SOME FACETS OF THE POLYTOPE OF EVEN PERMUTATION MATRICES

JEFFREY HOOD AND DAVID PERKINSON

Abstract. We describe a class of facets of the polytope of convex combinations of the collection of even $n \times n$ permutation matrices. As a consequence, we prove the conjecture of Brualdi and Liu [4] that the number of facets of the polytope is not bounded by a polynomial in n.

1. Introduction

The convex hull of the collection of all $n \times n$ permutation matrices is the Birkhoff polytope, $P(\mathcal{S}_n)$. Its structure is well-understood: Birkhoff [2] showed that a real $n \times n$ matrix is in $P(\mathcal{S}_n)$ if and only if it is doubly stochastic, i.e., its entries are non-negative and its row and columns sums all 1. The the face lattice of $P(\mathcal{S}_n)$ is easily described in terms of so-called elementary subgraphs of the complete bipartite graph with n vertices. Good entry points to the literature are [3] and [1].

In this note we consider the polytope $P(\mathcal{A}_n)$, which is the convex hull of just the *even* permutation matrices. In contrast to the closely related Birkhoff polytope, not much is known about its structure. Mirsky [8] is the first to have studied $P(\mathcal{A}_n)$; he considered the (still open) problem, first proposed by A. J. Hoffman, of deciding whether a given $n \times n$ matrix is in $P(\mathcal{A}_n)$. Mirsky provided necessary conditions later shown by von Below [9] to be insufficient if $n \geq 4$. Brualdi and Liu [4] produced several more necessary conditions, calculated the dimension of $P(\mathcal{A}_n)$, characterized its edges, and showed that any two vertices are connected by a path consisting of at most two edges. At the end of their paper, Brualdi and Liu make conjectures, the first of which is that the number of linear constraints necessary to define $P(\mathcal{A}_n)$ is not bounded by a polynomial in n. Our main result, Theorem 2.3 and Corollary 2.4, is to prove this conjecture by explicitly describing a large family of facets on $P(\mathcal{A}_n)$.

2. Results

We assume that the reader is familiar with the basic theory of polytopes as presented, for example, in [10]. In particular, a *polytope* $P \subset \mathbb{R}^d$ is the convex hull of a finite set of points; equivalently, it is the bounded intersection of a finite collection of linear halfspaces. The *dimension* of P, denoted $\dim P$,

Date: 8/18/03.

is defined to be the dimension of its affine span. If $c \in \mathbb{R}^d$ is any vector and c_0 is a real number such that $c \cdot x \leq c_0$ for all $x \in P$, then the set $\{x \in P \mid c \cdot x = c_0\}$ is called a *face* of P. Each face of P is a polytope. *Vertices* are faces of dimension 0, and *facets* are faces of dimension one less than that of P. Each face is the convex hull of its vertices, and these vertices are also vertices of P.

Let $\rho \colon \mathcal{S}_n \to \mathrm{GL}(\mathbb{R}^n)$ be the standard permutation representation of the symmetric group $\mathcal{S}_n$. We identify $\mathrm{GL}(\mathbb{R}^n)$ with the Euclidean space of $n \times n$ matrices, $\mathbb{R}^{n\times n} \approx \mathbb{R}^{n^2}$, so that $\rho(\sigma)_{ij} = 1$ if $\sigma(i) = j$ and $\rho(\sigma)_{ij} = 0$, otherwise. In the following, we identify each permutation with its corresponding matrix and drop mention of ρ. Let $\mathcal{A}_n$ denote the even permutations. Multiplying by the transposition (12), we get the odd permutations $(12)\mathcal{A}_n$. We find it convenient to work with the polytope $P((12)\mathcal{A}_n)$, i.e., the convex hull of the odd permutation matrices. It is isomorphic to $P(\mathcal{A}_n)$ in the sense that there is a linear function, induced by multiplication by (12), which maps $P(\mathcal{A}_n)$ bijectively onto $P((12)\mathcal{A}_n)$. Such a mapping preserves the facial structure of the polytopes.

With each k-subset $I \colon i_1 < \cdots < i_k$ of $[n] := \{1, \ldots, n\}$, associate the k-cycle $\sigma_I = (i_1 \ldots i_k) \in \mathcal{S}_n$. We say that σ_I is *monotone* since it has only one *descent*: there is only one $i \in [n]$ such that $\sigma_I(i) > \sigma_I(i+1)$. Let F denote the collection of all cycles associated with subsets of even cardinality—the collection of all odd monotone cycles.

Lemma 2.1. *The set F contains exactly the vertices of a face of $P((12)\mathcal{A}_n)$.*

Proof. The face-defining inequality is $D \cdot x \leq n - 1$ where D is the $n \times n$ upper-triangular matrix filled with 1s: $D_{ij} = 1$ if $i \leq j$, and $D_{ij} = 0$, otherwise. (Here, $D \cdot x$ denotes the dot-product of vectors D and x in $\mathbb{R}^{n^2}$, not matrix multiplication.) The inequality is obvious because every odd permutation moves points. Further, an odd permutation matrix σ satisfies the equality $D \cdot \sigma = n - 1$ if and only if it has a single descent. $\square$

Next, we wish to show that F is the set of vertices of a *facet* of $P((12)\mathcal{A}_n)$ provided $n \geq 6$. In [4], it is shown that the dimension of $P(\mathcal{A}_n)$ is the same as that of the corresponding Birkhoff polytope, $(n-1)^2$, provided $n \geq 4$. So we must show that the affine span of F has dimension $(n-1)^2 - 1$. As a preliminary step, for each $k < n$, define the *k-subset polytope* to be the convex hull of all permutation matrices corresponding to monotone k-cycles:

$$P\binom{n}{k} := \mathrm{conv}\{\sigma_I \in \mathcal{S}_n \mid I \text{ a } k\text{-subset of } [n]\} \subset \mathbb{R}^{n\times n}.$$

Lemma 2.2. $\dim P\binom{n}{2} = n(n-1)/2 - 1$ *and* $\dim P\binom{n}{k} = n(n-k) - 1$ *for* $k \geq 3$.

Proof. For the case of $k = 2$, note that the $n \times n$ transposition matrices are linearly independent. Since $0 \notin P\binom{n}{2}$, it follows $\dim P\binom{n}{2} = n(n-1)/2 - 1$.

Now consider the case where $k \geq 3$. Let V denote the complex linear span of $P\binom{n}{k}$ in $\mathbb{C}^{n\times n}$. It suffices to show that $\dim V = n(n-k)$. Let c be the linear automorphism of $\mathbb{C}^{n\times n}$ that cycles a matrix's columns to the left and its rows up: $c(X)_{uv} = X_{(u+1,v+1)}$ with indices modulo n. Then c generates a cyclic group of order n that acts on the space V. Let $\omega = e^{2\pi i/n}$ be a primitive n-th root of unity, and consider the eigenspaces $V_j = \{v \in V \mid c\,v = \omega^j v\}$. For each j there is a linear projection $\pi_j \colon V \twoheadrightarrow V_j$ defined by $\pi_j := \sum_{\ell=0}^{n-1} \omega^{j\ell} c^\ell$. Since $v = \sum_j \pi_j(v)/n$, it is clear that $V = \oplus_{j=0}^{n-1} V_j$. Moreover,

$$\begin{array}{rcl} V_0 & \rightarrow & V_j \\ v & \mapsto & \mathrm{diag}(1, \omega^j, \omega^{2j}, \ldots, \omega^{(n-1)j})\; v \end{array}$$

is a linear isomorphism. Hence, it suffices to show that $\dim V_0 = n-k$, which we now do by exhibiting a basis.

To describe the basis, we first associate with each monotone k-cycle of the form $\sigma = (1, i_2, i_3, \ldots, i_k)$ the partition of n into k parts,

$$p := (i_2 - 1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k + 1).$$

The following list of partitions of n then corresponds to a basis for V_0. The list occurs in two sections divided by the dotted line. Each of the $n-k$ rows in the list is a partition of n into and k parts, and 1^{k-3} denotes $k-3$ copies of 1:

p_1	1^{k-3}	1	1	$n-k+1$
p_2	1^{k-3}	1	2	$n-k$
p_3	1^{k-3}	1	3	$n-k-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p_{\lfloor (n-k)/2 \rfloor + 1}$	1^{k-3}	1	$\lfloor \frac{n-k}{2} \rfloor + 1$	$\lceil \frac{n-k}{2} \rceil + 1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$p_{\lfloor (n-k)/2 \rfloor + 2}$	1^{k-3}	2	2	$n-k-1$
$p_{\lfloor (n-k)/2 \rfloor + 3}$	1^{k-3}	2	3	$n-k-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_{n-k}	1^{k-3}	2	$\lfloor \frac{n-k+1}{2} \rfloor$	$\lceil \frac{n-k+1}{2} \rceil$

Since each matrix in V_0 is invariant under c, the mapping $\phi \colon V_0 \to \mathbb{C}^n$ that sends each matrix in V_0 to its first row is a linear isomorphism. For each partition p in the above list, define $X_p := \phi(\pi_0(\sigma_p))$. The first component of X_p is $n-k$, and for $i > 1$, the i-th component of X_p is the number of parts of p of size $i-1$. For example, if $n = 12$ and $k = 5$, we have the following data:

partition	k-cycle	X_p
11118	(12345)	740000001000
11127	(12346)	731000010000
11136	(12347)	730100100000
11145	(12348)	730011000000
...	...	...
11226	(12357)	722000100000
11235	(12358)	721101000000
11244	(12359)	721020000000

In the general case, suppose there were a linear relation $\sum_{\ell=1}^{n-k}\alpha_\ell X_{p_\ell} = 0$ among cycles in our putative basis, partitions labeled as above. Then $\alpha_1 = \alpha_2 = 0$, because X_{p_1} and X_{p_2} are the unique vectors corresponding to partitions with parts of sizes $n-k+1$ and $n-k$, respectively. We also have that

$$\alpha_3 = -\alpha_{\lfloor (n-k)/2\rfloor+2}$$

since the corresponding partitions are the only in our list with a part of size $n-k-1$. Next,

$$\alpha_{\lfloor (n-k)/2\rfloor+3} = -\alpha_3$$

since the corresponding partitions are the only in our list with a part of size 3. Similarly,

$$\alpha_4 = -\alpha_{\lfloor (n-k)/2\rfloor+3}$$

comparing parts of size $n-k-2$. Continuing in this way, back and forth between partitions above and below the dotted line, we see that all coefficients corresponding to partitions listed above the line are equal and all coefficients for partitions below the line are equal with opposite value. However, only partitions below the dotted line have parts of size 2. Therefore, all α_ℓ must be zero. This shows that the X_p for the listed partitions are linearly independent, and thus, $\dim V_0 \geq n-k$.

On the other hand, if X is the first row of any element of V_0 (recall the isomorphism, ϕ, mentioned earlier), then X satisfies the following k linear relations:

$$\begin{aligned} X_i &= 0, \qquad \text{for } i = n-k+3,\ldots,n;\\ X\cdot\left(-\tfrac{k}{n-k},1,1,\ldots,1\right) &= 0;\\ X\cdot\left(-\tfrac{n}{n-k},1,2,\ldots,n-1\right) &= 0. \end{aligned}$$

This shows that $\dim V_0 \leq n-k$, completing the proof. □

Theorem 2.3. *The collection F of odd monotone cycles is the set of vertices for a facet of $P((12)\mathcal{A}_n)$.*

Proof. Recall that by Lemma 2.1, F is the set of vertices for a face of $P((12)\mathcal{A}_n)$, so we just need to show that the dimension of this face is $\dim P((12)\mathcal{A}_n) - 1 = (n-1)^2 - 1$. From Lemma 2.2, the dimension of the affine span of the 4-cycles in F is $n^2 - 4n - 1$. The affine span of the 4-cycles in F satisfies the following equalities:

$$\sum_{i=1}^{n} X_{ij} = \sum_{i=1}^{n} X_{ji} = 1, \quad \text{for } j = 1, \ldots, n; \tag{1}$$

$$\sum_{i=1}^{n} \sum_{j=i}^{n} X_{ij} = n - 1; \tag{2}$$

$$\sum_{i=1}^{n} X_{ii} = n - 4; \tag{3}$$

$$X_{i,i-1} = X_{i,i-2} = 0, \quad \text{indices modulo } n. \tag{4}$$

The equations in (1) say that the row and column sums are 1; equation (2) assures that the cycles have one descent (cf. Lemma 2.1); equation (3) says a 4-cycle has $n - 4$ fixed points; and equation (4) says that the size of the descent in a monotone 4-cycle must be greater than two. For instance, there is no monotone 4-cycle of the form $(42ij)$ or $(43ij)$. The equations in (1) represent $2n-1$ independent conditions. The rest are clearly independent of these and each other since it is easy to find examples of permutation matrices satisfying some of the equations and not others. Since the dimension of the space defined by the equations is the same as the affine span of the 4-cycles, they define the affine span.

The cycle (14) is not in the affine span of the monotone 4-cycles since it does not satisfy (3). In addition, the 2-cycles of the form $(i-1, i)$ and $(i-2, i)$ each uniquely violate one equation in (4) and are linearly independent of each other. Hence, altogether, the monotone 4-cycles, (14), and the $2n$ transpositions just described, which are all members of F, span an affine space of dimension $(n - 1)^2 - 1$, as required. □

Having found one facet of $P((12)\mathcal{A}_n)$, the symmetries of $P((12)\mathcal{A}_n)$ can be used to produce several more. First, for each $\sigma \in \mathcal{S}_n$, conjugation by the corresponding permutation matrix defines a linear automorphism of $\mathbb{R}^{n\times n}$ mapping $P((12)\mathcal{A}_n)$ bijectively to itself. Therefore, $F_\sigma := \sigma F \sigma^{-1}$ is the set of vertices for a facet of $P((12)\mathcal{A}_n)$. Effectively, F_σ is obtained from F by permuting the labels $\{1, \ldots, n\}$ in each cycle. Next, multiplication on the left by any $\tau \in \mathcal{A}_n$ also gives a linear automorphism of $\mathbb{R}^{n\times n}$ fixing $P((12)\mathcal{A}_n)$. We now come to our main result.

Corollary 2.4. *Let $\mathcal{S}_n^1$ denote the stabilizer of 1 in $\mathcal{S}_n$. For $n \geq 6$ the collection $\{\tau F_\sigma\}$ for $\tau \in \mathcal{A}_n$ and for $\sigma \in \mathcal{S}_n^1$ consists of sets of vertices for $n!(n - 1)!/2$ distinct facets of $P((12)\mathcal{A}_n)$.*

Proof. Suppose $\tau F_\sigma = F_\mu$ for some $\tau \in \mathcal{A}_n$ and $\sigma, \mu \in \mathcal{S}_n^1$. We wish to show that $\tau = ()$, the identity permutation, and $\sigma = \mu$. To see $\tau = ()$, note that the sets F_σ and F_μ consist solely of cycles, each including all the transpositions. If $\tau \neq ()$, then τF_σ would contain a non-cycle, i.e., an element whose cycle decomposition contains more than one non-trivial cycle. If τ had at least two fixed points, then the product of τ with the transposition of those two points would be in τF_σ. If the cycle decomposition of τ consisted solely of transpositions and τ had fewer than two fixed points, then the number of transpositions in the decomposition would be at least 4 since $n \geq 6$ and τ is even. The product of τ with one of these transpositions would be a non-cycle in τF_σ. Finally, if the cycle decomposition of τ contained a cycle of length at least 3, then the product of τ with a transposition of two of the moved points of that cycle would be a non-cycle. Thus, $\tau = ()$.

The set F consists of monotone cycles, i.e., cycles fixing the cyclical ordering $1 \to 2 \to \cdots \to n \to 1$. The sets F_σ and F_μ are obtained from F by relabeling the elements $\{1, \ldots, n\}$. Any cyclical relabeling would leave F fixed. That is why we take σ and μ in the stabilizer, $\mathcal{S}_n^1$. Further, any two relabelings fixing 1 would be distinguished by the 4-cycles in F. Hence, $\sigma = \mu$ if $F_\sigma = F_\mu$. □

More generally, if $\sigma, \tau \in \mathcal{S}_n$ have the same parity, the mapping $x \mapsto \sigma x \tau^{-1}$ gives an automorphism of $\mathbb{R}^{n \times n}$ preserving $P((12)\mathcal{A}_n)$. The facet whose vertices are $\sigma F \tau^{-1}$ is included in the list above since $\sigma F \tau^{-1} = \sigma\tau F_\tau$. (The fact that τ may not fix 1 is not important, as noted in the proof, above.)

Recall that $P((12)\mathcal{A}_n)$ and $P(\mathcal{A}_n)$ are the same polytope up to a linear change of coordinates corresponding to multiplication by any $n \times n$ odd permutation matrix. Therefore, $\{\tau F_\sigma\}$ with $\tau \in \mathcal{S}_n \setminus \mathcal{A}_n$ and $\sigma \in \mathcal{S}_n^1$ is a set of facets of $P(\mathcal{A}_n)$.

3. Concluding Remarks

The polytope $P(\mathcal{A}_n)$ is an example of a *permutation polytope*: the convex hull of a finite group of permutation matrices. Our limited computer experimentation suggests that among these, $P(\mathcal{A}_n)$ is a complex object. Up to isomorphisms of permutation groups, there are 5 transitive permutation groups of degree 5: a cyclic group of order 5, a dihedral group of order 10, a Frobenius group of order 20, the alternating group $\mathcal{A}_n$, and the full cyclic group $\mathcal{S}_n$. Their corresponding polytopes have 5, 25, 625, 8665, and 25 facets, respectively.

The facets of most permutation polytopes corresponding to transitive permutation groups of degree 6 were easily computed using the programs *lrs* [7] or *porta* [5]. However, we were unable to compute the facets of $P(\mathcal{A}_6)$. The program *lrs*, using the reverse search algorithm of Avis and Fukuda uses little space and can run indefinitely, slowly calculating bounding hyperplanes for facets of $P(\mathcal{A}_6)$. Reed College student, Dag Arneson, computed orbits of these facets under symmetries of $\mathcal{A}_n$ (cf. proof of Corollary 2.4). As of this writing, over 4 million facets have been computed in this way. Of these, there are 36 facets with 300 vertices each. These are remnants of the facets of the Birkhoff polytope, having supporting hyperplanes $x_{ij} = 0$ for $i, j \in \{1, \ldots, 6\}$. The number of vertices on the other facets we computed ranged from 26 to 57. Corollary 2.4 accounts for only $43,200$ of these facets (each with 31 vertices).

While the number of facets of $P(\mathcal{A}_n)$ is not bounded by a polynomial in n, it is still an open question whether there is a polynomial time algorithm for determining if a given matrix lies in $P(\mathcal{A}_n)$. Brualdi and Liu [4] conjecture that no such algorithm exists and show that the standard algorithm for writing a doubly stochastic matrix as a convex combination of permutation matrices does not readily generalize. In a note added in proof, Brualdi and Liu mention an unpublished manuscript of A. B. Cruse. In that work, Cruse reformulated the problem of determining whether a given matrix is in $P(\mathcal{A}_n)$ into the problem of computing vertices of certain blocking polyhedra (whose complexity is not determined). Cruse shows how blocking polyhedra can be used to construct $P(\mathcal{A}_n)$ by removing points from the Birkhoff polytope one at a time, taking the convex hull of the remaining vertices. It was this approach that inspired our work. By computer calculation, we noted that if a single vertex (permutation matrix), v, is removed from $P(\mathcal{S}_n)$ in this way, remnants of the original n^2 facets of $P(\mathcal{S}_n)$ remain and $(n-1)!$ new facets are formed. Remnants of these facets remain even after removing all vertices with the same parity as v. Corollary 2.4 shows that in this way, we account for $n!(n-1)!/2$ new facets, $(n-1)!$ for each removed vertex. We did not explore the facets that arise from $P(\mathcal{S}_n)$ after removing pairs of vertices.

The open problem of most interest to us is to describe all of the facets and perhaps even the face structure of $P(\mathcal{A}_n)$. In addition, the k-subset polytopes that are introduced in this paper seem natural to us and worthy of further consideration. Finally, we mention that in addition to *lrs* and *porta*, we made extensive use of the computational group theory program GAP [6] in obtaining our results.

References

[1] Louis J. Billera and A. Sarangarajan. The combinatorics of permutation polytopes. In *Formal power series and algebraic combinatorics (New Brunswick, NJ, 1994)*, volume 24 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 1–23. Amer. Math. Soc., Providence, RI, 1996.

[2] Garrett Birkhoff. Three observations on linear algebra. *Univ. Nac. Tucumán. Revista A.*, 5:147–151, 1946.

[3] Richard A. Brualdi and Peter M. Gibson. Convex polyhedra of doubly stochastic matrices. I. Applications of the permanent function. *J. Combinatorial Theory Ser. A*, 22(2):194–230, 1977.

[4] Richard A. Brualdi and Bo Lian Liu. The polytope of even doubly stochastic matrices. *J. Combin. Theory Ser. A*, 57(2):243–253, 1991.

[5] Thomas Christof. *PORTA – a polyhedron representation transformation algorithm, (revised by Andreas Loebel and Mechthild Stoer).* ZIB electronic library eLib. (http://www.zib.de/Optimization/Software/Porta/).

[6] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.3*, 2002. (http://www.gap-system.org).

[7] *lrs.* (http://cgm.cs.mcgill.ca/~avis/C/lrs.html).

[8] L. Mirsky. Even doubly-stochastic matrices. *Math. Ann.*, 144:418–421, 1961.

[9] Joachim von Below. On a theorem of L. Mirsky on even doubly-stochastic matrices. *Discrete Math.*, 55(3):311–312, 1985.

[10] Günter M. Ziegler. *Lectures on polytopes.* Springer-Verlag, New York, 1995.

Reed College, Portland Oregon