TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

BIGRAPHICAL ARRANGEMENTS

SAM HOPKINS AND DAVID PERKINSON

Abstract. We define the bigraphical arrangement of a graph and show that the Pak-Stanley labels of its regions are the parking functions of a closely related graph, thus proving conjectures of Duval, Klivans, and Martin [4] and of Hopkins and Perkinson [5]. A consequence is a new proof of a bijection between labeled graphs and regions of the Shi arrangement first given by Stanley in [8]. We also give bounds on the number of regions of a bigraphical arrangement.

Throughout this paper G will be a simple graph (no multiedges or loops, but not necessarily connected) with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. A *(real) hyperplane arrangement* is a finite collection of affine hyperplanes in Euclidean space. Our object of study is the *bigraphical arrangement* of G, so called because it associates two hyperplanes to each edge of the graph.

Definition 0.1. For each edge $\{v_i, v_j\} \in E$, choose real numbers a_{ij} and a_{ji} such that there exists $x \in \mathbb{R}^n$ with $x_i - x_j < a_{ij}$ and $x_j - x_i < a_{ji}$ for all $\{v_i, v_j\} \in E$. We call these numbers *parameters* and we call $A := \{a_{ij}\}$ a *parameter list*. The *bigraphical arrangement* $\Sigma_G(A)$ is the set of $2|E|$ hyperplanes,

$$\Sigma_G(A) := \{x_i - x_j = a_{ij} \colon \{v_i, v_j\} \in E\}.$$

The *regions* of $\Sigma_G(A)$ are the connected components of $\mathbb{R}^n \setminus \Sigma_G(A)$. The *central region* is the region defined by $x_i - x_j < a_{ij}$ for all $\{v_i, v_j\} \in E$. The above condition on A guarantees that the central region is nonempty.

Several authors have connected hyperplane arrangements to graphs in various ways. The *graphical arrangement* [9, p. 414] of G,

$$\{x_i - x_j = 0 \colon \{v_i, v_j\} \in E \text{ with } i < j\},$$

associates a single hyperplane to each edge of G. One interesting property of the graphical arrangement is that its characteristic polynomial is the chromatic polynomial of G. Thus, the graphical arrangement encodes information about colorings of G. As we shall see, the bigraphical arrangement encodes information about the parking functions of $G_\bullet$, the graph obtained from G by adding a special sink vertex v_0 and an edge between v_0 and each vertex $v \in V$. For background on parking functions, see §2.

J.-Y. Shi [7], in his investigation of the Kazhdan-Lusztig cells of affine Weyl groups of type $\tilde{A}_{n-1}$, introduced the *Shi arrangement*,

$$\{x_i - x_j = 0, 1 \colon 1 \leq i < j \leq n\}.$$

He proved that the number of regions of this arrangement is $(n+1)^{n-1}$, Cayley's formula for the number of spanning trees of the complete graph K_{n+1}. Stanley [8],

2010 *Mathematics Subject Classification.* Primary 52C35; Secondary 05C25.

in collaboration with Pak, was the first to give a bijective proof of this result by labeling the regions of the Shi arrangement with parking functions. (There are several well-known bijections between parking functions and spanning trees.) Stanley and Pak's procedure labels the central region of the Shi arrangement with the parking function $00\dots0$. It then inductively labels the other regions by moving outwards and increasing the ith coordinate of a region's label whenever a hyperplane is crossed that corresponds to an increase in x_i. We call the resulting labels the *Pak-Stanley labels* of the regions of an arrangement. Duval, Klivans, and Martin [4] defined the *G-Shi arrangement*,

$$\{x_i - x_j = 0, 1\colon \{v_i, v_j\} \in E \text{ with } i < j\},$$

and conjectured that the Pak-Stanley labels of the G-Shi arrangement are the parking functions of $G_\bullet$.[1] We prove this conjecture as a consequence of Corollary 2.8. The G-Shi arrangement is in fact a special kind of bigraphical arrangement.

In [5], Hopkins and Perkinson studied the *G-semiorder arrangement*,

$$\{x_i - x_j = 1\colon \{v_i, v_j\} \in E\},$$

another special kind of bigraphical arrangement. They showed that the Pak-Stanley labels of the G-semiorder arrangement are the $G_\bullet$-parking functions sought by Duval, Klivans, and Martin. It was also conjectured in [5] that if one were to slide the hyperplanes of the G-semiorder arrangement along their normals, although some regions are destroyed and others are created, so long as the central region is preserved the set of parking function labels remains the same. In this way, one could deform the G-semiorder arrangement into the G-Shi arrangement and show that the G-Shi arrangement has the expected set of labels. Figure 1 depicts this sliding procedure when $G = K_3$.

Our Corollary 2.8 establishes the sliding conjecture: the Pak-Stanley labels of any bigraphical arrangement, $\Sigma_G(A)$, are the parking functions of $G_\bullet$. (Remark 2.9 indicates how, in addition, all the parking functions of G with respect to each of its vertices are encoded in the regions of $\Sigma_G(A)$.) Actual sliding does not appear in the proof of Corollary 2.8, although it was our original motivation. Moreover, in general, our results do not depend on those in [5].

In proving Corollary 2.8, we generalize a result of Benson, Chakbarty, and Tetali [2], who show that acyclic total orientations of G correspond to maximal parking functions of $G_\bullet$. We show that certain families of partial orientations of G defined in §1 correspond to all of the parking functions of $G_\bullet$. If G is the complete graph K_n and the parameter list A corresponds to the Shi arrangement, Corollary 2.8 provides an alternate proof of the bijection of Pak and Stanley between regions of the Shi arrangement and parking functions.

Example 0.2. The following are examples of bigraphical arrangements:

(1) Setting $a_{ij} = 1$ for all i, j gives the *G-semiorder* arrangement, studied in [5]. We will use SEMI to denote the parameter list of the G-semiorder arrangement and thus denote the G-semiorder arrangement by $\Sigma_G(\text{SEMI})$.

[1]The G-Shi arrangement is not to be confused with what Armstrong and Rhoades [1] call the *deleted Shi arrangement* and denote Shi(G):

$$\text{Shi}(G) := \{x_i - x_j = 0\colon 1 \le i < j \le n\} \cup \{x_i - x_j = 1\colon \{v_i, v_j\} \in E \text{ with } i < j\}.$$

While the G-Shi arrangement has $2|E|$ hyperplanes and is a bigraphical arrangement, the Shi(G) arrangement has $\binom{n}{2} + |E|$ hyperplanes and is therefore not a bigraphical arrangement.

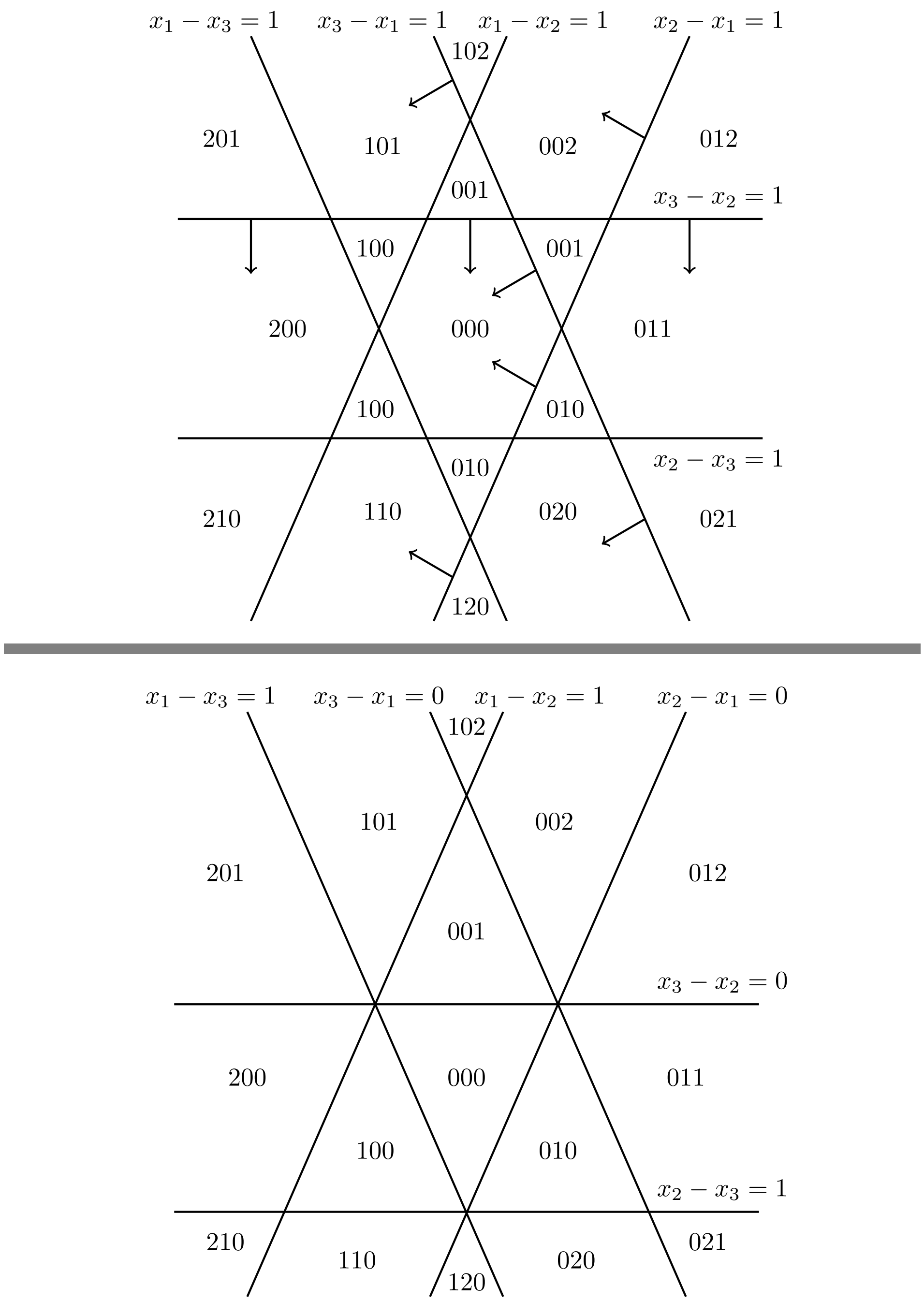


Figure 1. Hyperplanes in the K_3-semiorder arrangement (above) are slid in the directions indicated by the arrows to yield the K_3-Shi arrangement (below). The set of Pak-Stanley labels, strings $c_1c_2c_3$ inside each region, remains the same.

(2) Setting $a_{ij} = 1$ if $i < j$ and 0 otherwise gives the *G-Shi* arrangement, the subject of a conjecture in [4] that we establish as a consequence of Corollary 2.8. We will use SHI to denote the parameter list of the G-Shi arrangement and thus denote the G-Shi arrangement by $\Sigma_G(\mathrm{SHI})$.

(3) Let $\eta = (\ell_1, \dots, \ell_n) \in \mathbb{Z}^n_{>0}$. Setting $a_{ij} = \ell_i$ for all i, j gives what we call the (G, η)*-interval order arrangement.*

Taking G to be the complete graph K_n recovers the normal semiorder, Shi, and interval order arrangements. See [9] for definitions of these arrangements, as well as for basic concepts from the theory of hyperplane arrangements, in particular, that of the *characteristic polynomial.* □

From now on we assume we have fixed some parameter list A. Note that $\Sigma_G(A)$ having a nonempty central region is essentially equivalent to A having only positive entries. If the a_{ij} are all positive, then the origin satisfies $x_i - x_j < a_{ij}$ for all $\{v_i, v_j\} \in E$. On the other hand, suppose $\Sigma_G(A)$ has a nonempty central region and that p is a point in this region. Then the translation $x \mapsto x - p$ maps $\Sigma_G(A)$ to a bigraphical arrangement whose parameter list has positive entries.

In §1, we develop a correspondence between regions of $\Sigma_G(A)$ and partial orientations of G. In §2, we prove our main result, Corollary 2.8, which says that the Pak-Stanley labeling of any $\Sigma_G(A)$ yields the set of parking functions of $G_\bullet$. In §3, we bound the number of regions of $\Sigma_G(A)$ for arbitrary A and we find its characteristic polynomial when A is generic. The characteristic polynomial of a generic $\Sigma_G(A)$ turns out to be related to the Tutte polynomial of G.

Acknowledgements: We thank Art Duval, Caroline Klivans, and Jeremy Martin for introducing us to the G-Shi conjecture and for helpful comments. We thank students who participated in the geometry class taught by the second author at Reed College during the fall semester of 2012: Max Carpenter, Nadir Hajouji, Austin Humphrey, Mikhail Lepilov, Alex Perusse, Yotam Reshef, Marcus Robinson, Darko Trifunovski, Chris Vittal, and Austin Young. We also thank Collin Perkinson for help with proofreading.

1. Partial orientations and the regions of $\Sigma_G(A)$

Definition 1.1. A *partial orientation* of G is a choice of directions for a subset of the edges of G. Formally, a *step* is an ordered pair $(u, v) \in V \times V$ such that $\{u, v\} \in E$, and a partial orientation $\mathcal{O}$ is a set of steps with the property that if $(u, v) \in \mathcal{O}$, then $(v, u) \notin \mathcal{O}$. We say $\mathcal{O}$ is *acyclic* if it does not contain a cycle of steps.

Definition 1.2. Let $\mathcal{O}$ be a partial orientation. If $e = \{u, v\} \in E$ and $(u, v) \in \mathcal{O}$, then despite the ambiguity, we write $e \in \mathcal{O}$ and say e is *oriented.* In that case, we think of e as an arrow from u to v and write $e^- = u$ and $e^+ = v$. If neither (u, v) nor (v, u) is in $\mathcal{O}$, we write $e \notin \mathcal{O}$ and say that e is an unoriented or *blank* edge. The *indegree* of $u \in V$ relative to $\mathcal{O}$, denoted $\mathrm{indeg}_{\mathcal{O}}(u)$, is the number of edges $e \in \mathcal{O}$ such that $e^+ = u$. Similarly, the *outdegree* of the vertex $u \in V$ relative to $\mathcal{O}$ is the number of edges $e \in \mathcal{O}$ such that $e^- = u$. The *degree* of u is the number of $e \in E$ containing u.

Notation. Partial orientations naturally serve as labels for the regions of bigraphical arrangements. Suppose R is a region of $\Sigma_G(A)$. Define $\mathcal{O}_R$ to be the partial orientation obtained by (i) $(v_i, v_j) \in \mathcal{O}_R$ if $\{v_i, v_j\} \in E$ and $x_j > x_i + a_{ji}$ in R, and (ii) all other edges are blank. We now classify exactly which partial orientations are labels of regions of $\Sigma_G(A)$.

Definition 1.3. Let $\mathcal{O}$ be a partial orientation. A step (u, v) is *compatible* with $\mathcal{O}$ if $(v, u) \notin \mathcal{O}$. In other words, a step e is compatible with $\mathcal{O}$ if $e \in \mathcal{O}$ or e is a blank edge of $\mathcal{O}$. A *potential cycle for* $\mathcal{O}$ is a set $C = \{(u_1, u_2), (u_2, u_3), \ldots, (u_k, u_1)\}$ of steps compatible with $\mathcal{O}$.

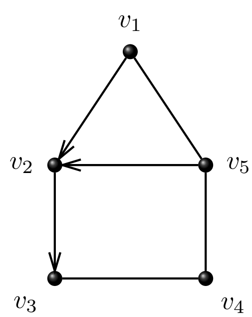


FIGURE 2. A partial orientation $\mathcal{O}$ of a graph with 5 vertices. Consider the potential cycle $C := \{(v_5, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$ for $\mathcal{O}$. We have $\nu_{\mathrm{SEMI}}(C, \mathcal{O}) = 0$, so $\mathcal{O}$ is not SEMI-admissible. However, the partial orientation $\mathcal{O}$ is SHI-admissible.

Definition 1.4. Let $\mathcal{O}$ be a partial orientation. The *score* of a step $e = (v_i, v_j)$ compatible with $\mathcal{O}$ is

$$\nu_A(e, \mathcal{O}) = \begin{cases} a_{ij} & \text{if } \{v_i, v_j\} \notin \mathcal{O}, \\ -a_{ji} & \text{if } (v_i, v_j) \in \mathcal{O}. \end{cases}$$

The *score* of a potential cycle C for $\mathcal{O}$ is

$$\nu_A(C, \mathcal{O}) = \sum_{e \in C} \nu(e, \mathcal{O}).$$

When the parameter list is clear from context, we omit the subscript and we write $\nu(C, \mathcal{O}) := \nu_A(C, \mathcal{O})$.

Definition 1.5. Let $\mathcal{O}$ be a partial orientation. A potential cycle for $\mathcal{O}$ is *bad* if it has a nonpositive score. We say $\mathcal{O}$ is *A-admissible* if it has no bad potential cycles.

Figure 2 gives an example of a partial orientation that is SHI-admissible but not SEMI-admissible.

Theorem 1.6. *The regions of $\Sigma_G(A)$ are in bijection with the A-admissible partial orientations of G. The bijection is given by $R \mapsto \mathcal{O}_R$.*

Proof: Let $\mathcal{O}$ be an A-admissible partial orientation, and define R to be the region of $\Sigma_G(A)$ determined by the following inequalities: for each edge $e = \{v_i, v_j\}$ of G:

- $x_i - x_j < a_{ij}$ and $x_j - x_i < a_{ji}$ if $e \notin \mathcal{O}$;
- $x_i - x_j < -a_{ji}$ if $(v_i, v_j) \in \mathcal{O}$.

We must show that R is nonempty. Encode the system of inequalities defining R in a $k \times n$ real matrix M and a column vector b in $\mathbb{R}^k$ such that $x \in R$ if and only if $Mx < b$. We take the rows of M to correspond with steps of G compatible with $\mathcal{O}$: a row of M with 1 in the ith entry and -1 in the jth entry corresponds to a step (v_i, v_j). By Farkas' lemma the insolvability of $Mx < b$ is equivalent to the existence of a row vector $y = (y_1, ..., y_k)$ satisfying:

$$(1) \qquad y_i \geq 0 \ \ \forall i, \quad y \neq 0, \quad yM = 0, \quad y \cdot b \leq 0.$$

Exactly the same argument as in the proof of Theorem 14 of [5] shows that such a y cannot exist. The linear dependencies of M are spanned by sums of rows corresponding to a cycle of G. A $y \geq 0$ satisfying $yM = 0$ would correspond to a sum of potential cycles of $\mathcal{O}$, but each potential cycle has a positive score, so $y \cdot b > 0$. By construction, $\mathcal{O}_R = \mathcal{O}$.

It remains to be shown that for any region R, we have that $\mathcal{O}_R$ is A-admissible. Let R be a region of $\Sigma_G(A)$. Encode the inequalities defining R as $Mx < b$. A bad cycle in $\mathcal{O}_R$ corresponds to a vector y satisfying condition (1) above. Thus, no bad cycles for $\mathcal{O}_R$ can exist. □

Notation. From now on, for an A-admissible partial orientation $\mathcal{O}$, we will use $\mathrm{rg}(\mathcal{O})$ to denote the unique region of $\Sigma_G(A)$ satisfying $\mathcal{O} = \mathcal{O}_{\mathrm{rg}(\mathcal{O})}$.

Definition 1.7. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ and let W be the subspace of $\mathbb{R}^n$ spanned by the normals of the hyperplanes in $\mathcal{A}$. We say that a region R of $\mathcal{A}$ is *relatively bounded* if $R \cap W$ is bounded. The *essentialization* of $\mathcal{A}$ is $\mathcal{A} \cap W$ considered as a hyperplane arrangement in $W \simeq \mathbb{R}^k$, where $k = \dim(W)$.

Theorem 1.8. *The relatively bounded regions of $\Sigma_G(A)$ are in bijection with the A-admissible partial orientations $\mathcal{O}$ of G for which every step in $\mathcal{O}$ belongs to some potential cycle. The bijection is given by $R \mapsto \mathcal{O}_R$.*

Proof: If G is the disjoint union of graphs G_1 and G_2, then the regions of G are precisely the Cartesian products of regions of G_1 with regions of G_2. Also, no potential cycle can contain both an edge of G_1 and of G_2. Thus, we may assume that G is connected and that the span of the normals of $\Sigma_G(A)$ is the space perpendicular to $\mathrm{Span}(\vec{1})$ where $\vec{1} := (1, 1, \ldots, 1) \in \mathbb{R}^n$.

Let R be a region of $\Sigma_G(A)$ and encode the system of inequalities for R as $Mx < b$ as in the proof of Theorem 1.6. Recall that the rows of M correspond to steps compatible with $\mathcal{O}_R$. We first need the following:

> **Claim**: The region R is relatively bounded if and only if for all vectors $z \lneq 0 \in \mathbb{R}^k$, there exists $y \geq 0 \in \mathbb{R}^k$ such that $yM = 0$ and $y \cdot z < 0$.

Proof of claim: By (another version of) Farkas' lemma, the existence of a y as in the claim is equivalent to the nonexistence of a solution x to $Mx \leq z$. So we will show that R is not relatively bounded if and only if there exists $z \lneq 0$ and x such that $Mx \leq z$.

The region R is not relatively bounded if and only if there exists $x \notin \mathrm{Span}(\vec{1})$ and $x_0 \in \mathbb{R}^n$ such that $M(x_0 + tx) < b$ for all $t \geq 0$, i.e., $tMx < b - Mx_0$ for all $t \geq 0$. In this case, $Mx \leq 0$. So if R is not relatively bounded, take x as above and let $z := Mx$. We cannot have $Mx = 0$ since for all $\{v_i, v_j\} \in E$, either (v_i, v_j) or (v_j, v_i) corresponds to a row of M (or both do), and so $\ker(M) = \mathrm{Span}(\vec{1}) \not\ni x$.

Conversely, suppose there exists $z \lneq 0$ and x such that $Mx \leq z$. It follows that $x \notin \ker(M) = \mathrm{Span}(\vec{1})$ and choosing any point $x_0 \in R$, we have

$$Mx_0 < b \Rightarrow 0 < b - Mx_0 \Rightarrow tMx \leq tz < b - Mx_0,$$

for all $t \geq 0$. Hence, R is not relatively bounded. □

Suppose that R is relatively bounded and let $e \in \mathcal{O}_R$. Define $z \lneq 0 \in \mathbb{R}^k$ by

$$z_{ij} = \begin{cases} -1 & \text{if } e^- = v_i \text{ and } e^+ = v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then there must exist y satisfying the condition in the claim. Since, as in the proof of Theorem 1.6, the potential cycles of $\mathcal{O}_R$ span the linear dependencies of the rows of M, the support of y contains a potential cycle of $\mathcal{O}_R$ containing the step e.

Conversely, let $\mathcal{O}$ be an A-admissible partial orientation where each step in $\mathcal{O}$ is part of a potential cycle. Encode the inequalities of $\mathrm{rg}(\mathcal{O})$ as $Mx < b$. Let $z \lneq 0$ be a vector in $\mathbb{R}^k$ and suppose $z_l < 0$ where the lth row of M corresponds to a step $e = (v_i, v_j)$. Let C be a potential cycle containing e (and note that if $e \notin \mathcal{O}$, then $\{(v_i, v_j), (v_j, v_i)\}$ is a potential cycle for $\mathcal{O}$ containing e). The vector corresponding to the steps of C satisfies the condition on y in the claim, and thus $\mathrm{rg}(\mathcal{O})$ is relatively bounded. □

2. Parking functions and the regions of $\Sigma_G(A)$

In this section, we explain how the indegree sequences of partial orientations of G are closely related to the parking functions of $G_\bullet$, the graph obtained from G by adding a vertex v_0 and an edge between v_0 and each $v \in V$. We will use $V_\bullet$ and $E_\bullet$ to denote the vertex and edge set, respectively, of $G_\bullet$. Our goal is to show that a natural set of labels for the regions of $\Sigma_G(A)$ are the set of parking functions of $G_\bullet$. Let $\mathbb{Z}V$ denote the free abelian group on the vertices in V.

Definition 2.1. Let $\mathcal{O}$ be a partial orientation of G. The *indegree sequence* of $\mathcal{O}$, denoted $\mathrm{indeg}(\mathcal{O})$, is $\sum_{i=1}^n \mathrm{indeg}_{\mathcal{O}}(v_i)\, v_i \in \mathbb{Z}V$.

Definition 2.2. A *parking function* $c = \sum_{i=1}^n c_i v_i$ of $G_\bullet$ with respect to v_0 is an element of $\mathbb{Z}V$ such that for every non-empty subset $W \subseteq V$, there exists $v_i \in W$ with $0 \leq c_i < d_W(v_i)$, where $d_W(v_i)$ is the number of edges $e = \{v_i, u\} \in E_\bullet$ with $u \in V_\bullet \setminus W$. For $c, c' \in \mathbb{Z}V$, we write $c \leq c'$ if $c_i \leq c'_i$ for all $0 \leq i \leq n$. A parking function c' is *maximal* if $c' \leq c$ for any parking function c implies $c = c'$.

Graphical parking functions were first formally introduced in [6]. However, the essentially equivalent notion of *superstable configurations* has been studied for longer in the context of the abelian sandpile model; see [5, §2.4] for a definition of these and their connection to parking functions. One easy observation from the above definition is that if c' is a maximal parking function and $c \in \mathbb{Z}V$ with $0 \leq c \leq c'$, then c is a parking function as well. The following characterization of maximal parking functions is Theorem 3.1 of Benson, Chakrabarty, and Tetali [2]:

Theorem 2.3. *A* total orientation *of a graph is a partial orientation where every edge is oriented. A* source *of a total orientation is a vertex whose outdegree equals its degree. The acyclic total orientations of $G_\bullet$ with unique source v_0 are in bijection with the maximal parking functions of $G_\bullet$ with respect to v_0. The bijection is given by $\mathcal{O} \mapsto \mathrm{indeg}(\mathcal{O}) - \sum_{i=1}^n v_i$.*

Proposition 2.4. *The set*

$$\{\mathrm{indeg}(\mathcal{O})\colon \mathcal{O} \textit{ is an acyclic partial orientation of } G\}$$

is the set of parking functions of $G_\bullet$ with respect to v_0.

Proof: If $\mathcal{O}$ is an acyclic total orientation of G, then $\mathcal{O}' := \mathcal{O} \cup \{(v_0, v_i)\}_{i=1}^n$ is an acyclic total orientation of $G_\bullet$ with unique source v_0, and $\mathrm{indeg}(\mathcal{O})$ is equal to $\mathrm{indeg}(\mathcal{O}') - \sum_{i=1}^n v_i$. Conversely, any acyclic total orientation $\mathcal{O}'$ of $G_\bullet$ with unique source v_0 restricts to an acyclic total orientation $\mathcal{O}$ of G, and the same indegree sequence identity holds. So maximal parking functions of $G_\bullet$ correspond to acyclic total orientations of G.

Consider any parking function c of $G_\bullet$. There exists a maximal parking function c' with $c \leq c'$. Let $\mathcal{O}'$ be the acyclic total orientation such that $\mathrm{indeg}(\mathcal{O}') = c'$. We can easily find $\mathcal{O} \subseteq \mathcal{O}'$ so that $\mathrm{indeg}(\mathcal{O}) = c$. Conversely, consider some acyclic partial orientation $\mathcal{O}$ of G. Take any edge $e = \{v_i, v_j\}$ that is blank in $\mathcal{O}$. Suppose both $\mathcal{O} \cup \{(v_i, v_j)\}$ and $\mathcal{O} \cup \{(v_j, v_i)\}$ have directed cycles. Then there was a directed path from v_i to v_j in $\mathcal{O}$ and a directed path from v_j to v_i in $\mathcal{O}$, so $\mathcal{O}$ already had a directed cycle, yielding a contradiction. Thus, we can repeatedly orient blank edges in $\mathcal{O}$ and arrive at an acyclic total orientation $\mathcal{O}' \supseteq \mathcal{O}$. Then $0 \leq \mathrm{indeg}(\mathcal{O}) \leq \mathrm{indeg}(\mathcal{O}')$ and $\mathrm{indeg}(\mathcal{O}')$ is a maximal parking function of $G_\bullet$. Therefore, $\mathrm{indeg}(\mathcal{O})$ is indeed a parking function. □

Theorem 1.6 makes clear that an A-admissible partial orientation is necessarily acyclic, as follows. Suppose $\mathcal{O}$ is a partial orientation that contains a directed cycle of steps $C = \{(u_1, u_2), (u_2, u_3), \ldots, (u_k, u_1)\}$. Then let

$$C^{\,\mathrm{rev}} := \{(u_1, u_k), (u_k, u_{k-1}), \ldots, (u_2, u_1)\}.$$

Since $\Sigma_G(A)$ has a central region R_0, the partial orientation $\mathcal{O}_{R_0} = \emptyset$ is A-admissible. So we have $\nu(C, \mathcal{O}) = -\nu(C^{\,\mathrm{rev}}, \mathcal{O}_{R_0}) < 0$ and thus $\mathcal{O}$ is not A-admissible. Thus, if $\mathcal{O}$ is A-admissible, Proposition 2.4 implies that $\mathrm{indeg}(\mathcal{O})$ is a parking function of $G_\bullet$. What remains to be proven is that any parking function of $G_\bullet$ can be realized as $\mathrm{indeg}(\mathcal{O})$ for some A-admissible $\mathcal{O}$. We are therefore interested in building A-admissible partial orientations with particular indegree sequences. The following "topological" lemma will allow us to build up A-admissible partial orientations from other A-admissible partial orientations with some control over the resulting indegrees.

Lemma 2.5. *Let $\mathcal{O}$ be an A-admissible partial orientation, and let $W \subseteq V$ be a subset of the vertices of G satisfying:*

(1) *there do not exist $u \in W^c$, $w \in W$ with $(w, u) \in \mathcal{O}$;*
(2) *there is some $u \in W^c$, $w \in W$ such that $\{u, w\}$ is a blank edge of $\mathcal{O}$.*

Then there exists $u \in W^c$ and $w \in W$ such that $\mathcal{O} \cup \{(u, w)\}$ is also A-admissible.

Proof: In this proof, all u_k are elements of W^c and all w_k are elements of W. By (2), we may choose a blank edge $\{u_1, w_1\}$ of $\mathcal{O}$. If $\mathcal{O}_1 := \mathcal{O} \cup \{(u_1, w_1)\}$ is not A-admissible, there exists some bad potential cycle C_1 containing (u_1, w_1) and some (w_2, u_2), where $\{u_2, w_2\}$ is necessarily blank by (1). Next we consider $\mathcal{O}_2 := \mathcal{O} \cup \{(u_2, w_2)\}$; if this not admissible, we get a bad cycle C_2 containing (u_2, w_2) and (w_3, u_3), and so on. Either we arrive at an admissible partial orientation, or this process goes on forever. Suppose it goes on forever. Because there are only a finite number of blank edges between W^c and W, eventually we

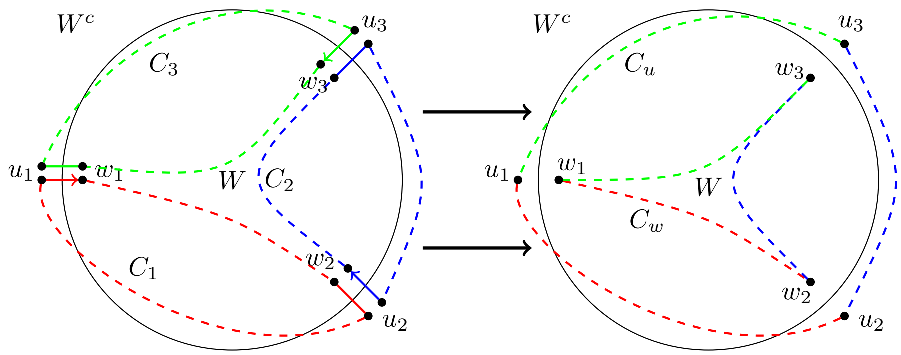


FIGURE 3. Diagram explaining the equality of the score sums in the proof of Lemma 2.5 with $i = 1$ and $j = 4$.

obtain $i < j$ where $(u_i, w_i) = (u_j, w_j)$. Consider $\sum_{k=i}^{j-1} \nu(C_k, \mathcal{O}_k)$. In this sum, the contribution of the step $(u_k, w_k) \in \mathcal{O}_k$ in C_k cancels with the contribution of the step $(w_k, u_k) \notin \mathcal{O}_{k-1}$ in C_{k-1}, and what remains is the sum of the scores relative to $\mathcal{O}$ of a cycle C_u joining u_{j-1} to u_{j-2} and so on to u_i and back to u_{j-1} and of a cycle C_w joining w_i to w_{i+1} and so on to w_{j-1} and back to w_i. Neither of the cycles C_u or C_w contain any of the directed edges (u_k, w_k). Figure 3 gives a diagrammatic explanation of the equality of these cycle score sums. But then $0 < \nu(C_u, \mathcal{O}) + \nu(C_w, \mathcal{O}) = \sum_{k=i}^{j-1} \nu(C_k, \mathcal{O}_k) \leq 0$, a contradiction. □

Theorem 2.6. *Let $\mathcal{O}$ be an acyclic partial orientation of G. Then there exists an A-admissible partial orientation $\mathcal{O}'$ such that* $\text{indeg}(\mathcal{O}) = \text{indeg}(\mathcal{O}')$.

Proof: If $\mathcal{O} = \emptyset$, there is nothing to prove. Otherwise, set $\mathcal{O}_0 := \emptyset$ and recursively define $\mathcal{O}_i$ from $\mathcal{O}_{i-1}$ as follows. Let W_i be the set of $v \in V$ such that $\text{indeg}_{\mathcal{O}_{i-1}}(v) < \text{indeg}_{\mathcal{O}}(v)$. If $W_i \neq \emptyset$, apply Lemma 2.5 to W_i and $\mathcal{O}_{i-1}$ and let $\mathcal{O}_i$ be the resulting partial orientation, i.e., set $\mathcal{O}_i := \mathcal{O}_{i-1} \cup \{(u, w)\}$ for the appropriate $u \in W_i^c$ and $w \in W_i$. We claim that the lemma may indeed be applied at every iteration. Suppose not. The first condition of the lemma clearly applies by construction. So suppose there exists $1 \leq i \leq |\mathcal{O}|$ such that every edge between some $u \in W_i^c$ and some $w \in W_i$ is already oriented as (u, w) in $\mathcal{O}_{i-1}$. Then for each $w \in W_i$, we have $\text{indeg}_{\mathcal{O}}(w) > |\{\{u, w\} \in E : u \in W_i^C\}|$, so w must have an arrow in $\mathcal{O}$ coming into it from some other vertex in W_i. But this forces $\mathcal{O}$ to contain a cycle of steps involving the vertices of W_i, which is a contradiction. Therefore Lemma 2.5 applies at every iteration as claimed. Setting $\mathcal{O}' := \mathcal{O}_{|\mathcal{O}|}$, we arrive at an A-admissible partial orientation with the desired indegree sequence. □

We are now prepared to prove the main result of this section, Corollary 2.8, which establishes a conjecture of Hopkins and Perkinson [5]. Indeed, Corollary 2.8 subsumes the main result of that paper (which was proved in a different way using the abelian sandpile model).

Definition 2.7. The following procedure is called the *Pak-Stanley labeling* of a bigraphical arrangement. It labels each region with an element of $\mathbb{Z}V$. Label the central region of $\Sigma_G(A)$ with 0. Put the central region in a queue, Q. Then, as long as Q is not empty:

(1) Remove the first region R from Q.
(2) For each unlabeled region R' bordering R:
   (a) Determine the unique indices $i \neq j$ such that $x_j - x_i < a_{ji}$ in R but $x_j > x_i + a_{ji}$ in R'.
   (b) If R is labeled by $c = \sum_{k=1}^{n} c_k v_k$, then label R' by $c' = c + v_j$.
   (c) Add R' to the end of Q.

Let $\lambda(R)$ denote the Pak-Stanley label of the region R.

Corollary 2.8. *The set*

$$\{\lambda(R)\colon R \text{ is a region of } \Sigma_G(A)\}$$

is the set of all parking functions of $G_\bullet$ with respect to v_0.

Proof: We first check inductively that $\lambda(R) = \mathrm{indeg}(\mathcal{O}_R)$. This identity clearly holds for the central region. So suppose $\lambda(R) = \mathrm{indeg}(\mathcal{O}_R)$ and that R' borders R with $x_j - x_i < a_{ji}$ in R but $x_j > x_i + a_{ji}$ in R'. Then we have

$$\lambda(R') = \lambda(R) + v_j = \mathrm{indeg}(\mathcal{O}_R) + v_j = \mathrm{indeg}(\mathcal{O}_{R'}).$$

By Proposition 2.4, for any parking function c of $G_\bullet$, there is some acyclic partial orientation $\mathcal{O}$ such that $\mathrm{indeg}(\mathcal{O}) = c$. Finally, by Theorem 2.6, we can find an A-admissible orientation $\mathcal{O}'$ with the same indegree sequence as $\mathcal{O}$, so we have $\lambda(\mathrm{rg}(\mathcal{O}')) = c$. □

Remark 2.9. In [5, §4], it is shown that

$$\{c \in \mathbb{Z}V : c = \mathrm{indeg}(\mathcal{O}) - \sum_{i=1}^{n} v_i \text{ for some acyclic partial orientation } \mathcal{O} \text{ of } G,$$
$$c_i = -1, \text{ and } c_j \geq 0 \text{ for all } j \neq i\}$$

is the set of parking functions of G with respect to v_i. Thus, in light of Theorem 1, Proposition 2.4, and Theorem 2.6, the parking functions of G with respect to each of its vertices are encoded in the regions of $\Sigma_G(A)$.

Corollary 2.10. *The number of regions of $\Sigma_G(A)$ is at least the number of spanning trees of $G_\bullet$.*

Proof: The number of spanning trees of a graph equals the number of parking functions of that graph with respect to any vertex (see [3] or [6]), so Corollary 2.8 implies this lower bound. □

3. Number of regions of $\Sigma_G(A)$

We have already seen that the number of regions of $\Sigma_G(A)$ is at least the number of spanning trees of $G_\bullet$. In this section we give further bounds on the number of regions of $\Sigma_G(A)$. The graph G remains fixed, but we will allow the parameter list A to vary (while always maintaining a central region).

Notation. We will denote the number of regions of $\Sigma_G(A)$ by $r(\Sigma_G(A))$ and the number of relatively bounded regions by $b(\Sigma_G(A))$.

Definition 3.1. Let $\mathcal{A}$ be a hyperplane arrangement. The hyperplanes $H_1, \ldots, H_k$ in $\mathcal{A}$ are *linearly independent* if their normals are linearly independent. The arrangement $\mathcal{A}$ is *generic* if

$$H_1 \cap \cdots \cap H_k \neq \emptyset \quad \Leftrightarrow \quad H_1, \ldots, H_k \text{ are linearly independent}$$

for all subsets $\{H_1, \ldots, H_k\} \subseteq \mathcal{A}$.

For instance, $\Sigma_G(A)$ is generic when the a_{ij} are algebraically independent. We will use GEN to denote the parameter list of an arbitrary generic bigraphical arrangement.

The *Tutte polynomial* of G is defined by

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{\kappa(A)-\kappa}(y-1)^{\kappa(A)+|A|-n}, \tag{2}$$

where $\kappa(A)$ is the number of components of the graph (V, A) and κ is the number of components of G.

Theorem 3.2. *For any generic bigraphical arrangement, the characteristic polynomial of* $\Sigma_G(\text{GEN})$ *is given by*

$$\chi_{\Sigma_G(\text{GEN})}(t) = (-2)^{n-\kappa}\, t^{\kappa}\, T_G(1-t/2, 1),$$

where $T_G(x,y)$ *is the Tutte polynomial of* G*. Consequently,*

$$r(\Sigma_G(\text{GEN})) = 2^{n-\kappa}\, T_G(3/2, 1);$$
$$b(\Sigma_G(\text{GEN})) = 2^{n-\kappa}\, T_G(1/2, 1).$$

Proof: As Stanley shows in [9, p. 412], the characteristic polynomial of a generic arrangement $\mathcal{A}$ is given by

$$\chi_{\mathcal{A}}(t) = \sum_{\mathcal{B}} (-1)^{|\mathcal{B}|} t^{n-|\mathcal{B}|},$$

where the sum is over all linearly independent subsets $\mathcal{B}$ of $\mathcal{A}$. So we must find the linearly independent subsets of $\Sigma_G(\text{GEN})$. Let the hyperplanes of $\Sigma_G(\text{GEN})$ be H_e with linear parts $L_e := x_i - x_j$ for steps $e = (v_i, v_j)$ of G. For a step $e = (v_i, v_j)$ of G, define $\pi(e) = \{v_i, v_j\}$. For $\mathcal{B} \subseteq \Sigma_G(\text{GEN})$, let $\pi(\mathcal{B})$ be the multiset $\{\pi(e)\}_{H_e \in \mathcal{B}}$. We claim that there exists a linear dependence among $\mathcal{B}$ if and only if there exists a cycle of undirected edges $\{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_k, u_1\}\} \subseteq \pi(\mathcal{B})$. Suppose there exists such a cycle and $H_{e_1}, \ldots, H_{e_k}$ are the corresponding hyperplanes. Then set $\lambda_{e_i} = 1$ if $e_i = (u_i, u_j)$ and $\lambda_{e_i} = -1$ if $e_i = (u_j, u_i)$. We see that $\sum_{i=1}^{k} \lambda_{e_i} L_{e_i} = 0$. Conversely, suppose there exists a linear dependence in some subset $\mathcal{B} \subseteq \Sigma_G(\text{GEN})$. Similarly to the proof of Theorem 1.6, the undirected cycles of G span the linear dependencies of $\Sigma_G(\text{GEN})$, so $\mathcal{B}$ must contain such a cycle. Thus, $\mathcal{B}$ is linearly independent if and only if $\pi(\mathcal{B})$ is a forest of G. For each edge $e = \{v_i, v_j\}$ in such forest, there are two hyperplanes $H_{(v_i,v_j)}$ or $H_{(v_j,v_i)}$ we could include in $\mathcal{B}$. A forest F thus corresponds to $2^{|F|}$ linearly independent subsets $\mathcal{B} \subseteq \Sigma_G(\text{GEN})$. Therefore, the characteristic polynomial of $\Sigma_G(\text{GEN})$ is

$$\chi_{\Sigma_G(\text{GEN})}(t) = \sum_{F} (-2)^{|F|} t^{n-|F|},$$

where the sum is over all forests F of G. A straightforward calculation from (2) shows that

$$\sum_{i=0}^{n-\kappa} f_i\, t^i = t^{n-\kappa}\, T_G(1+1/t,1),$$

where f_i is the number of forests of G of size i. So we have

$$\begin{aligned}\chi_{\Sigma_G(\mathrm{GEN})}(t) &= \sum_{i=0}^{n-\kappa} f_i\,(-2)^i\, t^{n-i}\\ &= t^n \sum_{i=0}^{n-\kappa} f_i \left(\frac{-2}{t}\right)^i\\ &= (-2)^{n-\kappa}\, t^\kappa\, T_G(1-t/2,1).\end{aligned}$$

A classical result in the theory of hyperplane arrangements, Zaslavsky's theorem [11], relates the number of regions and number of relatively bounded regions of a hyperplane arrangement $\mathcal{A}$ to its characteristic polynomial:

$$\begin{aligned} r(\mathcal{A}) &= |\chi_{\mathcal{A}}(-1)|;\\ b(\mathcal{A}) &= |\chi_{\mathcal{A}}(1)|.\end{aligned}$$

Zavlasky's theorem establishes that the formulas for the number of regions and bounded regions are as claimed. □

Corollary 3.3. *Suppose G is planar and G^* is its dual graph. Then the following are equal:*

(1) *the probability that after removing each edge from G^* with probability* $2/3$*, the resulting graph remains connected;*
(2) *the probability that a partial orientation of G chosen uniformly at random is* GEN*-admissible.*

Proof: The first probability is given by $R_{G^*}(2/3)$, where $R_{G^*}(p)$ is the all-terminal reliability polynomial of G^*. Let V^* be the vertex set of G^*, and let E^* be its edge set. A well-known formula connecting the reliability and Tutte polynomials (see [10, p. 1335]) is

$$R_{G^*}(p) = (1-p)^{|V^*|-1} p^{|E^*|-|V^*|+1} T_{G^*}(1,1/p).$$

Suppose G has f faces. Then the probability of the first event is

$$\begin{aligned} R_{G^*}(2/3) &= (1-2/3)^{|V^*|-1}(2/3)^{|E^*|-|V^*|+1} T_{G^*}(1,3/2)\\ &= \frac{2^{|E|-f+1} T_G(3/2,1)}{3^{|E|}}\\ &= \frac{2^{n-\kappa} T_G(3/2,1)}{3^{|E|}}\\ &= \frac{r(\Sigma_G(\mathrm{GEN}))}{3^{|E|}},\end{aligned}$$

where in the second line we have used the formula $T_{G^*}(y,x) = T_G(x,y)$ (see [10, p.1131]), in the third line we have used Euler's formula $n-|E|+f = 1+\kappa$, and in the last line we applied Theorem 3.2. The result now follows from Theorem 1.6. □

Definition 3.4. Let $\mathcal{O}$ be a partial orientation. A potential cycle for $\mathcal{O}$ is *very bad* if has a negative score. We say $\mathcal{O}$ is *almost-A-admissible* if it is not A-admissible but it has no very bad potential cycles. We say $\mathcal{O}$ is *far-from-A-admissible* if it has a very bad potential cycle.

Proposition 3.5. *For an almost-A-admissible partial orientation $\mathcal{O}$, define $w(\mathcal{O})$ to be the number of steps e of G, including blanks of $\mathcal{O}$, belonging to some potential cycle C for $\mathcal{O}$ with $\nu_A(C, \mathcal{O}) = 0$. Define $z(\mathcal{O})$ to be the maximum k such that there exists disjoint potential cycles $C_1, \dots, C_k$ for $\mathcal{O}$ with $\nu_A(C_i, \mathcal{O}) = 0$ for all i. Then*

$$\sum_{\mathcal{O}} \frac{1}{2^{w(\mathcal{O})}} \leq r(\Sigma_G(\text{GEN})) - r(\Sigma_G(A)) \leq \sum_{\mathcal{O}} \frac{1}{2^{z(\mathcal{O})}},$$

where the sum is over all almost-A-admissible partial orientations $\mathcal{O}$.

Proof: Let S denote the set of $2|E|$ steps of G. For each $(v_i, v_j) \in S$, let $\varepsilon_{ij} > 0$ be a real number. For each sign pattern $\sigma \in \{-1, 1\}^S$, let $\sigma_{ij} := \sigma(v_i, v_j)$, and define the parameter list A^σ with parameters $a^\sigma_{ij} := a_{ij} + \sigma_{ij}\,\varepsilon_{ij}$ for all steps $(v_i, v_j) \in S$. Take the real numbers ε_{ij} small enough and generic so that for all choice of σ, each $\Sigma_G(A^\sigma)$ is generic, each A-admissible partial orientation is A^σ-admissible, and each far-from-A-admissible partial orientation is far-from-A^σ-admissible. No partial orientations are almost-admissible for a generic arrangement. Let $\text{Gain}(\sigma)$ be the set of almost-A-admissible partial orientations that are A^σ-admissible. Note that $|\text{Gain}(\sigma)| = r(\Sigma_G(A^\sigma)) - r(\Sigma_G(A))$. But then by Theorem 3.2, we have $r(\Sigma_G(A^\sigma)) = r(\Sigma_G(\text{GEN}))$ for any $\sigma \in \{-1, 1\}^S$. So in order to bound $r(\Sigma_G(\text{GEN})) - r(\Sigma_G(A))$, we bound $|\text{Gain}(\sigma)|$.

Define $X = \{(\sigma, \mathcal{O}) : \mathcal{O} \in \text{Gain}(\sigma)\}$. Because $|\text{Gain}(\sigma)|$ is independent of σ, we have $|X| = 2^{|S|}|\text{Gain}(\sigma)|$. Fix some almost-$A$-admissible partial orientation $\mathcal{O}$ and suppose $w(\mathcal{O}) = k$. Let $S' \subseteq S$ be the set of steps of G belonging to some potential cycle C for $\mathcal{O}$ with $\nu_A(C, \mathcal{O}) = 0$. Suppose σ is such that for each $e = (v_i, v_j) \in S'$, we have $\sigma_{ij} = 1$ if $e \notin \mathcal{O}$ and $\sigma_{ji} = -1$ if $e \in \mathcal{O}$. Then $\mathcal{O}$ is A^σ admissible: let C be any potential cycle for $\mathcal{O}$ with $\nu_A(C, \mathcal{O}) = 0$; then,

$$\begin{aligned}\nu_{A^\sigma}(C, \mathcal{O}) &= \nu_A(C, \mathcal{O}) + \sum_{\substack{(v_i,v_j)\in C \\ \{v_i,v_j\}\notin \mathcal{O}}} \sigma_{ij}\varepsilon_{ij} - \sum_{\substack{(v_i,v_j)\in C \\ (v_i,v_j)\in \mathcal{O}}} \sigma_{ji}\varepsilon_{ji} \\ &= \sum_{\substack{(v_i,v_j)\in C \\ \{v_i,v_j\}\notin \mathcal{O}}} \varepsilon_{ij} + \sum_{\substack{(v_i,v_j)\in C \\ (v_i,v_j)\in \mathcal{O}}} \varepsilon_{ji} > 0.\end{aligned}$$

Since we are free to choose the sign of σ associated to any step not in S', there are at least $2^{|S|-k}$ sign patterns σ with $\mathcal{O} \in \text{Gain}(\sigma)$. We have $|\{\sigma : (\sigma, \mathcal{O}) \in X\}| \geq 2^{|S|-k}$. So,

$$|X| = \sum_{\mathcal{O}} |\{\sigma : (\sigma, \mathcal{O}) \in X\}| \geq \sum_{\mathcal{O}} 2^{|S|-w(\mathcal{O})},$$

where the sum is over all almost-A-admissible partial orientations $\mathcal{O}$.

Now fix some almost-A-admissible partial orientation $\mathcal{O}$ and suppose $z(\mathcal{O}) = k$. Let $\mathcal{C} := \{C_1, \dots, C_k\}$ be a set of disjoint potential cycles for $\mathcal{O}$ with $\nu_A(C_i, \mathcal{O}) = 0$ for all i. For each $\sigma \in \{-1, 1\}^S$ and each subset $\mathcal{D} \subseteq \mathcal{C}$, define $\text{flip}(\sigma, \mathcal{D})$ to be the same sign pattern as σ except that for any step $e = (v_i, v_j)$ in a cycle $C \in \mathcal{D}$, we have $\text{flip}(\sigma, \mathcal{D})_{ij} = -\sigma_{ij}$ if $e \notin \mathcal{O}$ and $\text{flip}(\sigma, \mathcal{D})_{ji} = -\sigma_{ji}$ if $e \in \mathcal{O}$. Write $\sigma \sim \tau$ if there exits $\mathcal{D} \subseteq \mathcal{C}$ such that $\tau = \text{flip}(\sigma, \mathcal{D})$. The relation $\sim$ defines an equivalence relation. Further, $\mathcal{O}$ belongs to at most one of the $\mathrm{Gain}(\sigma)$ among all σ in some equivalence class. Indeed, suppose $\mathcal{O} \in \mathrm{Gain}(\sigma)$ and $\tau \sim \sigma$ but $\tau \neq \sigma$. Then let $\mathcal{D}$ be such that $\tau = \mathrm{flip}(\sigma, \mathcal{D})$. Note that $\mathcal{D}$ is nonempty. So for any potential cycle $C \in \mathcal{D}$, we have

$$\begin{aligned}\nu_{A^\tau}(C,\mathcal{O}) &= \nu_A(C,\mathcal{O}) + \sum_{\substack{(v_i,v_j)\in C\\ \{v_i,v_j\}\notin\mathcal{O}}} \tau_{ij}\varepsilon_{ij} - \sum_{\substack{(v_i,v_j)\in C\\ (v_i,v_j)\in\mathcal{O}}} \tau_{ji}\varepsilon_{ji} \\ &= -\sum_{\substack{(v_i,v_j)\in C\\ \{v_i,v_j\}\notin\mathcal{O}}} \sigma_{ij}\varepsilon_{ij} + \sum_{\substack{(v_i,v_j)\in C\\ (v_i,v_j)\in\mathcal{O}}} \sigma_{ji}\varepsilon_{ji} \\ &= -\nu_{A^\sigma}(C,\mathcal{O}) < 0.\end{aligned}$$

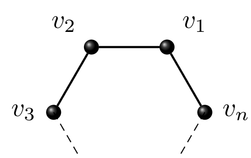
Thus, $\mathcal{O} \notin \mathrm{Gain}(\tau)$. Since $\mathcal{O}$ is in at most one of the $\mathrm{Gain}(\sigma)$ among σ in an equivalence class and each class has 2^k members, we have $|\{\sigma : (\sigma,\mathcal{O}) \in X\}| \leq 2^{|S|-k}$. So,

$$|X| = \sum_{\mathcal{O}} |\{\sigma : (\sigma,\mathcal{O}) \in X\}| \leq \sum_{\mathcal{O}} 2^{|S|-z(\mathcal{O})},$$

where the sum is over all almost-A-admissible partial orientations $\mathcal{O}$. But then recall that $|\mathrm{Gain}(\sigma)| = (1/2^{|S|})|X|$. Therefore,

$$\sum_{\mathcal{O}} \frac{1}{2^{w(\mathcal{O})}} \leq |\mathrm{Gain}(\sigma)| \leq \sum_{\mathcal{O}} \frac{1}{2^{z(\mathcal{O})}}. \ \square$$

Example 3.6. Consider the cycle graph C_n, labeled as below:



The right inequality from Proposition 3.5 becomes an equality for C_n: for any almost-A-admissible $\mathcal{O}$, there is only one potential cycle C for $\mathcal{O}$ for which we have $\nu_A(C,\mathcal{O}) = 0$. Thus, using the notation from the proof of Proposition 3.5, we have $\mathcal{O} \in \mathrm{Gain}(\sigma)$ if and only if $\mathcal{O} \notin \mathrm{Gain}(\mathrm{flip}(\sigma,\{C\}))$. Further, any $\mathcal{O} \in \mathrm{Gain}(\sigma)$ is relatively bounded by the condition of Theorem 1.8. So,

$$\begin{aligned} r(\Sigma_{C_n}(A)) &= r(\Sigma_{C_n}(\mathrm{GEN})) - |\{\mathcal{O}\colon \mathcal{O} \text{ is almost-}A\text{-admissible}\}|/2; \\ b(\Sigma_{C_n}(A)) &= b(\Sigma_{C_n}(\mathrm{GEN})) - |\{\mathcal{O}\colon \mathcal{O} \text{ is almost-}A\text{-admissible}\}|/2. \end{aligned}$$

This lets us compute exact formulas for $r(\Sigma_{C_n}(A))$. For instance, we now show

$$\begin{aligned} r(\Sigma_{C_n}(\mathrm{SEMI})) &= \begin{cases} 3^n - 2^n & \text{if } n \text{ is odd,} \\ 3^n - 2^n - \binom{n}{n/2} & \text{if } n \text{ is even.} \end{cases} \\ b(\Sigma_{C_n}(\mathrm{SEMI})) &= \begin{cases} 2^n - 1 & \text{if } n \text{ is odd,} \\ 2^n - 1 - \binom{n}{n/2} & \text{if } n \text{ is even.} \end{cases} \\ r(\Sigma_{C_n}(\mathrm{SHI})) &= 3^n - 2^n - n. \\ b(\Sigma_{C_n}(\mathrm{SHI})) &= 2^n - 1 - n. \end{aligned}$$

To see this, first note that $T_{C_n}(x,y) = y + \sum_{i=1}^{n-1} x^i$, so

$$r(\Sigma_{C_n}(\mathrm{GEN})) = 2^{n-1} T_{C_n}(3/2, 1) = 3^n - 2^n;$$

$$b(\Sigma_{C_n}(\text{GEN})) = 2^{n-1}T_{C_n}(1/2,1) = 2^n - 1.$$

When n is odd, there are no almost-SEMI-admissible partial orientations. When n is even, there are $2\binom{n}{n/2}$ almost-SEMI-admissible partial orientations: choose half of the edges of C_n and orient them all the same way. Regardless of the parity of n, there are $2n$ almost-SHI-admissible partial orientations; there are four cases:

(1) Orient $\{v_1, v_n\}$ as (v_n, v_1) and leave one of the edges $\{v_i, v_{i+1}\}$ blank for $1 \le i \le n-1$ while orienting the rest as (v_i, v_{i+1}).
(2) Orient $\{v_1, v_n\}$ as (v_1, v_n) and leave all the other edges blank.
(3) Leave $\{v_1, v_n\}$ blank and orient one of the $\{v_i, v_{i+1}\}$ for $1 \le i \le n-1$ as (v_{i+1}, v_i) while leaving leaving the rest blank.
(4) Leave $\{v_1, v_n\}$ blank and orient all of the $\{v_i, v_{i+1}\}$ for $1 \le i \le n-1$ as (v_i, v_{i+1}). □

Corollary 3.7. *The maximum number of regions of $\Sigma_G(A)$ over all parameter matrices A is $2^{n-1}T_G(3/2,1)$. This maximum is achieved by $\Sigma_G(\text{GEN})$.*

Proof: This follows from Theorem 3.2 and Proposition 3.5. □

We can slightly refine the lower bound for the number of regions given by Corollary 2.10 by considering the degrees of the parking functions of $G_\bullet$.

Definition 3.8. The *degree* of a parking function $c = \sum_{i=1}^n c_i v_i$ of $G_\bullet$ with respect to v_0 is

$$\deg(c) := \sum_{i=1}^{n} c_i.$$

Proposition 3.9. *Let $g := |E_\bullet| - |V_\bullet| + 1 = |E|$ be the genus of $G_\bullet$. Define the vector $h(G_\bullet) := (h_0, h_1, \ldots, h_g)$, where*

$$h_i := |\{c\colon c \text{ is a parking function of } G_\bullet \text{ with respect to } v_0 \text{ and } \deg(c) = i\}|.$$

Define $p(G,A) := (p_0, p_1, \ldots, p_g)$, where

$$p_i := |\{\mathcal{O}\colon \mathcal{O} \text{ is an } A\text{-admissible partial orientation of } G \text{ and } |\mathcal{O}| = i\}|.$$

Then $h(G_\bullet) \le p(G,A)$.

Proof: This is immediate from the proof of Corollary 2.8: for any region R of $\Sigma_G(A)$, we have $\deg(\lambda(R)) = |\mathcal{O}_R|$. □

The lower bound from Corollary 2.10 is sometimes sharp, as in the case of the complete graph K_n.

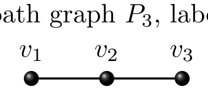
Proposition 3.10. *The minimum number of regions of $\Sigma_{K_n}(A)$ over all parameter matrices A is $(n+1)^{(n-1)}$. This minimum is achieved by $\Sigma_{K_n}(\text{SHI})$.*

Proof: Shi [7] proved that the number of regions of the Shi arrangement is given by $(n+1)^{(n-1)}$, the number of spanning trees of K_{n+1}. □

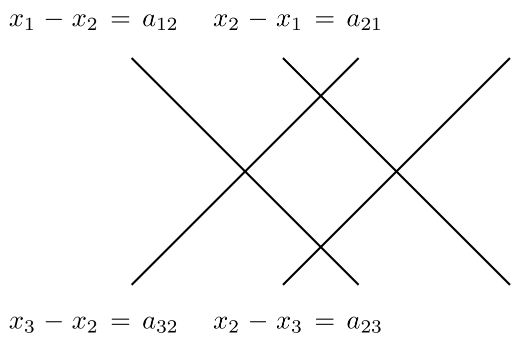
Remark 3.11. Our Corollary 2.8 provides an alternative proof of the bijection of Stanley and Pak [8] between regions of the Shi arrangement and parking functions.

However, the lower bound from Corollary 2.10 is in general not sharp:

Example 3.12. Consider the path graph P_3, labeled as below:



Then the essentialization of any $\Sigma_{P_3}(A)$ looks like



So $r(\Sigma_{P_3}(A)) = 9$ for any A, but there are only 8 parking functions of $P_{3\bullet}$ with respect to v_0: 0, v_1, v_2, v_3, $v_1 + v_2$, $v_1 + v_3$, $2v_2$, and $v_2 + v_3$. □

Also, $\Sigma_G(\text{SHI})$ does not in general achieve the minimum number of regions of $\Sigma_G(A)$:

Example 3.13. Consider the cycle graph C_4. As was shown in Example 3.6, we have $r(\Sigma_{C_4}(\text{SHI})) = 61$ while $r(\Sigma_{C_4}(\text{SEMI})) = 59$. □

Computing the minimum number of regions of $\Sigma_G(A)$ for arbitrary G and finding a parameter list A that achieves this minimum remain open problems. Proposition 3.5 suggests that maximizing the number of almost-A-admissible partial orientations may minimize the number of regions of $\Sigma_G(A)$, but the exact relationship between almost-admissible orientations and the number of regions remains unclear.

References

[1] Drew Armstrong and Brendon Rhoades. The Shi arrangement and the Ish arrangement. *Trans. Amer. Math. Soc.*, 364(3):1509–1528, 2012.

[2] Brian Benson, Deeparnab Chakrabarty, and Prasad Tetali. G-parking functions, acyclic orientations and spanning trees. *Discrete Math.*, 310(8):1340–1353, 2010.

[3] Denis Chebikin and Pavlo Pylyavskyy. A family of bijections between G-parking functions and spanning trees. *J. Comb. Theory Ser. A*, 110(1):31–41, 2005.

[4] Art Duval, Caroline Klivans, and Jeremy Martin. The G-shi arrangement, and its relation to G-parking functions. `http://www.math.utep.edu/Faculty/duval/papers/nola.pdf`, January 2011.

[5] Sam Hopkins and David Perkinson. Orientations, semiorders, arrangements, and parking functions. *Elec. J. of Combin.*, 19(4), 2012.

[6] Alexander Postnikov and Boris Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. *Trans. Amer. Math. Soc.*, 356(8):3109–3142 (electronic), 2004.

[7] Jian Yi Shi. *The Kazhdan-Lusztig cells in certain affine Weyl groups*, volume 1179 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[8] Richard P. Stanley. Hyperplane arrangements, interval orders, and trees. *Proc. Nat. Acad. Sci.*, 93(6):2620–2625, 1996.

[9] Richard P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 389–496. Amer. Math. Soc., Providence, RI, 2007.

[10] D. J. A. Welsh and C. Merino. The Potts model and the Tutte polynomial. *J. of Math. Phys.*, 41(3):1127–1152, 2000.

[11] Thomas Zaslavsky. Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. *Mem. Amer. Math. Soc.*, 1(154):102, 1975.

E-mail address: shopkins@mit.edu

E-mail address: davidp@reed.edu

Reed College, Portland OR, 97202