COUNTING WEIGHTED MAXIMAL CHAINS IN THE CIRCULAR BRUHAT ORDER

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Abstract. The totally nonnegative Grassmannian $\mathrm{Gr}(k,n)_{\geq 0}$ is the subset of the real Grassmannian $\mathrm{Gr}(k,n)$ consisting of points with all nonnegative Plücker coordinates. The circular Bruhat order is a poset isomorphic to the face poset of Postnikov's positroid cell decomposition of $\mathrm{Gr}(k,n)_{\geq 0}$ [Pos05]. We provide a closed formula for the sum of its weighted chains in the spirit of Stembridge [Ste02].

1. Introduction.

Let S_n be the symmetric group on $[n] := \{1,\dots,n\}$. An *inversion* of $\pi \in S_n$ is a pair $i, j \in [n]$ such that $i < j$ and $\pi(i) > \pi(j)$. The number of inversions of π is its *length*, denoted $\ell(\pi)$. The *Bruhat order* on S_n is a partial ordering on S_n, graded by length. It arises in geometry as the face poset for the Schubert decomposition of the variety of complete flags in $\mathbb{C}^n$. Its cover relations have the form $\pi s_{ij} \lessdot \pi$ where $s_{ij} := (i,j)$ is a transposition such that $\ell(\pi) = \ell(\pi s_{ij}) + 1$. The maximal element of the Bruhat order, written in row notation, is $\pi_{\text{top}} = [n, n-1, \dots, 1]$ of length $r := \binom{n}{2}$ and the smallest element is the identity permutation $\mathrm{id} = [1, 2, \dots, n]$ of length 0. In the Bruhat order, each maximal chain has the form $\mathrm{id} = \pi_0 \lessdot \pi_1 \lessdot \dots \lessdot \pi_r = \pi_{\text{top}}$. Let $\alpha_1, \dots, \alpha_n$ be indeterminates. Define the *weight* of a covering $\pi s_{ij} \lessdot \pi$ with $i < j$ to be $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$, and then define the weight of a maximal chain to be the product of the weights of its cover relations. In a result that extends to all Weyl groups, Stembridge [Ste02] shows that the sum of the weights of the maximal chains is

$$\frac{\binom{n}{2}!}{1^{n-1}2^{n-2}\cdots(n-1)^1}\prod_{1\leq i<j\leq n}(\alpha_i+\cdots+\alpha_{j-1}).$$

For instance, this formula reduces to $\binom{n}{2}!$ after setting all weights $\alpha_i = 1$.

The *totally nonnegative Grassmannian* $\mathrm{Gr}(k,n)_{\geq 0}$ is introduced in [Pos05] as the subset of points in the real Grassmannian $\mathrm{Gr}(k,n)$ which have all nonnegative Plücker coordinates. It is related to areas as diverse as cluster algebras [GL], electrical networks [Lam18], solitons [KW11], scattering amplitudes in Yang-Mills theory [AHBC+16], and the mathematical theory of juggling [KLS13]. Postnikov gave a decomposition of $\mathrm{Gr}(k,n)_{\geq 0}$ into *positroid cells* defined by setting certain Plücker coordinates equal to zero, and he conjectured that this decomposition forms a regular CW-complex. A generalization of that conjecture due to Williams [Wil07] was proved by Galashin, Karp, and Lam [GKL20]. Our object of interest is the face poset of this complex, known as the *circular Bruhat order* [Pos05, Section 17]. Postnikov's work provides characterizations in terms of many different combinatorial objects, e.g., decorated permutations, Grassmannian necklaces, Le-diagrams, and equivalence classes of certain plabic (planar, bi-colored) graphs. The list is extended by Knutson, Lam, and Speyer [KLS13] to include bounded affine permutations, bounded juggling patterns, and equivalence classes of intervals in the k-Bruhat order for S_n.

Our purpose is to give a Stembridge-like formula for the circular Bruhat order. We define "circular" analogues of Stembridge's weights (Definition 3.1) and our main result, Theorem 3.2, provides a closed formula for the sum of the weights of the maximal chains in the circular Bruhat order:

$$\tau(k,n)(\alpha_1+\cdots+\alpha_n)^{k(n-k)},$$

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where $\tau(k,n)$ is the number of Young tableaux for the $k \times (n-k)$ rectangle (cf. Example 3.3). Using the hook formula for $\tau(k,n)$, the above expression becomes

$$(k(n-k))! \left[\prod_{i=1}^{k} \frac{(k-i)!}{(n-i)!} \right] (\alpha_1 + \cdots + \alpha_n)^{k(n-k)}.$$

Section 2 provides background and notation. The poset of bounded affine permutations, $\mathrm{Bound}(k,n)$, is isomorphic to a lower order ideal in the Bruhat order for the $\tilde{A}_{n-1}$ affine Coxeter group and is dual to the circular Bruhat order [KLS13]. Throughout our paper, we use the language of bounded affine permutations, including in the statement of our main result, Theorem 3.2. Section 3 states and proves Theorem 3.2. The proof relies on two technical lemmas whose proofs are relegated to Section 4. These proofs rely on the interpretation of $\mathrm{Bound}(k,n)$ in terms of intervals in the k-Bruhat order for S_n developed in [KLS13]. We also use a result of Bergeron and Sottile [BS98, Corollary 1.3.1] on cyclic shifts of k-Bruhat intervals. Their proof is a consequence of a symmetry they find for Littlewood-Richardson coefficients using geometry. It would be nice to have a purely combinatorial proof of their cyclic shift result.

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2. Bounded affine permutations and the circular Bruhat order

We recall ideas and notation introduced in [KLS13], which built on earlier work by Postnikov on the totally nonnegative Grassmannian [Pos05]. Our reference for the affine symmetric group is [BB96]. Let $\tilde{S}_n$ denote the group of *affine permutations* consisting of bijections $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying $f(i+n) = f(i) + n$ for all $i \in \mathbb{Z}$. We use the standard *window* notation $f = [f(1), f(2), \dots, f(n)]$ to represent $f \in \tilde{S}_n$. Define the averaging function on $\tilde{S}_n$ by $\mathrm{av}(f) = \frac{1}{n}\sum_{i=1}^{n}(f(i) - i)$, and for $0 \le k \le n$, let $\tilde{S}_n^k := \mathrm{av}^{-1}(k)$. In particular, $\tilde{S}_n^0$ is the *affine symmetric group*.

The affine symmetric group is generated by its *simple reflections*:

$$s_i = \begin{cases} [0, 2, 3, \dots, n-1, n+1] & \text{if } i = 0, \\ [1, 2, \dots, i-1, i+1, i, i+2, \dots, n] & \text{if } 0 < i \le n-1. \end{cases}$$

For instance,

$$[f(1), \dots, f(n)]s_0 = [f(0), f(2), \dots, f(n-1), f(n+1)] = [f(n) - n, f(2), \dots, f(n-1), f(1) + n].$$

Then $\tilde{S}_n^0$ is Coxeter group of type $\tilde{A}_{n-1}$ and is thus a graded poset under the Bruhat order. The *reflections* for $\tilde{S}_n^0$, i.e., the conjugates of the simple reflections, are

$$[1, 2, \dots, i-1, j - rn, i+1, \dots, j-1, i + rn, j+1, \dots, n] \tag{1}$$

for $1 \le i < j \le n$ and $r \in \mathbb{Z}$.

The mapping $[f(1), \dots, f(n)] \mapsto [f(1) - k, \dots, f(n) - k]$ is a bijection $\tilde{S}_n^k \to \tilde{S}_n^0$, and thus the Bruhat order on $\tilde{S}_n^0$ induces a graded poset structure on $\tilde{S}_n^k$ for which we now give an explicit description. A pair $(i,j) \in \mathbb{Z}^2$ is an *inversion* for $f \in \tilde{S}_n^k$ if $i < j$ and $f(j) > f(i)$. Define an equivalence relation on the set of inversions by $(i,j) \sim (i',j')$ if $i' = i + rn$ and $j' = j + rn$ for some integer r. Then the *length* of f, denoted $\ell(f)$, is the number of equivalence classes of inversions of f. (This notion of length coincides with that inherited from the Bruhat order [BB96, Proposition 4.1].) In general, if $f \in \tilde{S}_n^{k'}$ and $g \in \tilde{S}_n^k$, then $fg \in \tilde{S}_n^{k'+k}$. In particular, $\tilde{S}_n^0$ acts on $\tilde{S}_n^k$. If $f, g \in \tilde{S}_n^k$, then f covers g, denoted $g \lessdot f$, exactly when $g = ft$ for some reflection t in $\tilde{S}_n^0$ and $\ell(f) = \ell(g) + 1$.

A permutation $f \in \tilde{S}_n$ is *bounded* if $i \le f(i) \le i + n$ for all $i \in \mathbb{Z}$. For each $0 \le k \le n$ the bounded elements of $\tilde{S}_n^k$ are denoted

$$\mathrm{Bound}(k,n) = \left\{ f \in \tilde{S}_n^k : i \le f(i) \le i + n \text{ for all } i \in \mathbb{Z} \right\}.$$

By Lemma 3.6 of [KLS13], $\mathrm{Bound}(k,n)$ is a lower order ideal in $\tilde{S}_n^k$ and thus forms a graded poset with rank function given by length.

The *circular Bruhat order* $\mathrm{CB}(k,n)$ was originally defined in [Pos05] in terms of *decorated permutations*. These are permutations $\pi \in S_n$ for which each fixed point is assigned a color—either black or white. The *anti-excedances* of a decorated permutation π are $i \in [n]$ for which either $\pi^{-1}(i) > i$ or i is a white fixed point. Then $\mathrm{CB}(k,n)$ is the set of decorated permutations with k anti-excedances and with a poset structure determined by *alignments* and *crossings* in *chord diagrams*. See [Pos05] for details. As posets, $\mathrm{Bound}(k,n)$ and $\mathrm{CB}(k,n)$ are anti-isomorphic, i.e. $\mathrm{Bound}(k,n)$ is isomorphic to the dual of $\mathrm{CB}(k,n)$. To go from a bounded affine permutation f to a decorated permutation π, reduce the window of f modulo n, and then color each fixed point i in the resulting permutation black if $f(i) = i$ or white if $f(i) = i + n$.

We translate the notion of an anti-excedance from decorated permutations to bounded affine permutations:

Definition 2.1. The *anti-excedances* of $f \in \mathrm{Bound}(k,n)$ are the integers $f(i) - n$ such that $i \in [n]$ and $f(i) > n$.

One may check that the elements of $\mathrm{Bound}(k,n)$ are exactly the bounded affine permutations with k anti-excedances.

To describe the poset structure on $\mathrm{Bound}(k,n)$ in detail, note that for a reflection of an element of $\mathrm{Bound}(k,n)$ to remain in $\mathrm{Bound}(k,n)$, it is necessary (but not sufficient) that the integer r in (1) be 0 or 1. Thus, for $i, j \in [n]$ with $i \neq j$, we define

$$t_{ij} = \begin{cases} [1,2,\dots,i-1,j,i+1,\dots,j-1,i,j+1,\dots,n] & \text{if } i < j, \\ [1,2,\dots,j-1,i-n,j+1,\dots,i-1,j+n,i+1,\dots,n] & \text{if } i > j. \end{cases}$$

The cover relations in $\mathrm{Bound}(k,n)$ are given by $g \lessdot f$ if and only if there exists t_{ij} such that $g = ft_{ij}$ and $\ell(f) = \ell(g) + 1$.

By Lemma 17.6 of [Pos05], the unique minimal element of $\mathrm{Bound}(k,n)$ is

$$f_{\min} = [1+k, 2+k, \dots, n+k].$$

The maximal elements are in bijection with $\binom{[n]}{k}$. Given $\lambda \in \binom{[n]}{k}$, the corresponding maximal element is

$$f_{\max,\lambda}(i) = \begin{cases} i+n & \text{if } i \in \lambda, \\ i & \text{otherwise.} \end{cases}$$

We have $\ell(f_{\min}) = 0$, and $\ell(f_{\max,\lambda}) = k(n-k)$ for any maximal element. By Proposition 23.1 of [Pos05], the exponential generating function for the cardinality of $\mathrm{Bound}(k,n)$ is

$$\sum_{0 \le k \le n} |\,\mathrm{Bound}(k,n)|\, x^k \frac{y^n}{n!} = e^{xy} \frac{x-1}{x - e^{y(x-1)}}.$$

For the rank generating function of $\mathrm{Bound}(k,n)$, see [Wil05].

3. Main theorem

Definition 3.1. Let $\alpha_1, \dots, \alpha_n$ be indeterminates. The *weight* of a covering $ft_{ij} \lessdot f$ in $\mathrm{Bound}(k,n)$ is the sum of α_i through α_{j-1} in cyclic order:

$$\mathrm{wt}(ft_{ij} \lessdot f) = \begin{cases} \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} & \text{if } i < j, \\ \alpha_i + \cdots + \alpha_n + \alpha_1 + \alpha_2 + \cdots + \alpha_{j-1} & \text{if } i > j. \end{cases}$$

The *weight* of a saturated chain in $\mathrm{Bound}(k,n)$ is the product of the weights of its cover relations (the empty chain is assigned weight 1).[1] For $r \in [n]$, a covering is *r-good* if α_r appears in its weight. A saturated chain in $\mathrm{Bound}(k,n)$ is *r-good* if all of its cover relations are r-good. For arbitrary $r \in \mathbb{Z}$, we define r-good covers and chains by replacing r with its representative in $[n]$ modulo n.

Our main theorem is the following.

Theorem 3.2. *The sum of the weights of the maximal chains in* $\mathrm{Bound}(k,n)$ *is*

$$\tau(k,n)(\alpha_1 + \cdots + \alpha_n)^{k(n-k)}$$

where $\tau(k,n)$ is the number of standard Young tableaux of a $k \times (n-k)$ rectangle.

[1] Since $\mathrm{Bound}(k,n)$ and the circular Bruhat order $\mathrm{CB}(k,n)$ are anti-isomorphic, their covers are in bijection, as are their maximal chains. Thus, Theorem 3.2 could be stated as giving the sum of the weights of maximal chains in $\mathrm{CB}(k,n)$.

Example 3.3. Figure 1 illustrates Bound$(2,3)$ with its cover weights. The sum of the weights of its six maximal chains is

$$\alpha_1\alpha_2+\alpha_1(\alpha_1+\alpha_3)+\alpha_2(\alpha_1+\alpha_2)+\alpha_2\alpha_3+\alpha_3\alpha_1+\alpha_3(\alpha_2+\alpha_3)=\tau(2,3)(\alpha_1+\alpha_2+\alpha_3)^2,$$

where $\tau(2,3)=1$ since there is only one standard Young tableau for the 2×1 rectangle.

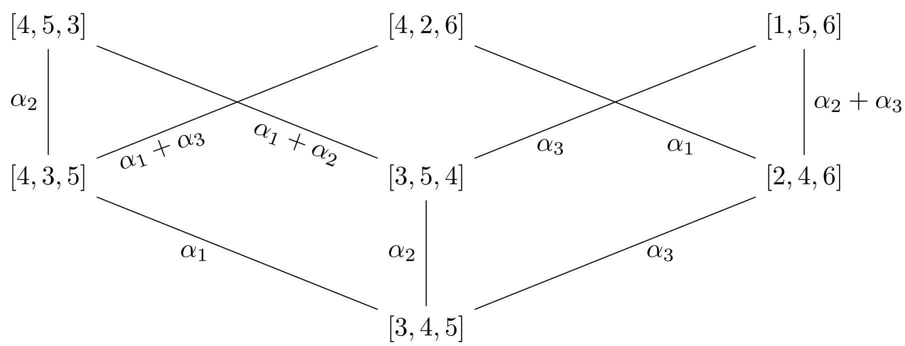


FIGURE 1. Bound$(2,3)$ with edge weights.

We say a saturated chain $f_0 \lessdot \cdots \lessdot f_\ell$ in a poset is *upward-saturated* if f_ℓ is maximal. The proof of Theorem 3.2 follows from two lemmas whose proofs appear in the next section.

Lemma 3.4. *Let $f \in$ Bound(k,n). Then the number of r-good upward-saturated chains in* Bound(k,n) *with minimal element f is independent of r.*

Lemma 3.5. *The number of n-good maximal chains in* Bound(k,n) *is $\tau(k,n)$.*

Proof of Theorem 3.2. Let $\delta(f)$ be the number of r-good upward-saturated chains with minimal element $f \in$ Bound(k,n). This number is independent of r by Lemma 3.4. It follows that

$$\sum_{g:f\lessdot g}\delta(g)\,\mathrm{wt}(f\lessdot g)=\sum_{r=1}^{n}\alpha_r\sum_{\substack{f\lessdot g\\ r-\mathrm{good}}}^{n}\delta(g)=\delta(f)(\alpha_1+\cdots+\alpha_n). \tag{2}$$

Let $\mathcal{C}(m)$ denote the set of maximal chains C in Bound(k,n) such that $\min(C)=f_{\min}$ and $\ell(\max(C))=m$. We now show by induction on m that

$$\sum_{C\in\mathcal{C}(m)}\delta(\max(C))\,\mathrm{wt}(C)=\delta(f_{\min})(\alpha_1+\cdots+\alpha_n)^m$$

for $0\le m\le k(n-k)$. In the case $m=k(n-k)$, we are summing over maximal chains C of Bound(k,n). For these $\delta(\max(C))=1$, and Theorem 3.2 will then follow from Lemma 3.5. The case $m=0$ is a tautology since the only element of Bound(k,n) with length 0 is $f_{\min}$. To proceed with induction, fix some m with $0\le m<k(n-k)$. Then

$$\begin{aligned}\sum_{C\in\mathcal{C}(m+1)}\delta(\max(C))\,\mathrm{wt}(C)&=\sum_{C'\in\mathcal{C}(m)}\sum_{\max(C')\lessdot f}\delta(f)\,\mathrm{wt}(\max(C')\lessdot f)\,\mathrm{wt}(C')\\&=\sum_{C'\in\mathcal{C}(m)}\delta(\max(C'))(\alpha_1+\cdots+\alpha_n)\,\mathrm{wt}(C') &&\text{(by (2))}\\&=\delta(f_{\min})(\alpha_1+\cdots+\alpha_n)^{m+1} &&\text{(by induction).}\end{aligned}$$

$\square$

4. Proofs of lemmas

4.1. **k-Bruhat order.** Our references for the k-Bruhat order are [BS98] and [KLS13].

Definition 4.1. The *k-Bruhat order* $\leq_k$ on the symmetric group S_n is given by $u \leq_k v$ if

(1) $u(i) \leq v(i)$ for $1 \leq i \leq k$;
(2) $u(j) \geq v(j)$ for $k < j \leq n$;
(3) $u(i) < u(j)$ implies $v(i) < v(j)$ if $1 \leq i < j \leq k$ or if $k < i < j \leq n$

The cover relations for the k-Bruhat order have the form $u \lessdot_k v$ if $u \lessdot v$ (in the ordinary Bruhat order) and $\{u(1), \ldots, u(k)\} \neq \{v(1), \ldots, v(k)\}$. Each interval $[u, w]_k$ in the k-Bruhat order is a graded poset of rank $\ell(w) - \ell(u)$.

Definition 4.2. A permutation $w \in S_n$ is *k-Grassmannian* if $w(1) < \cdots < w(k)$ and $w(k+1) < \cdots < w(n)$. These are in bijection with $\lambda \in \binom{[n]}{k}$ by letting w_λ be the unique k-Grassmannian permutation such that $\{w(1), \ldots, w(k)\} = \lambda$.

Denote the positions of the anti-excedances of $f \in \mathrm{Bound}(k, n)$ by

$$\Lambda(f) = \{i \in [n] : f(i) - n \text{ is an anti-excedance of } f\} = \{i \in [n] : f(i) > n\}.$$

Then associate a k-Grassmannian permutation to f by

$$w_f = w_{\Lambda(f)}.$$

Fixing $\lambda \in \binom{[n]}{k}$, we define two posets. The first is the principal order ideal in the k-Bruhat order generated by w_λ:

$$S_{n,\lambda} = \{u \in S_n : u \leq_k w_\lambda\}.$$

As sets, $S_{n,\lambda}$ and the usual Bruhat interval $[e, w_\lambda]$ are identical [KLS14, Proposition 2.5], although their poset structures differ, in general. The second is

$$\mathrm{Bound}(k, n)_\lambda = \{f \in \mathrm{Bound}(k, n) : \Lambda(f) = \lambda\}$$

with partial order $\leq_\gamma$ defined by its cover relations: $f \lessdot_\gamma g$ if f is covered by g in $\mathrm{Bound}(k, n)$ and the covering $f \lessdot g$ is n-good (γ is a mnemonic for "good"). Note that a covering $ft_{ij} \lessdot f$ in $\mathrm{Bound}(k, n)$ is n-good if and only if $i > j$, in which case $\Lambda(ft_{ij}) = \Lambda(f)$.

Embed S_n in $\tilde{S}_n$ via $u \mapsto [u(1), \ldots, u(n)]$ and define the *translation element* $t_k = [1+n, 2+n, \ldots, k+n, k+1, k+2, \ldots, n] \in \tilde{S}_n^k$. Taking our lead from [KLS13], for each $u \in S_{n,\lambda}$ define $f_u = f_{u,\lambda} = ut_k w_\lambda^{-1}$. Therefore,

$$f_u(w_\lambda(i)) = \begin{cases} u(i) + n & \text{if } 1 \leq i \leq k \\ u(i) & \text{if } k < i \leq n. \end{cases}$$

Since w_λ is k-Grassmannian and $u \leq_k w_\lambda$, we have $1 \leq u(i) \leq w_\lambda(i) \leq n$ for $1 \leq i \leq k$, and $w_\lambda(i) \leq u(i) \leq n$ for $k < i \leq n$. Therefore, $i \leq f(i) \leq i + n$ for all i. Further, $\Lambda(f_u) = \lambda$. Hence, $f_u \in \mathrm{Bound}(k, n)_\lambda$.

For each $f \in \mathrm{Bound}(k, n)_\lambda$, define $u_f = u_{f,\lambda} = f w_\lambda t_k^{-1}$ so that

$$u_f(i) = \begin{cases} f(w_\lambda(i)) - n & \text{if } 1 \leq i \leq k \\ f(w_\lambda(i)) & \text{if } k < i \leq n. \end{cases}$$

Since $\lambda = \Lambda(f)$, it follows that $1 \leq u_f(i) \leq n$ for $i \in [n]$. To see that $u \leq_k w_\lambda$, first note that $w_\lambda(i) \leq f(w_\lambda(i)) \leq w_\lambda(i) + n$ for all i since $f \in \mathrm{Bound}(k, n)$. Properties (1) and (2) of Definition 4.1 then follow. Property (3) holds since w_λ is a k-Grassmannian element and, therefore, $w_\lambda(i)$ is increasing for $1 \leq i \leq k$ and for $k < i \leq n$.

Example 4.3. Let $\lambda = \{2, 4, 5\} \in \binom{[5]}{3}$ and $f = [3, 6, 5, 9, 7]$. Then $w_\lambda = [2, 4, 5, 1, 3]$, and $f \in \mathrm{Bound}(3, 5)_\lambda$ since its anti-excedances appear in positions $2, 4$, and 5. We have $u_f = [1, 4, 2, 3, 5]$, which is formed by first listing the anti-excedances of f, reduced modulo 5, as they appear in order by position in f, i.e., $1 = 6 - 5$, $4 = 9 - 5$, and $2 = 7 - 5$, and then listing the non-anti-excedances, 3 and 5. Reversing this process yields $f_{u_f} = f$.

Proposition 4.4. *Let* $\lambda \in \binom{[n]}{k}$*. Then the mapping*

$$(S_{n,\lambda}, \leq_k) \to (\mathrm{Bound}(k,n)_\lambda, \leq_\gamma)$$
$$u \mapsto f_u$$

is an isomorphism of posets with inverse $f \mapsto u_f$*.*

Proof. It is clear that $u \mapsto f_u$ and $f \mapsto u_f$ are inverses. We must show they preserve cover relations. Let $u, v \in S_{n,\lambda}$ with corresponding $f := f_u$ and $g := f_v$ in $\mathrm{Bound}(k,n)$. The condition that $u \lessdot_k v$ is equivalent to:

1. There exists $p \leq k < q$ such that $v = us_{p,q}$ where $s_{p,q} = (p,q)$ is the transposition swapping p and q, and
2. $\ell(v) = \ell(u) + 1$, i.e.,
   (i) $u(p) < u(q)$, and
   (ii) there is no integer r such that $p < r < q$ and $u(p) < u(r) < u(q)$.

On the other hand, the condition that $f \lessdot_\gamma g$ is equivalent to:

1*. There exists $i < j$ such that $f(i)$ is a non-anti-excedance, $f(j)$ is an anti-excedance, $g = ft_{ji}$, and
2*. $\ell(g) = \ell(f) + 1$, i.e.,
   (i) $f(j) < f(i) + n$, and
   (ii) there is no integer a such that $j < a < i + n$ and $f(j) < f(a) < f(i) + n$.

To show equivalence of these two sets of conditions, first suppose that $u \lessdot_k v \leq_k w_\lambda$. We will show that $f \lessdot_\gamma g$. Take $p \leq k < q$ as in condition 1, and let $i := w_\lambda(q)$ and $j := w_\lambda(p)$. It follows from the k-Bruhat order that $i < j$:

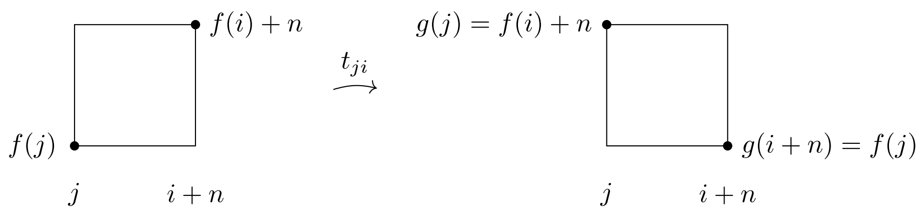
$$i = w_\lambda(q) \leq v(q) = u(p) \leq v(p) \leq w_\lambda(p) = j.$$

We have that $f(r) = g(r)$ for $r \in [n] \setminus \{i, j\}$, and

$$g(i) = g(w_\lambda(q)) = v(q) = u(p) = f(w_\lambda(p)) - n = f(j) - n$$
$$g(j) = g(w_\lambda(p)) = v(p) + n = u(q) + n = f(w_\lambda(q)) + n = f(i) + n.$$

Therefore, condition 1* holds, and 2*(i) follows from 2(i).

Condition 2*(ii) says that the graph of f has no points inside a certain box:



To verify condition 2*(ii) holds, it helps to divide the set of integers strictly between j and $i+n$ into two parts: $X := \{a \in \mathbb{Z} : j < a \leq n\}$, and $Y := \{a \in \mathbb{Z} : n < a < i + n\}$. If $a \in X$ and $f(a)$ is not an anti-excedance, then $f(a) < n < f(j)$ and, hence, condition 2*(ii) is not violated. Similarly, if $a \in Y$ and $f(a - n)$ *is* an anti-excedance, then $f(i) + n \leq 2n < f(a - n) + n = f(a)$ and, hence, 2*(ii) is again not violated.

It remains to check anti-excedances whose positions are in X and non-anti-excedances whose positions are in $-n + Y$. Take $a \in X$ and suppose that $f(a)$ is an anti-excedance. Since $a > j = w_\lambda(p)$, there exists r with $p < r \leq k < q$ such that $f(a) = u(r) + n$. By condition 2(ii), $u(r)$ is not between $u(p)$ and $u(q)$, which implies that $f(a)$ is not between $f(i) + n = u(q) + n$ and $f(j) = u(p) + n$ in accordance with 2*(ii). Now, instead, take $a \in Y$ and suppose that $f(a - n)$ is not an anti-excedance. Since $a - n < i = w_\lambda(q)$, there exists r with $p \leq k < r < q$ such that $f(a - n) = u(r)$. As above, $u(r)$ is not between $u(p)$ and $u(q)$, and this implies that $f(a) = u(r) + n$ is not between $f(i) + n = u(q) + n$ and $f(j) = u(p) + n$.

We have shown that the mapping $u \mapsto f_u$ preserves cover relations. The proof that its inverse $f \mapsto u_f$ preserves cover relations is similar. ∎

Remark. Theorem 3.16 of [KLS13] gives an isomorphism between $\mathrm{Bound}(k,n)$ and a poset $\mathcal{Q}(k,n)$ consisting of classes of k-Bruhat intervals with respect to a certain equivalence relation. Identifying $u \in S_{n,\lambda}$ with the equivalence class for the interval $[u, w_\lambda]_k$, Proposition 4.4 then shows that the isomorphism of [KLS13] restricts to an isomorphism of the weak subposets $S_{n,\lambda} \subseteq \mathcal{Q}(k,n)$ and $\mathrm{Bound}(k,n)_\lambda \subseteq \mathrm{Bound}(k,n)$.

Corollary 4.5. *Let $f \in \mathrm{Bound}(k,n)$ with corresponding k-Grassmannian permutation w_f. Then the n-good upward-saturated chains in $\mathrm{Bound}(k,n)$ with minimal element f are in bijection with the maximal chains in the k-Bruhat interval $[u_f, w_f]_k$.*

Proof. Let $\lambda = \Lambda(f)$. Since an n-good covering preserves anti-excedance positions, upward-saturated n-good chains in $\mathrm{Bound}(k,n)$ with minimal element f are exactly upward saturated chains in $\mathrm{Bound}(k,n)_\lambda$ with minimal element f. By Proposition 4.4 these are in bijection with upward saturated chains in $S_{n,\lambda}$ with minimal element u_f. Since $S_{n,\lambda}$ has unique maximal element $w_\lambda = w_f$, these upward-saturated chains are exactly the maximal chains in $[u_f, w_f]_k$. ☐

Example 4.6. Let $f = [2,5,4,7] \in \mathrm{Bound}(2,4)$. Then $w_f = [2,4,1,3] \in S_n$. We have $u_f = [1,3,2,4]$ and $f_{w_f} = [1,6,3,8]$. As seen in Figure 2, the interval $[u_f, w_f]_2$ has two maximal chains:

$$[1,3,2,4] \lessdot_2 [1,4,2,3] \lessdot_2 [2,4,1,3] \quad \text{and} \quad [1,3,2,4] \lessdot_2 [2,3,1,4] \lessdot_2 [2,4,1,3].$$

Under the isomorphism of Proposition 4.4, these correspond to the two 4-good upward-saturated chains in $\mathrm{Bound}(2,4)$ with minimal element f:

$$[2,5,4,7] \lessdot [2,5,3,8] \lessdot [1,6,3,8] \quad \text{and} \quad [2,5,4,7] \lessdot [1,6,4,7] \lessdot [1,6,3,8].$$

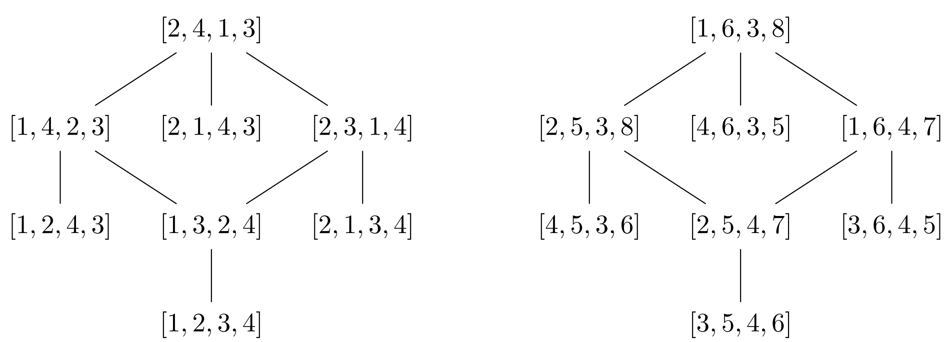


FIGURE 2. The posets $S_{4,\lambda}$ and $\mathrm{Bound}(2,4)_\lambda$ in the case $\lambda = \{2,4\} \in \binom{[4]}{2}$. They are isomorphic in accordance with Proposition 4.4.

4.2. **Cyclic shifts.** Define the *cyclic shift* of $f \in \mathrm{Bound}(k,n)$ by

$$\chi(f) = [f(0)+1, f(1)+1, \ldots, f(n-1)+1].$$

The following properties of χ are immediate: (i) $i \le \chi(f)(i) \le i+n$ for all $i \in [n]$, (ii) $\chi(ft_{ij}) = \chi(f)t_{i+1,j+1}$ for all reflections $t_{i,j}$ (with indices taken modulo n), and (iii) $(i,j) \in \mathbb{Z}^2$ represents an inversion for f if and only if $(i+1, j+1)$ represents an inversion for $\chi(f)$. Therefore, χ is an automorphism of the graded poset $\mathrm{Bound}(k,n)$ and induces a faithful action of the cyclic group of order n on $\mathrm{Bound}(k,n)$. The following is an immediate implication of property (ii).

Proposition 4.7. *A saturated chain C in $\mathrm{Bound}(k,n)$ is r-good if and only if $\chi(C)$ is $(r+1)$-good.*

The following result of Bergeron and Sottile is an important step in the proof of Lemma 3.4.

Theorem 4.8 ([BS98, Corollary 1.3.1])**.** *Let $u \le_k v$ and $x \le_k y$ in the k-Bruhat order on S_n, and suppose that $cvu^{-1}c^{-1} = yx^{-1}$ where $c = [2,3,\ldots,n,1] = (1,2,\ldots,n)$. Then the intervals $[u,v]_k$ and $[x,y]_k$ have the same number of maximal chains.*

4.3. Proofs of lemmas.

Proof of Lemma 3.4. For general $g \in \mathrm{Bound}(k,n)$, let $\delta_r(g)$ denote the number of r-good upward-saturated chains in $\mathrm{Bound}(k,n)$ with minimal element g. Fix $f \in \mathrm{Bound}(k,n)$. We first show that Theorem 4.8 applies to the pair of intervals $[u_f, w_f]_k$ and $[u_{\chi(f)}, w_{\chi(f)}]_k$ by checking that $cu_f w_f^{-1} = u_{\chi(f)} w_{\chi(f)}^{-1} c$ where $c = [2,3,\dots,n,1]$. Let $i \in [n]$. Working modulo n,

$$cu_f w_f^{-1}(i) = cu_f t_k w_f^{-1}(i) = f(i) + 1 = \chi(f)(i+1) = u_{\chi(f)} t_k w_{\chi(f)}^{-1} c(i) = u_{\chi(f)} w_{\chi(f)}^{-1} c(i),$$

and the conclusion follows. Therefore, the number of maximal chains in $[u_f, w_f]_k$ is the same as the number of maximal chains in $[u_{\chi(f)}, w_{\chi(f)}]_k$ and, by iteration, as the number of maximal chains in $[u_{\chi^r(f)}, w_{\chi^r(f)}]_k$ for all $r \in [n]$. Applying Corollary 4.5 and Proposition 4.7,

$$\delta_n(f) = \delta_n(\chi^r(f)) = \delta_r(f)$$

for all $r \in [n]$. □

Proof of Lemma 3.5. Let $w_{\max} = w_{\{n-k+1, n-k+2, \dots, n\}}$. Then $[\mathrm{id}, w_{\max}]_k$ is an interval of rank $k(n-k)$ consisting exactly of the k-Grassmannian elements of S_n. By Corollary 4.5 its maximal chains are in correspondence with maximal n-good chains in $\mathrm{Bound}(k,n)$.

Let $L(k, n-k)$ denote the poset of Young diagrams fitting inside a $k \times (n-k)$ rectangle, ordered by containment, as usual. There is a well-known correspondence between $\binom{[n]}{k}$ and $L(k,n-k)$: given $\lambda \in \binom{[n]}{k}$ with $\lambda = \{\lambda_1 < \cdots < \lambda_k\}$, let $Y_{p(\lambda)}$ be the Young diagram corresponding to the partition $p(\lambda) = \{p_1 > \cdots > p_k\}$ where $p_i := (n-k) - \lambda_i + i$. In English notation, $Y_{p(\lambda)}$ is the diagram determined by the left-down walk in $\mathbb{Z}^2$ from $(k, n-k)$ to $(0,0)$ whose λ_i-th step is its i-th *vertical* step. This correspondence yields an anti-isomorphism $w_\lambda \mapsto Y_{p(\lambda)}$ from the interval $[\mathrm{id}, w_{\max}]_k$ with its k-Bruhat order and $L(k,n-k)$. The result now follows since Young tableaux for the $k \times (n-k)$ rectangle are in bijection with maximal chains in $L(k,n-k)$. □

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