with corresponding set of coparking functions P^*. Let $k \in \mathring{C}_{[g]} \cap \mathring{C}_i$ for some i.*

1. *Let $M' = M \setminus k$ with cycle system $\mathcal{C}'$ as in Lemma 2.1.2 and corresponding coparking functions $P^{*\prime}$. Then, there is an injection $P^{*\prime} \hookrightarrow P^*$ given by*

$$(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_g) \to (a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_g).$$

The image is the set $\{(a_1, \ldots, a_g) \in P^* : a_i = 0\}$. If k is a loop, this injection is a bijection.

2. *If k is a non-loop, let $M'' = M/k$ with cycle system $\mathcal{C}''$ as in Lemma 2.1.2 and corresponding coparking functions $P^{*\prime\prime}$. Then $P^{*\prime\prime} = \{(a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_g) : a = (a_1, \ldots, a_g) \in P^* \text{ and } a_i > 0\}$. Then, there is an injection $P^{*\prime\prime} \hookrightarrow P^*$ given by*

$$(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_g) \to (a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_g).$$

The image is the set $\{(a_1, \ldots, a_g) \in P^* : a_i > 0\}$.

Proof. Without loss of generality, assume $i = g$.

Part (1): First, we need to show that the image is as described by showing containment in both directions. Take $a' = (a_1, \ldots, a_{g-1}) \in P^{*\prime}$ and let $a = (a_1, \ldots, a_{g-1}, 0)$. To show $a \in P^*$, take a nonempty subset $S \subseteq [g]$. If $g \notin S$, then $\mathring{C}_S = \mathring{C}'_S$. Because $a' \in P^{*\prime}$, there exists $j \in S$ such that $a_j < |\mathring{C}'_S \cap \mathring{C}'_j| = |\mathring{C}_S \cap \mathring{C}_j|$. On the other hand, if $g \in S$, then we can simply use $a_g = 0 < |\mathring{C}_S \cap \mathring{C}_g|$. So, $a \in P^*$.

For the opposite direction, consider $a = (a_1, \ldots, a_{g-1}, 0) \in P^*$. Take a nonempty subset $S \subseteq [g-1]$. Since $a \in P^*$, there exists $j \in S$ such that $a_j < |\mathring{C}_S \cap \mathring{C}_j| = |\mathring{C}'_S \cap \mathring{C}'_j|$. Then, $a' = (a_1, \ldots, a_{g-1})$ also has the same a_j. So $a' \in P^{*\prime}$.

We will now show that if k is a loop, the map is a bijection. It is clear from the definition that the map is an injection, so we only need to show that the map is surjective. We will do this by showing that, if k is a loop, $a_i = 0$. We have assumed that $i = g$, so our goal is $a_g = 0$.

Because k is a loop, there exists some $f = (0, \ldots, 0, 1, 0, \ldots, 0) \in \text{Cyc}(M)$ with a 1 in the k-th entry. Since $\mathcal{C}$ is a basis for $\text{Cyc}(M)$,

$$f = \sum_{j=1}^{g} s_j C_j,$$

with $s_j \in K$. But, we assumed in the theorem's statement that $k \in \mathring{C}_{[g]} \cap \mathring{C}_g$, so by the unique intersection properties, the k-th component of any C_j is 0 unless $j = g$. Hence, in order for this summation to equal f as defined, we need $s_g \neq 0$. So, we can define $J = \{j \in [g] : s_j \neq 0,\ j \neq g\}$ and write

$$f = s_g C_g + \sum_{j \in J} a_j C_j.$$

Then, because f has zeroes everywhere except its k-th entry, we know that for all C_j with nonzero non-k entries, there must exist another C_h with a nonzero value in the same entry. So, by the unique intersection property,

$$\mathring{C}_{J\cup\{g\}} = \{k\}.$$

Take a coparking function $a = (a_1, \ldots, a_g)$. There exists $\ell \in J \cup \{g\}$ such that

$$a_\ell < |\mathring{C}_{J\cup\{g\}} \cap \mathring{C}_\ell| = |\{k\} \cap \mathring{C}_\ell|.$$

But,

$$|\{k\} \cap \mathring{C}_\ell| = 0 \qquad \text{if } \ell \neq g$$

and

$$|\{k\} \cap \mathring{C}_\ell| = 1 \qquad \text{if } \ell = g.$$

Coparking functions have only nonnegative entries, so $a_\ell \not< 0$, which leaves only $a_\ell = 0$ with $\ell = g$.

Part (2): Let $j \in [g]$. Recall that C''_j is C_j with the k-th entry removed.

$$\mathring{C}_j = \begin{cases} \mathring{C}''_j & \text{if } j \neq g, \text{ and} \\ \mathring{C}''_j \cup \{k\} & \text{if } j = g. \end{cases}$$

Also, for all nonempty $S \subseteq [g]$,

$$\mathring{C}_S = \begin{cases} \mathring{C}''_S & \text{if } g \notin S, \text{ and} \\ \mathring{C}''_S \cup \{k\} & \text{if } g \in S. \end{cases}$$

Therefore,

$$|\mathring{C}_S \cap \mathring{C}_j| = \begin{cases} |\mathring{C}''_S \cap \mathring{C}''_j| & \text{if } j \neq g \text{ or } g \notin S, \text{ and} \\ |\mathring{C}''_S \cap \mathring{C}''_j| + 1 & \text{if } j = g \text{ and } g \in S. \end{cases}$$

The definition of coparking function requires that, for all S, there exists j such that $a_j < |\mathring{C}_S \cap \mathring{C}_j|$. The result follows. □

Theorem 2.1.1. *The h-vector conjecture holds for linear matroids with cycle systems: Suppose a linear matroid M has a cycle system $\mathcal{C}$ with corresponding coparking functions P^*. Then, the degree vector $\deg(P^*)$ is equal to the h-vector of M, omitting the h-vector's zero entries.*

Proof. We use proof by induction on the number of non-loop, non-bridge edges in E.
Base Case: E consists of bridges and loops only.

Suppose the matroid has b bridges and g loops. In that case, there is a cycle system $\mathcal{C} = \{C_i \mid i = 1, \ldots, g\}$, where C_i is the i-th standard basis vector in K^n. Then, for nonempty $S \subseteq [g]$, we have $\mathring{C}_S = S$. So, $|\mathring{C}_S \cap \mathring{C}_i| = 1$ for all i. We therefore know that a coparking function $(a_1, \ldots, a_g)$ has $a_i < 1$ for all i, so the only coparking function is the zero vector, which on its own is a pure multicomplex.

Note that, when $g = 0$ we have $K^g = \{()\}$. The degree $()$ is 0, and we have one element of this degree, hence $\deg(P_M*) = 1$.

Inductive Step: Let $k \in \mathring{C}_{[g]} \cap \mathring{C}_i$ be a non-loop, non-bridge edge. Assume the theorem holds for $M \setminus k$ and M/k.

We will work through deletion and contraction using notation from Lemma 2.1.3. Let $\ell = |E| - g - \#\{\text{bridges}\}$. Then, for M, M', and M'', we have h-vectors, omitting any final zeros,

$$h = (h_0, \ldots, h_\ell), \quad h' = (h'_0, \ldots, h'_\ell), \quad h'' = (h''_0, \ldots, h''_{\ell-1}),$$

and degree vectors

$$\deg(P^*) = (d_0, \ldots, d_\ell), \quad \deg(P^{*\prime}) = (d'_0, \ldots, d'_\ell), \quad \deg(P^{*\prime\prime}) = (d''_0, \ldots, d''_{\ell-1}).$$

From Lemma 2.1.3 and the definition of multicomplex degree, we see that

$$\deg(P^*) = (d'_0, d'_1 + d''_0, \ldots, d'_\ell + d''_{\ell-1}).$$

(The entries are "offset" here because moving from M'' to M increases the degree of each coparking function by 1, as seen in Lemma 2.1.3.)

Let $T(x, y)$, $T'(x, y)$, and $T''(x, y)$ denote the Tutte polynomials of M, M', and M'', respectively. By the rules of deletion and contraction, we have

$$T(x, y) = T'(x, y) + T''(x, y),$$

which expands as

$$\sum_{i=0}^{\ell} h_{\ell-i} x^i = \sum_{i=0}^{\ell} h'_{\ell-i} x^i + \sum_{i=0}^{\ell-1} h''_{\ell-1-i} x^i$$

and shows us that

$$(h_0, \ldots, h_\ell) = (h'_0, h'_1, \ldots, h'_\ell) + (0, h''_0, \ldots, h''_{\ell-1}).$$

Our inductive step assumes that $h' = \deg(P^{*\prime})$ and $h'' = \deg(P^{*\prime\prime})$, hence,

$$\begin{aligned} h &= (h_0, \ldots, h_\ell) = (h'_0, h'_1, \ldots, h'_\ell) + (0, h''_0, \ldots, h''_{\ell-1}) \\ &= (h'_0, h'_1 + h''_0, \ldots, h'_\ell + h''_{\ell-1}) = (d'_0, d'_1 + d''_0, \ldots, d'_\ell + d''_{\ell-1}) = \deg(P^*). \end{aligned}$$

□

2.2 The Deletion-Contraction Tree

For a linear matroid M, a deletion-contraction diagram is constructed as a binary tree (see Figure 1.1). At each node N, beginning with $N = M$, a non-loop, non-bridge edge e is selected. The left child of that node is $N \setminus e$ and the right child is N/e. As shown in Lemma 2.1.2, for linear matroids with cycle systems, the children $N \setminus e$ and N/e are also linear matroids with cycle systems. In Theorem 2.2.1, we define a method of selecting which edge to delete/contract by and a method of labelling the branches to create a bijection between bases and coparking functions.

Before we can create this bijection, we have to show that deleting or contracting by an edge creates a partition of bases. (We already showed in Lemma 2.1.3 that this creates a partition of coparking functions.)

Lemma 2.2.1. *Let $M = (E, I)$ be a matroid.*

1. *For a non-bridge edge e of M, the bases of M not containing e are exactly the bases of $M \setminus e$.*
2. *For a non-loop edge e of M, the bases of M containing e are exactly the bases of M/e with e added.*

Proof. Part (1): Let B be a basis for $M \setminus e$. Since e is non-bridge, $\text{rank}(M \setminus e) = \text{rank}(M)$. So, B is an independent subset of M with $|B| = \text{rank}(M)$, and thus is a basis for M.

For the other direction, note that deleting edges does not create new dependence relations and use rank argument.

Part (2): Take B a basis for M containing e. Then, $B' = B \setminus \{e\}$ is independent in M/e by definition of contraction. If there exists $f \in E \setminus B$ such that $B' \cup \{f\}$ is independent in M/e, then $B' \cup \{e\} \cup \{f\}$ is a basis for M, which contradicts the fact that B is a maximal independent set. Therefore, B' is a maximal independent set in M/e.

For the other direction, take a basis B' in M/e and let $B = B' \cup \{e\}$. This B is independent in M. If there exists $f \in E \setminus B$ such that $B \cup \{f\}$ is independent, then $B \cup \{f\} \setminus \{e\} = B' \cup \{f\}$ must be independent in M/e. This contradicts the supposed maximality of B'. Therefore, no such f exists, and so B is a basis for M. □

Theorem 2.2.1. *Let M be a linear matroid with a cycle system. Then, a choice of a cycle system and a total ordering of the ground set for M determines a bijection between the bases of M and its coparking functions.*

Proof. Fix a cycle system D for M and a total ordering of the ground set (not necessarily the standard ordering). We use this to construct a uniquely determined deletion-contraction diagram for M. The tree is constructed iteratively at each node N with cycle system $\mathcal{C}$, beginning at $N = M$ with $\mathcal{C} = D$ and using the ground set order we fixed above.

Let $N = (E, I)$ be a linear matroid with $E = [n]$ and cycle system $\mathcal{C} = (C_1, \ldots, C_g)$.

Take $S = [g]$ and let $k \in E$ be the maximal element of $\mathring{C}_S$. Then, there is a unique $i \in S$ for which $k \in \mathring{C}_i$. If k is a loop, replace S with $S \setminus \{i\}$ and repeat until we arrive at some non-loop k or we reach $S = \emptyset$. If $S = \emptyset$, then E consists only of bridges and loops, and so N is a leaf of the deletion-contraction diagram. Otherwise, if we arrived at a non-loop k, it is a uniquely determined $k \in \mathring{C}_S$ for some uniquely determined $S \subseteq [g]$. We know that k cannot be a bridge because $k \in \mathring{C}_S$. Because k is not a loop or bridge, we can define the direct children of N as $N \setminus k$ on the left and N/k on the right. Recall that k was determined by some i for which $k \in \mathring{C}_S \cap \mathring{C}_i$.

We label the branch between N and N/k as k, C_i. Both $N \setminus k$ and N/k inherit cycle systems from $\mathcal{C}$, as described in Lemma 2.1.2. They also inherit the total ordering on the ground set.

Now that we have constructed the tree, we can assign each leaf a coparking function by tracing the path from the leaf up to the root M. Each contraction is labeled with some C_i. The leaf's associated coparking function is

$$a = (\#\{C_1\text{'s in path}\}, \ldots, \#\{C_g\text{'s in path}\})$$

(where $\{C_1, \ldots, C_g\}$ is the cycle system in M).

To show rigorously that a is a coparking function, we have to be more technical in our definition of a. For a leaf, trace the path up to the root M. The leaf is a linear matroid with a cycle system, and so has a coparking function a with all 0's (a may be empty). Each entry of a is associated with some C_j, and the entries are ordered according to the j's. (However, a_j may not be associated to C_j quite yet.) At each step up the tree, an edge was deleted or contracted from a cycle C_i. If the edge was deleted, add a new 0 entry to a such that the order of the entries according to their cycles is preserved. If the edge was contracted, simply add 1 to the i'th entry. By Lemma 2.1.3, a will remain a coparking function throughout the process. This cumbersome definition is directly equivalent to the simple one above; the only hiccup is to show that a ends with g entries. This is true because deleting an edge removes an element of the cycle system, whereas contracting an edge does not. So, the leaf had $g - d$ cycles, where d was the number of times an edge was deleted to create it. From the leaf, a starts with $g - d$ entries, and the d missing entries are added back as we move up the tree.

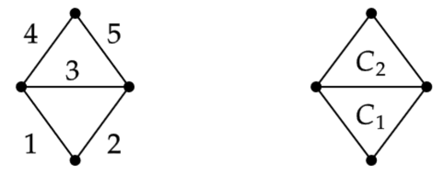
These coparking functions are unique for each leaf. When comparing the coparking functions associated with two leaves, we can count the C_i's in their paths, starting from the root. Say the leaves' most recent common ancestor is N. Upon reaching N, the leaves so far have the same coparking function, $a = (a_1, \ldots, a_g)$. Moving left (deleting) from N destroys a cycle $C_j \in \mathcal{C}$, locking a_j for the rest of the count. Moving right (contracting) from N retains C_j and adds 1 to a_j. So, any leaf to the left of N will have a different value for a_j than any leaf to the right of N.

We can also assign each leaf a basis for M. Starting with the set of all bridges in the leaf which is a basis for the leaf. Complete a basis for M by adding all edges which were contracted in the creation of the leaf. This is a basis by the definitions of independence under deletion and contraction.

To see that the bases are unique, find each node's common ancestor and apply Lemma 2.2.1.

By Lemma 2.1.3, we have accounted for all coparking functions of M. By Lemma 2.2.1, we have accounted for all bases of M. Thus, we have created a bijection. □

Example 2.2.1. Take the diamond graph with edges and cycles labeled as follows:



The process described in the above proof yields Figure 2.1.

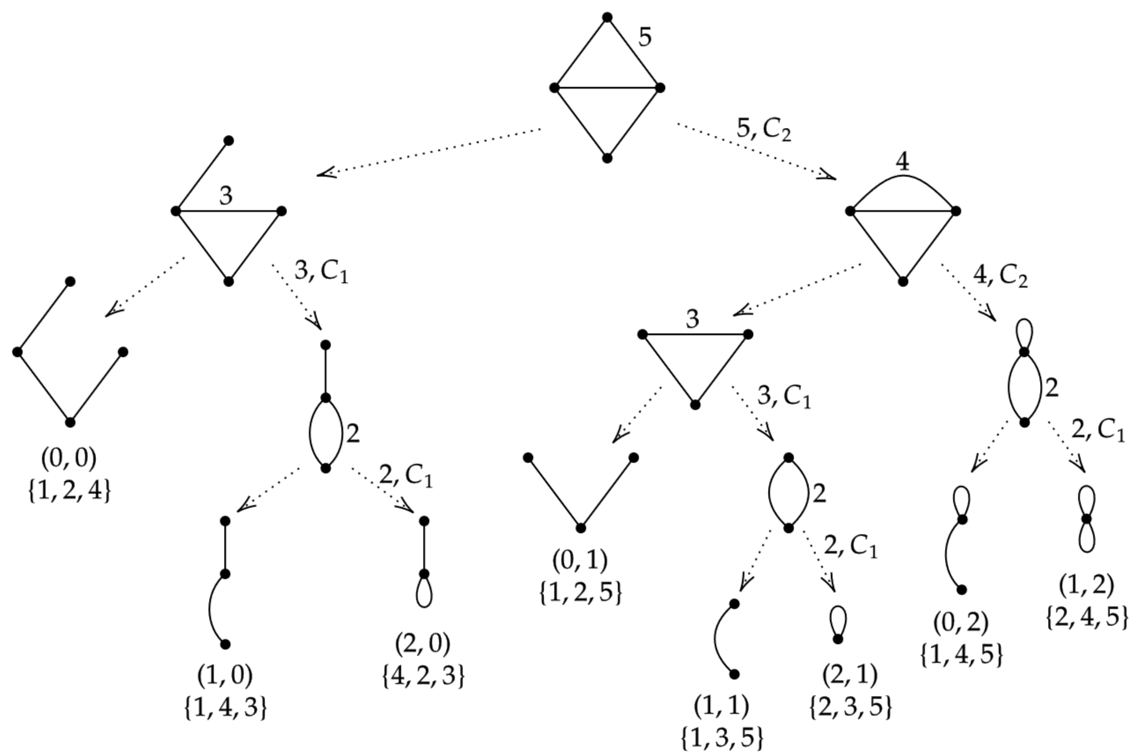


Figure 2.1: The diamond graph deleted and contracted according to the proof of Theorem 2.2.1, with leaves labeled by their corresponding coparking functions and bases.

Conclusion

Theorem 2.1.1 establishes the h-vector conjecture for linear matroids with cycle systems. From here, there are many open questions. Some of particular interest to us are:

1. It would be nice to see a characterization of linear matroids with cycle systems. Not every linear matroid has a cycle system. Also, there exist linear matroids with cycle systems that are not graphic, for example, $M^*_{K_{3,3}}$ and $M^*_{K_5}$.

2. Let M be a linear matroid with a cycle system. Does there exist an ordering of the groundset so that our bijection ϕ: Bases $\rightarrow$ Coparking Functions has the property
$$\mathrm{ip}(B) = \deg(\phi(B))$$
where $\mathrm{ip}(B)$ is the internal passivity of a basis B? We know of no counterexamples. (See TP III for definition of internal passivity.)

3. Is there a way to generalize the notion of cycle system in order to expand the proof to all linear matroids?

4. Can we generalize the notion of a cycle system to all matroids?

Appendix A

Exact Sequences and the Snake Lemma

This appendix provides a brief explanation of short exact sequences and the snake lemma, which are used in the proof of Lemma 2.1.2. These techniques are common throughout mathematics and will be stated without proof.

A sequence of mappings of vector spaces

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is *exact* at B if $f(A) = \ker(g)$. A sequence of linear mappings

$$\cdots \longrightarrow A_{i-1} \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

is exact if it is exact at each A_j.

A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

If f is injective, then g is surjective, and $C \cong \mathrm{cok}(g) = B/f(A) = B/\ker(g)$. We also then have

$$\dim(B) = \dim(A) + \dim(C).$$

The *snake lemma* describes a property of the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & & \end{array}$$

Consider expanding the diagram to

$$
\begin{array}{ccccccccc}
 & & \ker(a) & \longrightarrow & \ker(b) & \longrightarrow & \ker(c) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{cok}(a) & \longrightarrow & \mathrm{cok}(b) & \longrightarrow & \mathrm{cok}(c) & &
\end{array}
$$

Then, there exists a map $\ker(c) \to \mathrm{cok}(a)$ such that the sequence

$$\ker(a) \longrightarrow \ker(b) \longrightarrow \ker(c) \longrightarrow \mathrm{cok}(a) \longrightarrow \mathrm{cok}(b) \longrightarrow \mathrm{cok}(c)$$

is exact.

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