

Multi-person siteswaps
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Is it a siteswap? Is 51414 a valid siteswap pattern? There is a well-known criterion that answers this question: add 0 to the first number in the sequence, 1 to the second, 2 to the third, and so on:

$$\begin{array}{r} 5\ 1\ 4\ 1\ 4 \\ +\ 0\ 1\ 2\ 3\ 4 \\ \hline 5\ 2\ 6\ 4\ 8 \end{array}$$

Now subtract the period, in this case 5, from each digit of the resulting sequence until all the numbers appearing in the sequence are less than the period (but not negative):

$$\begin{array}{ccc} 5\ 2\ 6\ 4\ 8 & \text{subtract } 5\text{s} & 0\ 2\ 1\ 4\ 3 \\ & \longrightarrow & \end{array}$$

The resulting sequence is a rearrangement of 0, 1, 2, . . . , as above, if and only if the original sequence was a siteswap. As another example, consider the sequence 434. Adding 0, 1, and 2 gives 446; subtracting the period, 3, gives 110. Since 110 is not a rearrangement of 012, the sequence 434 is not a valid siteswap pattern.

Going backwards, we get a way of writing down every possible siteswap sequence of period p . Arrange the numbers 0, 1, . . . , $p - 1$ in some order. Now *subtract* 0, 1, 2, up to $p - 1$ and *add* multiples of the period to entries in the resulting sequence. With little more trouble, you can use the same idea to create multiple-person patterns, where the number of possibilities—even for jugglers of modest skill—is staggering. I'll present the idea below in the form of a card game.

Card Game. Here is a recipe for creating multiple-person siteswap passing patterns.

1. To create a passing pattern of period p among several jugglers, each juggler creates cards numbered 0, 1, 2, . . . , $p - 1$. The cards are combined, shuffled, then redistributed: p to each juggler. In the example below, $p = 3$.

$$\begin{array}{ccc} & \text{UNSHUFFLED} & & & \text{SHUFFLED} \\ \text{Alice:} & 0\ 1\ 2 & \longrightarrow & & 0\ 2\ 2 \\ \text{Bob:} & 0\ 1\ 2 & & & 1\ 1\ 0 \end{array}$$

2. Each juggler lays out his or her cards in some order. Subtract 0 from the first card, 1 from the second, 2 from the third, and so on up to subtracting $p - 1$ from the p -th card. Add p to any entries that are negative. Each juggler now has a siteswap pattern, but the number of balls probably needs to be adjusted, and we need to decide where each throw will be passed.

$$\begin{array}{ccccccc} \text{Alice:} & 0\ 2\ 2 & \text{subtract} & 0\ 1\ 2 & = & 0\ 1\ 0 & \text{add } 3\text{s} & \text{SITESWAP} \\ \text{Bob:} & 1\ 1\ 0 & - & 0\ 1\ 2 & = & 1\ 0\ -2 & \longrightarrow & 0\ 1\ 0 \\ & & & & & & & 1\ 0\ 1 \end{array}$$

3. To find out the number of balls in the pattern so far, add up all the numbers just calculated by all of the players and divide by the period. To determine the number of balls being juggled by any particular player, just add up that player's numbers and divide by the period. Don't be surprised if the number is fractional. You can adjust the number of balls by adding p , the period, to any entry (resulting in higher throws).

		SITESWAP
0 1 0	add five 3s	3 4 3
1 0 1	→	4 3 1
one ball		six balls

The resulting siteswap has six balls since $((3 + 4 + 3) + (4 + 3 + 1))/3 = 6$. The first juggler is juggling $(3 + 4 + 3)/3 = 3\frac{1}{3}$ balls and the second is juggling $2\frac{2}{3}$ balls (slightly easier than first juggler).

4. Now for the fun part. We need to decide which throws will be passes. There are lots of choices. First note that each throw in each juggler's siteswap pattern comes from one of his or her original cards. For example, Alice's third shuffled card is a 2, which corresponds with the third throw in her siteswap pattern, namely a 3.

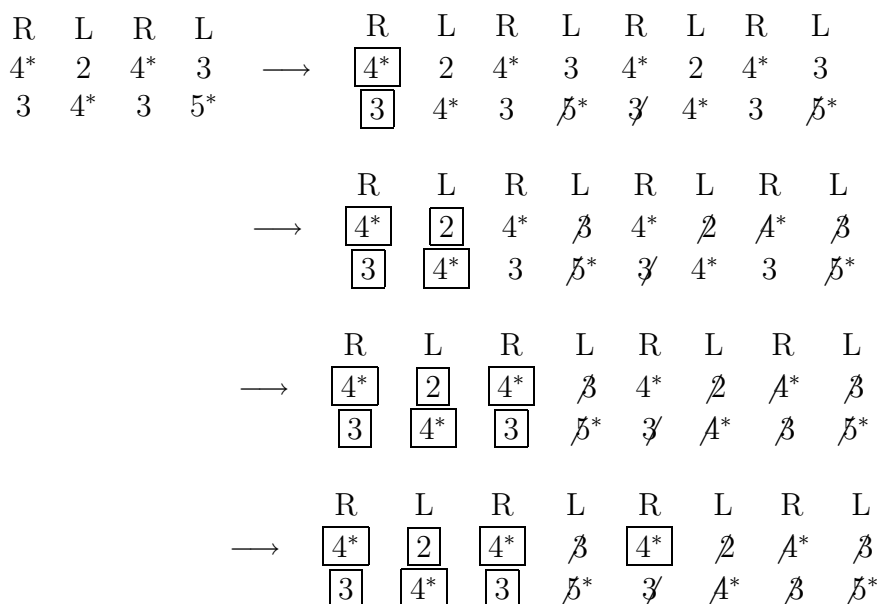
	SHUFFLED		SITESWAP
Alice:	0 2 2	↔	3 4 3
Bob:	1 1 0		4 3 1

Before starting the pattern, the rule is that jugglers must trade cards in anyway they want as long as in the end each juggler ends up with the numbers $0, 1, \dots, p - 1$ again but not necessarily with his or her original cards. Each trade you make tells you to whom to pass the corresponding throw in the siteswap: if player X gives a card to player Y , then player X 's siteswap throw corresponding to that card goes to player Y .

In the example above, Alice and Bob need to trade cards so that each has the cards 0, 1, and 2 again. That means that at least Alice must give a 2 to Bob and Bob must give a 1 to Alice. Both have two choices here. In addition, Alice and Bob could trade zeros. Let's say that Alice gives her first 2 (her second card) to Bob and Bob gives his first 1 (his first card) to Alice. These trades correspond to passes which we'll indicate with asterisks. (Since there are only two people, there is no confusion as to which person a pass is going.)

	SITESWAP
	3 4* 3
	4* 3 1

Starting the pattern. Before starting the pattern, the jugglers need to know how many balls to place in each hand. First write out a couple of cycles of the pattern, and circle each player's first ball. After being thrown, these balls will eventually land, and be thrown again. Cross off the throws in the pattern that are accounted for by these first balls. Now go to the second beat of the pattern. Circle any balls that are not crossed out, and cross out the throws that are accounted for by these balls. Above the pattern, write alternating R's and L's to denote right- and left-hand throws. The circles tell you how many balls to hold in each hand when starting the pattern. Here's an example using a 7-ball pattern of period 4.



So in this pattern, the first juggler starts with three balls in her right hand and one in her left and the other starts with two in his right and one in his left.

Miscellaneous pointers.

- If all jugglers start with their right hands, then passes with even numbers go across and passes with odd numbers stay on the same side. For instance, an even-numbered pass will go from a juggler's left hand to the receiver's left hand, while an odd-numbered pass will go from a juggler's left hand to the receiver's right hand, straight across. To get the opposite effect, say with two jugglers, have one juggler start RLRL... and the other start LRLR...
- The method described here is perfect for finding patterns between jugglers of different abilities. Just adjust the number of balls each player juggles by adding or subtracting the period from any entry in the siteswap pattern. I have found this especially useful for passing with children.
- To slow things down, pay attention to how much time you have before you need to catch the next pass. In general, don't pass until you are forced to do so.

- Start creating patterns with small periods. As the period gets larger, this method becomes more inflexible (and interesting). The problem is that in making adjustments, you add the period, which is more likely to result in a high throw. One way to cope with this problem, instead of starting over, is to just swap the offending card with some other card, and only adjust the corresponding part of the sequence.
- Practice your part of the pattern by itself, without the other jugglers, pretending to make the passes. It turns out that this is strictly possible if and only if the cards you receive after shuffling contain no repeated cards, i.e., you get the cards $0, 1, \dots, p - 1$ in some order. Even if this is not the case, you can usually find single-person patterns that are close to the one that is actually required.

Exercises.

- Make up your own multiple-person pattern and juggle it. Start with period 3.
- Here is a nice easy pattern for three people. The superscripts denote which throws are passes, and to whom.

	SITESWAP	STARTING
juggler A:	1 3 ^B 3	2 balls in R (right hand)
juggler B:	3 1 3 ^C	1 in each hand
juggler C:	3 ^A 3 1	2 balls in R, 1 in L (left)

The pattern uses 7 balls: two for each juggler, plus an extra to share.

- A challenging pattern for four people in which each juggler juggles 3-1/4 balls. (For mathematicians: note that this pattern and the previous both come from the identity matrix, mod the period.)

	SITESWAP	STARTING
juggler A:	1 4 ^B 4 4	2 in R, 1 in L
juggler B:	4 1 4 ^C 4	1 in R, 2 in L
juggler C:	4 4 1 4 ^D	2 in R, 1 in L
juggler D:	4 ^A 4 4 1	2 in R, 2 in L

- Systematically try all the period-2 patterns you can do. Among these, you'll find standard 7-ball juggling between two people: $4^* 3|3 4^*$, one juggler starting with his left and the other with his right.
- Some 2-person patterns with one beginning juggler.

SITESWAP	SITESWAP	SITESWAP	
4* 2 0	4 4 1 5*	4* 3 1	
3 3* 3	1 5* 0 0	0 1 3*	
SITESWAP	SITESWAP	SITESWAP	SITESWAP
1 2 4*	1 2 4*	1 2 4*	1 2 4*
1 4 3*	1 4* 3	4 4 3*	4 4* 3

- What are these?

SITESWAP					SITESWAP						
1	2	3*	4*	5*	4*	3	3	4*	2	3	3
3	3*	3*	3	3*	3*	2	3	3	3*	3	3

In the second pattern here, if the first juggler starts one-half beat late, both jugglers' passes become $3\frac{1}{2}$'s, resulting in identical patterns for both. This is known as *Jim's 3-count*.

- Our method of creating juggling patterns can be used in reverse. The idea is to start with two siteswap patterns and go in reverse to find which distribution of shuffled cards gave rise to the pattern. The utility in this is that knowing how the cards were shuffled reveals which throws can be made to splice together a pattern. It is easy to go back: just add 0, 1, 2, etc. where before you subtracted. For instance:

	SITESWAP							SHUFFLE
Alice:	4 4 1	add	0 1 2	=	4 5 3	subtract 3s		1 2 0
Bob:	3 3 3	+	0 1 2		3 4 5	→		0 1 2

Thus, for instance, Alice and Bob could trade the cards numbered 1. This corresponds to Alice throwing the first of her 4s in the 441 pattern and Bob throwing his second 3 in his 333 pattern. Try it.

Stay in touch! If you find a really nice pattern, please let me know about it via email: davidp@reed.edu.