# Mathematical aspects of the abelian sandpile model

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## 1 Introduction

In 1987, Bak, Tang and Wiesenfeld (BTW) introduced a lattice model of what they called "self-organized criticality". Since its appearance, this model has been studied intensively, both in the physics and in the mathematics literature. This model shows that a simple dynamics can lead to the emergence of very complex structures and drives the system towards a stationary state which shares several properties of equilibrium systems at the critical point, e.g. power law decay of cluster sizes and of correlations of the height-variables.

Some years later, Deepak Dhar generalized the model, discovered the "abelian group structure of addition operators" in it and called it "the abelian sandpile model"( abbreviated from now on ASM). He studied the self-organized critical nature of the stationary measure and gave an algorithmic characterization of recurrent configurations, the so-called "burning algorithm". This algorithm gives a one-to one correspondence between the recurrent configurations of the ASM and rooted spanning trees. The correspondence with spanning trees allowed Priezzhev to compute the height probabilities in dimension 2 in the infinite volume limit. Probabilities of certain special events -so-called "weakly allowed clusters"- can be computed exactly in the infinite volume limit using the "Bombay-trick". In the physics literature people studied critical exponents with scaling arguments, renormalization group method and conformal field theory (in $d = 2$), and it is argued that the critical dimension of the model is $d = 4$ [35]. Dhar and Majumdar studied the model on the Bethe lattice where they computed various correlation functions and avalanche cluster-size distributions exactly in the thermodynamic limit, using a transfer matrix approach.

Since the discovery of the abelian group structure of the set of of recurrent configurations, in the mathematics literature (especially combinatorics and algebraic combinatorics) one (re)introduces the model under the names "chip-firing game, Dirichlet

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game"[6], [7]. "The sandpile group of a graph" is studied [6], and the intriguing geometric structure of its neutral element is investigated e.g. in [23]. Asymmetric models are studied in [36], [15], and one finds a generalization of the "burning algorithm": the so-called "script algorithm".

In the meanwhile, several other models of "self-organized criticality" were introduced, such as e.g. forest fires, the Bak-Sneppen evolution model, and Zhang's model which is a (non-trivial) variation of the BTW-model with continuous heights (see [38], [3], [21] for a good overview of SOC-models). The paradigm of "self-organized criticality" became very popular and is now used in many areas of natural sciences such as geology, biology, cosmology. The underlying idea is always that some simple dynamical mechanism generates a stationary state in which complex behavior (manifested e.g. through power laws and large avalanches) appears in the limit of large system sizes. This complex structure appears "spontaneously", i.e., as a result of the dynamics, more precisely without having fine-tuned certain parameters (such as temperature or magnetic field). Let us remark however that the concept of SOC, especially as opposed to ordinary criticality, has also been criticized see e.g. [13]. From the mathematical point of view, one does not have a clear definition of SOC, but this is not really a problem because the models in which SOC appears such as the BTW-model are interesting and challenging, and are a starting point of many well-defined mathematical questions.

These notes provide an introduction to the ASM and focus on the problem of defining and studying the stationary measure of the ASM and its stationary dynamics *in the infinite volume limit*. There are several motivations to do this. First, all interesting properties such as power-law decay of avalanche sizes and height-correlations, emerge in the large volume limit. It is therefore a pertinent and basic question whether this large-volume limit actually exists, i.e., whether all expectations of local functions of height variables have a well-defined infinite volume limit. Although for special local functions this can be proved because the infinite volume limit expectations can be computed exactly using the "Bombay-trick", for simple functions such as the indicator that two neigboring sites have height two, it was not even known whether the infinite volume expectation exists.

Second, from the mathematical point of view, defining the infinite volume dynamics is a non-trivial and interesting problem because one has to deal with the strongly "non-local" nature of the sandpile dynamics (adding a grain at a particular site can influence the heights at sites far away). In that respect both the construction and the study of ergodic properties of such processes in infinite volume is very different from classical interacting particle system theory [24], where processes have the Feller property and are constructed via a semigroup on the space of continuous functions. Continuity in the product topology is related to "locality" (a continuos function of the height-configuration does uniformly weakly depend on height variables far away). In the sandpile model this construction and all its beautiful corollaries (such as the relation between stationary measures and the generator) break down. The non-locality can be controlled only for "typical configurations" i.e., the probability that the height at a fixed site is influenced by aditions far away is very small. This non-uniform control of the locality turns out to be sufficient to define a stationary dynamics (i.e., starting from typical configurations). Moreover the abelian group structure which can be recovered in infinite volume is of great help in the definition and the study of the ergodic properties of the dynamics.

The systematic study of the ASM in infinite volume began in [27] where the one-dimensional ASM is defined, using a coupling and monotonicity approach, in the spirit of [25]. The one-dimensional ASM has no non-trivial stationary dynamics (the infinite volume limit of the stationary measures is the Dirac measure on the all-two configuration), so the only interesting properties lie in the transient regime of the evolution of all initial height configurations to the all-two configuration. In [28] we studied the infinite limit of the ASM on the Bethe lattice (i.e., rootless binary tree), using the tranfer matrix method developed in [11]. In [29] we investigate the dissipative model, which is not critical anymore (the tail of the disribution of avalanche sizes is exponential) but still has some (better controllable) form of non-locality. In this context, one recovers the full group structure of the finite-volume ASM, and obtains unique ergodicity of the dynamics. In [1] the authors prove the existence of the infinite-volume limit $\mu$ of the stationary measures $\mu_V$ of the ASM for $\mathbb{Z}^d$ using the correspondence with spanning trees, and properties of the infinite volume limit of uniform spanning trees, as studied first by Pemantle [32] (see also [4] for an extended study of uniform spanning forests). The question of existence of the dynamics starting from $\mu$-typical configurations is solved in $d > 4$ in [18]. In [18] we also prove almost-sure finiteness of avalanches and ergodicity of the infinite volume dynamics in $d > 4$. Definition and properties of the dynamics in $d = 2$ remains an open and challenging problem.

These notes are organized as follows. In chapter two we study the one-dimensional model and introduce much of the material of the Dhar-formalism such as abelian group structure, burning algorithm, spanning trees, toppling numbers. In chapter three we introduce the Dhar-formalism in the general context, introduce "waves of topplings" and explain the Bombay-trick. In chapter four we introduce the basic questions about infinite volume limits of the stationary measure and of the dynamics. In chapter five we study the dissipative model and its ergodic properties. Finally in chapter 6 we come back to the critical model and summarize the results of [1], [18]. I have chosen to include only "simple" (i.e., short and non-trivial) proofs, whereas complicated proofs are scetched in such a way that the reader will (hopefully) be convinced of their correctness and is capable of reading them in full detail without difficulty.

# 2 Prelude: the one-dimensional model.

## 2.1 The crazy office

In the crazy office, $N$ persons are sitting at a rectangular table. Person 1 is sitting at the left end, person $N$ at the right end. We can e.g. think of them as commissioners who have to treat files (e.g. proposals for funding of projects). All of them can treat at most two files, and initially all of them have at least one file (to save the appearance). In fact, commissioners never treat the files (that is one of the reasons of the craziness of the office), so in normal circumstances these files are just laying there in front of them. From time to time however the director of the commission comes in. He doesn't treat files either, he distributes them. When he enters, he chooses a commissioner at random and gives him an extra file. If the resulting number of files is still below 2, then the commissioner accepts, otherwise he gets crazy, and gives one file to each of his neighbors (this is called a "toppling"). For the leftmost commissioner this means giving one file to his right neighbor, and throwing one file through the window, and analogously for the rightmost commissioner. This process goes on until all commissioners again have at most two files in front of them. The process is called "a period of activity" or "an avalanche". A little thought about the presence of the windows reveals that the process always stops (this would not be the case if it were a round table). We now only look at the stable configurations (without crazy commissioners) in this process.

A stable configuration of files (later we will call this a height configuration) is a map

$$\eta : V = \{1, \ldots, N\} \to \{1, 2\}$$

The set of all stable configurations is denoted by $\Omega$. A general (i.e., possibly unstable) configuration is a map

$$\eta : V = \{1, \ldots, N\} \to \{1, 2 \ldots\}$$

The set of all configurations is denoted by $\mathcal{H}$. We call $\mathcal{S}(\eta)$ the stable result of $\eta$, i.e., the configuration left when all activity of crazy commissioners has stopped. At this point one can (and should) doubt whether this map $\mathcal{S} : \mathcal{H} \to \Omega$ is well-defined, i.e., what to do if two commissioners get crazy, stabilize them both together, or in some order. It turns out that in whatever way you organize the activity of stabilization, you will always end up with the same stable configuration.

The configuration at time $n$ of the process just described is given by

$$\eta_n = \mathcal{S}\left(\eta_0 + \sum_{i=1}^{n} \delta_{X_i}\right)$$

where $X_i$ are i.i.d. uniformly distributed on $\{1, \ldots, N\}$, representing the places where the director adds a file, and where $\mathcal{S}$ denotes stabilization. This defines a Markov chain on the finite state space $\Omega$, and so in the long run we will only see the recurrent configurations of this Markov chain. The Markov chain is not irreducible. In particular there are transient configurations. Consider e.g. $N = 3$ and the configuration 211. Once the two neighboring ones disappear, they will never come back. Indeed, in this process a height one can only be created by a toppling, but then the neighboring site cannot have height one anymore. A little thought reveals what is the end-result of an addition in some configuration of type

$$222221\dot{2}2212222222$$

Suppose that the dotted two is the place where we add a file. Then think a mirror in the middle between the two closest sites where the height is one, to the left and to the right of the dotted site, say $e^+(x,\eta), e^-(x,\eta)$, where $x$ is the site where we added. After stabilization, $e^+(x,\eta), e^-(x,\eta)$ will have height two, and a height one is created at the mirror image of the place where the addition took place. In the example, this gives,

$$2222222212222222$$

If one adds now at the left-most two say, then one plays the same game where the mirror is in between the closest one to the right and an additional site to the left, i.e., the result is

$$2222222122222222$$

Finally, upon addition at

$$2222222222222222$$

the mirror is placed between an extra site to the left and an extra site to the right, e.g.,

$$\dot{2}2222222222222222$$

gives

$$222222222222222221$$

From this description, it is clear that in the long run we will be left with $N+1$ configurations that have at most one site with height one. Restricted to that unique class of recurrent configurations, which we denote by $\mathcal{R}$, the Markov chain is irreducible, i.e., every element of $\mathcal{R}$ can be reached from every other element of $\mathcal{R}$.

## 2.2 Rooted spanning trees

Let us now reveal some interesting properties of the (in our case) simple set $\mathcal{R}$. First $|\mathcal{R}| = N+1$, and it is not a coincidence that

$$N + 1 = det(\Delta) \tag{2.1}$$

where the matrix $\Delta$ is defined by $\Delta_{ii} = 2$, $\Delta_{ij} = -1$ for $|i-j| = 1$, $i,j \in \{1,\ldots,N\}$. Verifying (2.1) is a simple exercise. The matrix $\Delta$ is well-known in graph theory under the name discrete lattice Laplacian, which comes from the fact that $-\Delta = \partial^2$ is a discrete version of the Laplacian (with Dirichlet boundary conditions). Indeed, if we think of our set $\{1,\ldots,N\}$ as a lattice with lattice spacing $\epsilon$, and realize that

$$f(x+\epsilon) + f(x-\epsilon) - 2f(x) = \partial^2 f = \epsilon^2 f''(x) + O(\epsilon^3)$$

$\partial^2 f$ is an approximation of the second derivative. The determinant of $\Delta$ is equal to the number of rooted spanning trees on $\{1,\ldots,N\}$. In our simple setting a rooted spanning tree is defined as follows. Define the set $V^* = V \cup \{*\}$; the $*$ is an artificial site, called the root, added to $V$. By definition $*$, 1 and $*$, $N$ are neighbors in $V^*$. Formally, we are thus defining a graph $(V^*, E^*)$ where the edges are between neighboring sites. A spanning subgraph is a graph $(V^*, E')$ where $E' \subseteq E^*$, and a rooted spanning tree is a connected spanning subgraph that does not contain loops. The matrix tree theorem (see e.g. [5]) gives that the number of rooted spanning trees of a graph equals the determinant of $\Delta$. So we expect that a natural bijection can be found between the set $\mathcal{R}$ of recurrent configurations and the set of rooted spanning trees.

This bijection can be defined with the help of "the burning algorithm" introduced by Dhar [9]. The burning algorithm has as an input the set $V = \{1, \ldots, N\}$ a configuration $\eta \in \Omega$, and as an output a subset $V' \subseteq V$. It runs as follows. Initially, $V_0 = V$. In the first step remove ("burn") all sites $x$ from $V$ which have a height $\eta(x)$ strictly bigger than the number of neighbors of $x$ in $V$. Notice that by stability of the configuration, these sites are necessarily boundary sites. After the first burning one is left with the set $V_1$, and one then repeats the same procedure with $V$ replaced by $V_1$, etc. until no more sites can be burnt. The output $\mathcal{B}(\eta, V)$ is the set of sites left when the algorithm stops.

It is easy to see that in our example $\mathcal{B}(\eta, V) = \varnothing$ if and only if $\eta \in \mathcal{R}$. Since the burning algorithm applied to a recurrent configuration burns all sites, one can imagine that specifying the paths of burning of every site defines the genealogy of a tree, and since all sites are burnt, this tree is spanning. Let us now make this correspondence more precise. Start from a configuration in $\mathcal{R}$. We give to each site in $V^*$ a "burning" time. By definition the burning time of $*$ is zero. We assign burning time 1 to the boundary sites that can be burnt in the first step, burning time 2 to the sites that can be burnt after those, etc. The idea is that a site with burning time $k+1$ "receives his fire" (has as an ancestor) from a site with burning time $k$. However there can be ambiguity if a site $x$ with burning time $k+1$ has two neighbors with burning time $k$. In that case we choose one of the neighbors as the ancestor, according to a preference-rule defined by the height $\eta(x)$. Say the left neighbor has lower priority. Then in case of ambiguity one chooses the left neighbor if $\eta(x) = 1$, and otherwise the right neighbor.

As an example consider 222 and 212. The edges of the rooted spanning tree of 222 are $(*1)(23)(3*)$. Indeed, for the middle site the right (highest priority) neighbor is chosen as an ancestor because its height is two. For 212 we obtain $(*1)(12)(3*)$.

Given the preference rule we obtain a bijection between the set of rooted spanning trees and the recurrent configurations. E.g., given our left $<$ right rule, from $(*1)(23)(3*)$ we reconstruct the configuration 2?2 immediately (from the root site 1 and 3 could be burnt in the first step), and from the priority rule we conclude that the height of the middle site is two because the right neighbor has been chosen.

The choice of the preference-rule is quite arbitrary (one can even choose it depending on the site), and this makes the bijection between the set of rooted spanning trees and recurrent configurations not extremely elegant. It would be interesting to find a "natural bijection" where this arbitrariness of the preference-rule is not present (see also [8], p. 354).

## 2.3 Group structure

Besides its relation to spanning trees, there are some more fascinating properties of the set $\mathcal{R}$. Consider $N = 2$ for the sake of (extreme) simplicity. Define the operation $\oplus$ on $\mathcal{R}$ by

$$\eta \oplus \zeta = \mathcal{S}(\eta + \zeta)$$

where the ordinary $+$ means point-wise addition. This gives rise to the following table

| $\oplus$ | 21 | 12 | 22 |
|---|---|---|---|
| 21 | 12 | 22 | 21 |
| 12 | 22 | 21 | 12 |
| 22 | 21 | 12 | 22 |

We recognize here the Caley table of an abelian group, i.e., $(\mathcal{R}, \oplus)$ is an abelian group with neutral element 22. Remark that we can define $\oplus$ on the whole of $\Omega$, but $(\Omega, \oplus)$ is not a group.

We now introduce still another group (which is isomorphic to the preceding one, as we will see later). Let us introduce the addition operator $a_i : \Omega \to \Omega$

$$a_i(\eta) = \mathcal{S}(\eta + \delta_i)$$

for $i \in \{1, \ldots, N\}$. In words, $a_i\eta$ is the stable result of an addition at site $i$. Accept (or verify) for the moment that for all $i, j \in \{1, \ldots, N\}$,

$$a_i a_j = a_j a_i \tag{2.2}$$

Later we will prove this so-called abelian property in full detail and generality. By definition of recurrence, if a configuration $\eta$ is recurrent then there exist integers $n_i > 0$ such that

$$\prod_{i=1}^{N} a_i^{n_i}(\eta) = \eta \tag{2.3}$$

The product in (2.3) is well-defined by abelianness. The fact that $n_i$ can be chosen strictly positive derives from the fact that in the course of the Markov chain one adds to every site with strictly positive probability. Call $e = \prod_{i=1}^{N} a_i^{n_i}$ and consider

$$A = \{\zeta \in \mathcal{R} : e\zeta = \zeta\}$$

By definition $A$ is not empty ($\eta \in A$), and if $g = \prod_{i=1}^{N} a_i^{m_i}$ for some integers $m_i \geq 0$, then we have the implication "$\zeta \in A$ implies $g\zeta \in A$". Indeed, by abelianness, for $\zeta \in A$,

$$e(g\zeta) = g(e(\zeta)) = g(\zeta)$$

Therefore, $A$ is a "trapping set" for the Markov chain, i.e., a subset of configurations such that once the Markov chains enters it, it never leaves it. As a consequence $A \supset \mathcal{R}$, because the Markov chain has only one recurrent class which contains the maximal configuration. Since by definition we have $A \subseteq \mathcal{R}$, $A = \mathcal{R}$. Therefore, acting on $\mathcal{R}$, $e$ is neutral. Since $n_i > 0$, we can define

$$a_i^{-1} = a_i^{n_i - 1} \prod_{j=1}^{N} a_j^{n_j}$$

and we have the relation

$$a_i^{-1} a_i = a_i a_i^{-1} = e \tag{2.4}$$

From (2.4) we conclude that

$$G := \{\prod_{i=1}^{N} a_i^{k_i}, k_i \in \mathbb{N}\} \tag{2.5}$$

acting on $\mathcal{R}$ defines an abelian group.

Of course not all the products of addition operators defining $G$ are different. In fact, it is easily seen that the group is finite, and we will show that once again

$$|G| = N + 1 \tag{2.6}$$

For that, it is sufficient to show that the group acts transitively and freely on $\mathcal{R}$, i.e., for all $\eta \in \mathcal{R}$ the orbit $O_\eta = \{g\eta : g \in G\} = \mathcal{R}$ and if $g\eta = g'\eta$ for some $g, g' \in G$, then $g = g'$, i.e., $g\zeta = g'\zeta$ for all $\zeta \in \mathcal{R}$. For the first statement, if $\eta \in \mathcal{R}$ and $g \in G$, then $g\eta$ can be reached from $\eta$ in the Markov chain, hence $g\eta \in \mathcal{R}$, and $O_\eta$ is clearly a trapping set for the Markov chain, hence $O_\eta \supset \mathcal{R}$. To prove the second statement, consider for $g\eta = g'\eta$ the set

$$A = \{\zeta \in \mathcal{R} : g\zeta = g'\zeta\}$$

then $A = \mathcal{R}$ with the same kind of reasoning used in the definition of inverses. Therefore, for all $\eta$, the map

$$\Psi_\eta : G \to \mathcal{R} : g \mapsto g\eta$$

is a bijection between $G$ and $\mathcal{R}$.

However, there is still another way to see that $|G| = N + 1$. This way of reasoning will be useful because in the general case we will not so easily be able to count the recurrent configurations. The equality $|G| = |\mathcal{R}|$ is however completely general, and that will be useful to obtain $|\mathcal{R}|$. Counting the number of elements of a group can become an easy task if we find a treatable isomorphic group. For this, we have to look for closure relations in $G$. Here is an easy one. Suppose you add two files to some commissioner. Since he has at least one file (to save his face), he will certainly get crazy and give one file to each of his neighbors (modulo the boundary conditions of course). In symbols this means

$$a_i^2 = a_{i-1} a_{i+1} \tag{2.7}$$

for $i \in \{2, \dots N-1\}$ and

$$a_1^2 = a_2, \ a_N^2 = a_{N-1}$$

Using the toppling matrix $\Delta$ introduced in (2.1), this is summarized as

$$a_i^{\Delta_{ii}} = \prod_{j \in V, j \neq i} a_j^{-\Delta_{ij}} \tag{2.8}$$

for all $i \in V$. Acting on $\mathcal{R}$ we can bring the right hand site to the left, and obtain

$$\prod_{j \in V} a_j^{\Delta_{ij}} = e \tag{2.9}$$

for all $i \in V$. By abelianness, we infer from (2.9) that for all $n : V \to \mathbb{Z}$

$$\prod_{i \in V} \prod_{j \in V} a_j^{n_i \Delta_{ij}} = e \tag{2.10}$$

Using $\Delta_{ij} = \Delta_{ji}$ and the definition $(\Delta n)_i = \sum_{j \in V} \Delta_{ij} n_j$ , we obtain

$$\prod_{i \in V} a_i^{(\Delta n)_i} = e \tag{2.11}$$

for all $n : V \to \mathbb{Z}$. We will show now that, conversely, if

$$\prod_{i \in V} a_i^{m_i} = e$$

for some $m_i \in \mathbb{Z}$ then there exists $n : V \to \mathbb{Z}$ such that

$$m_i = (\Delta n)_i$$

In words, this closure relation means that the only "trivial additions" on $\mathcal{R}$ are (integer column) multiples of the matrix $\Delta$.

Suppose

$$\prod_{x \in V} a_x^{m_x} = e \tag{2.12}$$

where $m \in \mathbb{Z}^V$. Write $m = m^+ - m^-$ where $m^+$ and $m^-$ are non-negative integer valued. The relation (2.12) applied to a recurrent configuration $\eta$ yields

$$\prod_{x \in V} a_x^{m_x^+} \eta = \prod_{x \in V} a_x^{m_x^-} \eta \tag{2.13}$$

In words, addition of $m^+$ or $m^-$ to $\eta$ leads to the same final stable configuration, say $\zeta$. But then there exist $k^+, k^-$ non-negative integer valued functions on $V$ such that

$$\eta + m^+ - \Delta k^+ = \zeta = \eta + m^- - \Delta k^-$$

which gives

$$m = m^+ - m^- = \Delta(k^+ - k^-)$$

Arrived at this point, we can invoke a well-known theorem of elementary algebra. If you have a group $G$ and a group $H$ and a homeomorphism

$$\Psi : H \to G$$

then $G$ is isomorphic to the quotient $H/Ker(\Psi)$ where $Ker(\Psi)$ is the set of $h \in H$ which are mapped to the neutral element of $G$. In our case, define

$$H := \{n : V \to \mathbb{Z}\} = \mathbb{Z}^V$$

with group operation pointwise addition. Next

$$\Psi : H \to G : n \mapsto \prod_{i \in V} a_i^{n_i}$$

Then what we just discussed can be summarized in the equality

$$Ker(\Psi) = \Delta\mathbb{Z}^V = \{\Delta n : n \in \mathbb{Z}^V\}$$

and hence we have the isomorphism

$$G \simeq \mathbb{Z}^V / \Delta\mathbb{Z}^V$$

Therefore we have

$$|\mathcal{R}| = |G| = |\mathbb{Z}^V / \Delta\mathbb{Z}^V| = det(\Delta)$$

To see the last equality, note that $\mathbb{Z}^V$ is the $|V|$ dimensional hypercubic lattice, with a volume one unit cell. $\Delta\mathbb{Z}^V$ is another lattice with the columns of $\Delta$ as vectors defining the unit cell. The quotient of these two lattices can geometrically be viewed as the non-equivalent points $n \in \mathbb{Z}^V$ of the unit cell of the lattice $\Delta\mathbb{Z}^V$. Equivalence is here defined as

$$n \sim m$$

if there exists $k \in \mathbb{Z}^V$ such that

$$n - m = \Delta k$$

This number of non-equivalent points is precisely the volume of the unit cell of the lattice $\Delta\mathbb{Z}^V$, which is $det(\Delta)$ ( Puzzle this out in the case $N = 2$ to be convinced). In general, the equality $|\mathbb{Z}^V / A\mathbb{Z}^V| = det(A)$ (with $A$ a symmetric matrix with integer elements and non-negative determinant) is trivial for a diagonal matrix. Indeed , in that case $A_{ij} = a_{ii}\delta_{ij}$ and

$$\mathbb{Z}^V / A\mathbb{Z}^V \simeq \mathbb{Z}/a_{11}\mathbb{Z} \oplus \mathbb{Z}/a_{22}\mathbb{Z} \ldots \oplus \mathbb{Z}/a_{nn}\mathbb{Z}$$

an hence $|\mathbb{Z}^V / A\mathbb{Z}^V| = \prod_{i=1}^n a_{ii} = det(A)$. Since by row and column operations (i.e., addition and subtraction of columns, or permutation of columns) one can make every integer-valued matrix diagonal, see e.g. [22], we just have to remark that such operations do not change the determinant of a matrix, and do not change (up to isomorphism) the lattice $\mathbb{Z}^V/A\mathbb{Z}^V$.

Here is still another, geometrical proof. $|\mathbb{Z}^V/A\mathbb{Z}^V|$ is the number of non-equivalent points in the unit cell defined by $A$ (i.e., the parallellepipe spanned by the rows of $A$). We can cover $\mathbb{R}^{|V|}$ by disjoint copies of this unit cell. Consider now a large cube $C_n = [-n,n]^{|V|}$. Let $N_n$ denote the number of integer points (i.e., points of $\mathbb{Z}^V$) in the cube, let $x_n$ denote the number of unit cells (copies of $A$) in $C_n$, and let $y$ denote the number of non-equivalent points in one unit cell. Then we have

$$x_n y = N_n$$

The volume of the $x_n$ unit cells in $C_n$ is $x_n det(A)$, so we have

$$x_n det(A) = (2n+1)^d + o(n^d)$$

Dividing these two relations and taking the limit $n \to \infty$ gives

$$\frac{y}{det(A)} = \lim_{n\to\infty} \frac{N_n}{(2n+1)^d + o(n^d)} = 1$$

## 2.4 The stationary measure

The Markov chain which we defined has a unique recurrent class $\mathcal{R}$ and hence its stationary measure $\mu$ concentrates on $\mathcal{R}$. We show now that $\mu$ is simply uniform on $\mathcal{R}$. Consider

$$\mu = \frac{1}{|\mathcal{R}|} \sum_{\eta\in\mathcal{R}} \delta_\eta$$

Since for all $i$, $a_i$ is a bijection on $\mathcal{R}$ we have that the image measure $\mu \circ a_i$ is again uniform on $\mathcal{R}$. Therefore $\mu$ is invariant under the individual addition operators $a_i$, and hence under the Markov chain. In fact for every pair of functions $f, g : \Omega \to \mathbb{R}$ we have

$$\int f(\eta) g(a_i\eta) \mu(d\eta) = \int f(a_i^{-1}\eta) g(\eta) \mu(d\eta) \tag{2.14}$$

The transition operator of our Markov chain is given by

$$Pf(\eta) = \frac{1}{|V|} \sum_{i\in V} f(a_i\eta) \tag{2.15}$$

and so we obtain

$$\int gPf d\mu = \int fP^* g d\mu \tag{2.16}$$

with

$$P^* f(\eta) = \frac{1}{|V|} \sum_{i\in V} f(a_i^{-1}\eta) \tag{2.17}$$

Substituting $g \equiv 1$ in (2.16) gives the stationarity of $\mu$. Moreover, if we consider the reversed (stationary) Markov chain $\{\eta_{-n}, n \in \mathbb{Z}\}$ then its transition operator is given by (2.17). Therefore the Markov chain is not reversible, but quasi-reversible, meaning that $P$ and $P^*$ commute. Remember the group $(\mathcal{R}, \oplus)$. It is clear that our Markov chain restricted to $\mathcal{R}$ is an irreducible random walk on this group. Therefore, the Haar measure (here simply the uniform measure) is invariant. That gives another explanation for the invariance of the uniform measure on $\mathcal{R}$.

## 2.5 Toppling numbers

If $\eta \in \mathcal{H}$ is a (possibly) unstable configuration, then its stabilization consists in subtracting columns of $\Delta$ for the unstable sites. Indeed, the effect of a toppling at site $x$ can be written as

$$T_x(\eta)(y) = \eta(y) - \Delta_{xy}$$

for $\eta(x) > 2$. The stabilization $\mathcal{S}(\eta)$ is completely determined by the fact that $\mathcal{S}$ is a composition of "legal" topplings $T_x$ (legal meaning that only unstable sites are toppled) and that $\mathcal{S}(\eta)$ is stable. Moreover, we have the relation

$$\eta - \Delta m = \mathcal{S}(\eta) \tag{2.18}$$

where $m : V \to \mathbb{N}$ denotes the column vector collecting the number of topplings at each site needed to stabilize $\eta$. This equation (2.18) will play a crucial role in the whole theory of the abelian sandpile model.

If $\eta \in \Omega$ is a stable configuration, then after addition at $i \in V$, (2.18) specifies to

$$\eta + \delta_i - \Delta m^i_\eta = a_i(\eta) \tag{2.19}$$

In particular, integrating this equation over $\mu$, and using the invariance of $\mu$ under the action of $a_i$ gives

$$(\Delta \int m^i_\eta \mu(d\eta))_j = \delta_{ij} \tag{2.20}$$

i.e.,

$$(\int m^i_\eta \mu(d\eta))_j = \Delta^{-1}_{ij} \tag{2.21}$$

In words this means that the expected number of topplings at site $j$ upon addition at site $i$ is equal to the "Green function" $G_{ij} = \Delta^{-1}_{ij}$. Later on, we will see that this Green function has an interpretation in terms of simple random walk: up to a (multiplicative) constant it is equal to the expected number of visits at site $j$ of a simple random walk starting at site $i$ and killed upon leaving $V$.

Running ahead a little bit, we know that the Green function of random walk on $\mathbb{Z}^d$ decays (in the transient case, $d \geq 3$) as a power in the distance $|i - j|$, more precisely

$$\mathbb{E}_i \left(|\{k : S_k = j\}|\right) \simeq |i - j|^{-(d-2)}$$

where $S_k$ denotes simple random walk on $\mathbb{Z}^d$, $d > 2$, started at $i$ under the expectation $\mathbb{E}_i$. This is a first signature of something very interesting which is going on in this model. In the "thermodynamic limit" (larger and larger volumes) power laws show up: the influence (this time expressed as the number of topplings) of an addition at a site on a far-away site is not decaying exponentially. Physicists say that there is no “characteristic size” of avalanches, and connect this to “critical behavior”(absence of a “finite correlation length”). For those of you who are familiar with percolation, at the critical point there is no infinite cluster (expected for all $d$, but proved for $d = 1, 2$ and high $d$), but the expected cluster size (of the cluster of the origin say) is infinite. A similar thing is happening here, since

$$\sum_{j \in \mathbb{Z}^d, j \neq i} |i - j|^{-(d-2)} = \infty$$

Summarizing, we have treated a simple model, discovered connections with spanning trees, an abelian group, and random walks. This will be the typical ingredients of these lectures: elementary algebra, some random walk theory, and properties of spanning trees. It is remarkable that many properties which could be derived in a very elementary way in this particular model are valid in complete generality.

## 2.6 Exercises

1. Verify formula (2.1).

2. For $\eta \in \mathbb{Z}^V$ we define the recurrent reduction of $\eta$ (notation $red(\eta)$) to be the unique element in $\mathcal{R}_V$ which is equivalent to $\eta$ in the sense that there exist $m \in \mathbb{Z}^V$ such that $\eta = red(\eta) + \Delta m$. Show that this reduction is well-defined (i.e. that there exists a *unique* equivalent recurrent configuration), and find the recurrent reduction of 11111.

3. What is the neutral element for the group $(\mathcal{R}, \oplus)$ when $N = 5$ ?

4. Show that the toppling numbers defined in (2.18) are *monotone* in the configuration $\eta$. This means if $\eta$ and $\eta'$ are two configurations such that for all $x \in V$ $\eta_x \leq \eta'_x$, (notation $\eta \leq \eta'$), then for the corresponding topplings $m_\eta \leq m_{\eta'}$, i.e., for all $x \in V$, $m_\eta(x) \leq m_{\eta'}(x)$.

5. Show that if $m \equiv 1$, then $\Delta m$ is equal to zero except $(\Delta m)_1 = (\Delta m)_N = 1$. Derive from this fact that a configuration is recurrent if and only if upon addition of one grain at 1 and one grain at $N$, all sites topple once and the configuration remains the same.

6. Suppose now that the table is a round table, and there are no windows (=periodic boundary conditions). Then the game of redistributing files is possibly going on forever. What is the maximal number of files allowed in the system such that the game stops ?

# 3 General finite volume abelian sandpiles

In this section we will generalize our findings of the previous section. We consider a simply connected set $V \subseteq \mathbb{Z}^d$, think e.g. of $V = [-n, n]^d \cap \mathbb{Z}^d$. In general a simply connected subset of $\mathbb{Z}^d$ is a subset $V$ such that "filling the squares" of the lattice points in $V$ leads to a simply connected subset of $\mathbb{R}^d$. As a toppling matrix $\Delta$ we consider minus the lattice Laplacian $\Delta_{xx} = 2d$, $\Delta_{xy} = -1$, for $x, y \in V$, $|x - y| = 1$. Later on we will generalize this. A height configuration is a map $\eta : V \to \{1, 2, \ldots\}$, and the set of all height configurations is again denoted by $\mathcal{H}$. A height configuration $\eta \in \mathcal{H}$ is stable if for every $x \in V$, $\eta(x) \leq \Delta_{xx}$. A site where $\eta(x) > \Delta_{xx}$ is called an unstable site. The toppling of a site $x$ is defined by

$$T_x(\eta)(y) = \eta(y) - \Delta_{xy} \tag{3.1}$$

This means that the site loses $2d$ grains and distributes these to his neighbors in $V$. Those sites in $V$ which have less than $2d$ neighbors ("boundary site") lose grains upon a toppling. The toppling is called *legal* if the site is unstable, otherwise it is called *illegal*. It is elementary to see that

$$T_x T_y(\eta) = \eta - \Delta_{x,\cdot} - \Delta_{y,\cdot} = T_y T_x(\eta) \tag{3.2}$$

if $x, y$ are both unstable sites of $\eta$. More precisely, this identity means that if $T_x T_y$ is a sequence of legal topplings, then so is $T_y T_x$ and the effect is the same. This is called the "elementary abelian property". For a general height configuration we define its stabilization by

$$\mathcal{S}(\eta) = T_{x_1} \ldots T_{x_n}(\eta), \tag{3.3}$$

by the requirement that in the sequence of topplings of (3.3) every toppling is legal, and that $\mathcal{S}(\eta)$ is stable.

For a height configuration $\eta \in \mathcal{H}$ and a sequence $T_{x_1} \ldots T_{x_n}$ of legal topplings we define the toppling numbers (of that sequence) to be

$$n_x = \sum_{i=1}^{n} I(x_i = x) \tag{3.4}$$

The configuration resulting from that sequence of topplings can be written as $T_{x_1} \ldots T_{x_n}(\eta) = \eta - \Delta n$, where $n$ is the column indexed by $x \in V$, with elements $n_x$.

**Remark 3.5.** *In the algebraic combinatorics literature, the model we just introduced is called "chip-firing game", or "Dirichlet game", see e.g. [8]. The number of topplings is called "score-function". The fact that $\mathcal{S}$ is well-defined is then formulated as follows: at the end of every Dirichlet game, the score function and the final chip-configuration are the same.*

**Lemma 3.6.** *If $\eta \in \mathcal{H}$ is a height configuration and $T_{x_1} \ldots T_{x_n}$ a sequence of legal topplings such that the resulting configuration is stable, then the numbers $n_x, x \in V$ are maximal. I.e., for every sequence of legal topplings $T_{y_1} \ldots T_{y_m}$ the toppling numbers $n'$ satisfy $n'_x \leq n_x$ for all $x \in V$.*

Before proving the lemma, let us see that this is sufficient to prove the following

**Proposition 3.7.** *$\mathcal{S}$ is well-defined.*

Indeed, suppose that $T_{x_1} \dots T_{x_n}$ and $T_{y_1} \dots T_{y_m}$ are two legal sequences of topplings leading to a stable configuration. Then, the resulting stable configuration is a function of the toppling numbers only. By maximality, $n_x = m_x$ for all $x \in V$, for both sequences, and hence the resulting stable configurations are equal as well. So this statement implies that in whatever order you topple the unstable sites of some height configuration, at the end you will have toppled every site a fixed amount of times (independent of the order) and the resulting stable configuration will be independent of the chosen order.

We now give the proof of lemma 3.6.

*Proof.* We will prove the following. Suppose that we have the identity

$$\xi = \eta - \Delta n \tag{3.8}$$

with $\xi$ stable and $n_x \geq 0$ for all $x \in V$. Suppose that $x_1, \dots, x_n$ is a legal sequence of topplings with toppling numbers $m_x = \sum_{i=1}^{n} \delta_{x_i,x}$, then $m_x \leq n_x$ for all $x \in V$.

Suppose that a sequence of legal topplings has toppling numbers $m_x \leq n_x$ (this is always possible since we can choose $m_x = 0$ for all $x$), and for a site $j \in V$ an extra legal toppling can be performed. Put

$$\zeta = \eta - \Delta m \tag{3.9}$$

Since an extra legal toppling is possible at site $j$ in $\zeta$, $\zeta_j > \xi_j$. Combining (3.8) and (3.9) this gives

$$(m_j - n_j)\Delta_{jj} < \sum_{i \neq j} (n_i - m_i)\Delta_{ij} \leq 0 \tag{3.10}$$

In the last inequality we used that $\Delta_{ij} \leq 0$ and $n_i \geq m_i$ for $i \neq j$. This implies that $m_j + 1 \leq n_j$ (since $\Delta_{jj} > 0$). Therefore if $m'$ denotes the toppling vector where we legally topple the site $j$ once more (after the legal topplings with toppling number vector $m$), then the inequality $m'_j \leq n_j$ still holds. □

The fact that $\mathcal{S}$ is well-defined implies immediately that the addition operator $a_x\eta = \mathcal{S}(\eta + \delta_x)$ is well-defined and that abelianness holds, i.e., for $x, y \in V$, and all $\eta \in \Omega$,

$$a_x a_y \eta = a_y a_x \eta = \mathcal{S}(\eta + \delta_x + \delta_y)$$

The dynamics of the abelian sandpile model (ASM) is then defined as follows. Let $p = p(x)$ be a probability distribution on $V$, i.e., $p(x) > 0, \sum_{x \in V} p(x) = 1$. Starting from $\eta_0 = \eta \in \Omega$, the state at time $n$ is given by the random variable

$$\eta_n = \prod_{i=1}^{n} a_{X_i} \eta \tag{3.11}$$

where $X_1, \ldots, X_n$ are i.i.d. with distribution $p$. The Markov transition operator defined on functions $f : \Omega \to \mathbb{R}$ is then given by

$$Pf(\eta) = \sum_{x \in V} p(x) f(a_x \eta) \tag{3.12}$$

Note that since we add at each site with strictly positive probability, from every configuration $\eta \in \Omega$ the maximal configuration $\eta_{max}$ (defined by $\eta_{max}(x) = \Delta_{xx}$ for all $x \in V$) can be reached from $\eta$. Therefore the set of recurrent configurationss $\mathcal{R}$ is equal to the unique recurrent class containing the maximal configuration.

We denote by $\mathcal{A}$ the set of all finite products of addition operators working on $\Omega$. This is an abelian semigroup, and as before, on the set $\mathcal{R}$ of recurrent configurations of the Markov chain, $\mathcal{A}$ is a group, which we denote by $G$.

The following lemma gives some equivalent characterizations of recurrence.

**Lemma 3.13.** *Define*

$$\mathcal{R}_1 = \{\eta : \forall x \in V, \exists n_x \geq 1, a_x^{n_x} \eta = \eta\}$$

*and*

$$\mathcal{R}_2 = \{\eta : \exists x \in V, \exists n_x \geq 1, a_x^{n_x} \eta = \eta\}$$

*then* $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$.

*Proof.* Clearly, $\mathcal{R}_1 \subseteq \mathcal{R}$. Indeed starting from $\eta \in \mathcal{R}_1$, $\eta$ can be reached again with probability bounded from below by $p(0)^{n_0} > 0$, and hence will be reached infinitely many times with probability one.

To prove the inclusion $\mathcal{R} \subseteq \mathcal{R}_1$, remember that the set of products of addition operators working on $\mathcal{R}$ forms a finite abelian group $G$, with $|G| = |\mathcal{R}|$. Since every element of a finite group is of finite order, for every $a_x \in G$ there exists $n_x \in \{1, 2, \ldots\}$ such that $a_x^{n_x} = e$.

To prove that $\mathcal{R}_2 = \mathcal{R}$, we have to show that if there exists $x, n_x$ such that $a_x^{n_x} \eta = \eta$, then for all $y \in V$, there exist $n_y$ such that $a_y^{n_y}(\eta) = \eta$. Since the number of stable configurations is finite, there exist $p_y, n_y$ such that

$$a_y^{p_y} a_y^{n_y}(\eta) = a_y^{p_y} \eta \tag{3.14}$$

Since $a_x^{n_x} \eta = \eta$ by assumption we have $a_x^{k n_x} \eta = \eta$ for every non-negative integer $k$. Consider $k$ large enough, apply the closure relation (3.24), and the abelian property to rewrite the equality $a_x^{k n_x} \eta = \eta$ in the form

$$a_x^{k n_x} \eta = a_y^{p_y} (a_{x_1} \ldots a_{x_n})(\eta) = \eta \tag{3.15}$$

for some $x_1, \ldots, x_n \in V$ (possibly equal to $y$). Then, using (3.14) we obtain

$$\begin{aligned} a_y^{n_y}(\eta) &= a_y^{n_y} a_y^{p_y} (a_{x_1} \ldots a_{x_n})(\eta) \\ &= a_{x_1} \ldots a_{x_n} a_y^{p_y} a_y^{n_y}(\eta) \\ &= a_{x_1} \ldots a_{x_n} a_y^{p_y}(\eta) = \eta \end{aligned} \tag{3.16}$$

□

We now give a preparatory lemma in view of the "burning algorithm characterization" of recurrent configurations.

**Definition 3.17.** *1. Let $A \subseteq \Omega$, and $H \subseteq \mathcal{A}$. We say that $A$ is $H$-connected to $\mathcal{R}$ if for every $\eta \in A$, there exists $h \in H$ such that $h(\eta) \in \mathcal{R}$.*

2. *Let $A \subseteq \Omega$, and $H \subseteq \mathcal{A}$. We say that $H$ has the $A$-group property if $H$ restricted to $A$ is a group.*

3. *A subset $A \subseteq \Omega$ is said to be closed under the dynamics if $\eta \in A$ and $g \in \mathcal{A}$ implies $g\eta \in A$.*

Example: $\Omega$ is $\mathcal{A}$-connected to $\mathcal{R}$, and $\mathcal{A}$ has the $\mathcal{R}$-group property.

**Lemma 3.18.** *Suppose $A \subseteq \Omega$, such that $H \subseteq \mathcal{A}$ has the $A$-group property, is $H$-connected to $\mathcal{R}$, and suppose furthermore that $A$ is closed under the dynamics, then $A = \mathcal{R}$.*

*Proof.* Clearly, $A$ is a trap for the Markov chain, hence by uniqueness of the recurrent class $\mathcal{R}$, $A \supset \mathcal{R}$. Suppose $\eta \in A$, then by assumption there exists $g \in H$ such that $g\eta \in \mathcal{R}$. Since $H$ has the $A$ group property, the element $g \in H$ can be inverted, i.e., $\eta = g^{-1}(g\eta)$. Hence $\eta$ can be reached from a recurrent configuration in the Markov chain, and therefore $\eta \in \mathcal{R}$. $\square$

## 3.1 Allowed configurations

**Definition 3.19.** *Let $\eta \in \mathcal{H}$. For $W \subseteq V$, $W \neq \varnothing$, we call the pair $(W, \eta_W)$ a forbidden subconfiguration (FSC) if for all $x \in W$,*

$$\eta(x) \leq \sum_{y \in W \setminus \{x\}} (-\Delta_{xy}) \tag{3.20}$$

*If for $\eta \in \Omega$ there exists a FSC $(W, \eta_W)$, then we say that $\eta$ contains a FSC. A configuration $\eta \in \mathcal{H}$ is called allowed if it does not contain forbidden subconfigurations. The set of all stable allowed configurations is denoted by $\mathcal{R}'$.*

**Remark 3.21.** *1. Notice that the definition of allowed has a consistency property. If $V \subseteq V'$ and $\eta \in \Omega_{V'}$ is allowed, then its restriction $\eta_V \in \Omega_V$ is also allowed. This will later allow us to define allowed configurations on the infinite lattice $\mathbb{Z}^d$.*

2. *If $(W, \eta_W)$ is a forbidden subsconfiguration, then $|W| \geq 2$.*

**Lemma 3.22.** *$\mathcal{R}'$ is closed under the dynamics, i.e., for all $g \in \mathcal{A}$, $\eta \in \mathcal{R}'$, $g\eta \in \mathcal{R}'$.*

*Proof.* Let us call $\mathbf{A}$ the set of all (possibly unstable) allowed configurations. It is sufficient to prove that this set is closed under additions and legal topplings. Clearly, if $\eta \in \mathbf{A}$ and $x \in V$, then $\eta + \delta_x \in \mathbf{A}$. Therefore, it suffices to see that $\mathbf{A}$ is closed under toppling of unstable sites. Suppose the contrary, i.e., suppose $\eta \in \mathbf{A}$ and $T_x(\eta) \notin \mathbf{A}$, where $T_x$ is a legal toppling. Then $T_x(\eta)$ contains a FSC $(W, (T_x\eta)_W)$. Clearly, $W$ must contain $x$, otherwise $(W, \eta_W)$ is forbidden. Therefore we have

$$T_x(\eta)(y) = \eta(y) - \Delta_{xy} \leq \sum_{z \in W \setminus \{y\}} (-\Delta_{yz})$$

for all $y \in W$. This implies

$$\eta(y) \leq \sum_{z \in W \setminus \{x,y\}} (-\Delta_{yz})$$

i.e., $(W \setminus \{x\}, \eta_{W \setminus \{x\}})$ is a FSC. Indeed by remark 3.21 an FSC contains at least two points, hence $W \setminus \{x\}$ cannot be empty. □

This lemma immediately implies $\mathcal{R}' \supset \mathcal{R}$. To prove the inclusion $\mathcal{R}' \subseteq \mathcal{R}$ (which will deliver us from the primes), we will use lemma 3.18. In order to do so, let us first recall the burning algorithm. For $\eta \in \Omega$ we start removing all sites from $V$ that satisfy $\eta(x) > \sum_{y \in V, y \neq x} (-\Delta_{xy})$. This leaves us with a set $V_1$, and we continue the same procedure with $V_1, \eta_{V_1}$, etc. until the algorithm stops at some set $\mathcal{B}(V, \eta) \subseteq V$. It is clear from the definition of allowed configurations that $\eta \in \mathcal{R}'$ if and only if $\mathcal{B}(V, \eta) = \varnothing$.

The following lemma gives an equivalent characterization.

**Lemma 3.23.** *For $x \in V$ denote by $\alpha_V(x)$ the number of neighbors of $x$ in $V$. In particular $\alpha_V(x) \neq 2d$ if and only if $x$ is a boundary site, $x \in \partial V$. Then we have $\eta \in \mathcal{R}'$ if and only if*

$$\prod_{x \in \partial V} a_x^{2d - \alpha_V(x)} \eta = \eta \tag{3.24}$$

Before giving the proof, let us describe what is happening here. Imagine the two-dimensional case with $V$ a square. Then the special addition of (3.24) means "add two grains to each corner site and one grain to each boundary site which is not a corner site". The result of this addition on an allowed stable configuration will be that every site topples exactly once. If you accept that for a moment (we will prove that), then, after the topplings the resulting configuration is

$$\eta + \sum_{x \in \partial V} (2d - \alpha_V(x)) \delta_x - \Delta \bar{1}$$

where $\bar{1}$ is the column defined by $\bar{1}_x = 1$ for all $x \in V$. It is a trivial computation to show that the second and the third term of this expression cancel, and hence if it is true that upon the special addition every site topples exactly once, then we have proved (3.24).

*Proof.* We have to prove that every site topples exactly once if and only if the configuration is allowed. Suppose we add $s_V := \sum_{x\in V}(2d - \alpha_V(x))\delta_x$ to an allowed configuration. First there will be some boundary sites which will topple, these are exactly the sites where the height configuration satisfies the inequality

$$\eta(x) + 2d - \alpha_V(x) > 2d \tag{3.25}$$

i.e.,

$$\eta(x) > \alpha_V(x) = \sum_{y\in V\setminus\{x\}} (-\Delta_{xy}) \tag{3.26}$$

So these sites which topple are exactly the sites that can be burnt in the first step of the burning algorithm. After the topplings of these unstable boundary sites, it is again easy to see that the sites which are unstable are exactly those that would be burnt in the second step of the burning algorithm, etc. Therefore, if the configuration is allowed, every site topples at least once. To see that every site topples at most once, it suffices to prove that this is the case for the special addition applied to the maximal configuration. Indeed, by abelianness, we will have the maximal number of topplings at each site for that configuration. After the special addition, we topple all boundary sites once. The result is that the height of these boundary sites equals $2d - \alpha_V(x)$ and the result is the "special addition" $s_{V\setminus\partial V}$ so every boundary site of this new set $V_1 = V \setminus \partial V$ will topple once and these topplings will not cause unstable sites on $\partial V$, because there the height is $2d - \alpha_V(x)$ and the topplings at $\partial V_1$ can (and will) add at most $\alpha_V(x)$ grains to the site $x$. To visualize, one has to imagin that the "wave of topplings" goes to the inside of the volume and does not come back to places previously visited.

Repeating this argument with $V_1$, we see that every site topples exactly once. $\square$

**Theorem 3.27.** *A stable configuration $\eta \in \Omega$ is recurrent if and only if it is allowed.*

*Proof.* Consider

$$H = \{\prod_{x\in\partial V} a_x^{n_x}, n_x \geq 0\}$$

By lemma 3.23, $H$ restricted to $\mathcal{R}'$ is a group. Indeed the product defining the neutral element $e = \prod_{x\in\partial V} a_x^{2d-\alpha_V(x)}$ has strictly positive powers of every $a_x, x \in \partial V$. It is clear that $\mathcal{R}'$ is $H$-connected to $\mathcal{R}$, and $\mathcal{R}'$ is also closed under the dynamics by lemma 3.18. Hence by lemma 3.18, $\mathcal{R}' = \mathcal{R}$. $\square$

So we can now decide by running the burning algorithm whether a given stable configuration is recurrent or not. Here are some examples of forbidden subconfigurations in $d = 2$.

| | |
|---|---|
| 1 | 1 |

| | | |
|---|---|---|
| 1 | 2 | 1 |

| | | |
|---|---|---|
| 1 | 1 | |
| 2 | 2 | 1 |

Here are some examples of recurrent configurations also in $d = 2$.

| | | |
|---|---|---|
| 2 | 3 | 3 |
| 4 | 1 | 4 |
| 2 | 4 | 3 |

| | | |
|---|---|---|
| 1 | 2 | 4 |
| 4 | 2 | 4 |
| 3 | 3 | 1 |

## 3.2 Rooted spanning trees

As in the one-dimensional case of the previous section, also in the general case there is a one-to-one correspondence between recurrent configurations and rooted spanning trees. The construction is analogous. One first "extends" the graph by adding $*$, an extra site which will be the root. The extended graph $(V^*, E^*)$ is then defined by adding extra edges from the boundary sites to the root, for $x \in \partial V$, $2d - \alpha_V(x)$ edges go from $x$ to the root. In $(V^*, E^*)$ every site in $V$ has exactly $2d$ outgoing edges. We order the edges in some way say, e.g., in $d = 2$, $N < E < S < W$. Given the recurrent configuration, we give burning times to every site. The burning time of $*$ is zero by definition, and the burning time 1 is given to the boundary sites which can be burnt in the first stage of the algorithm, boundary time 2 to the sites which can be burnt after those, etc. The edges in the spanning tree are between sites with burning time $t$ and $t + 1$, with the interpretation that the site with burning time $t + 1$ receives his fire from the neighboring site with burning time $t$. In case of ambiguity, we choose the edge according to the preference rule, depending on the height. As an example, consider

| | | |
|---|---|---|
| 4 | 4 | 4 |
| 4 | 1 | 4 |
| 4 | 4 | 3 |

The corresponding burning time configuration is

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 1 |

Therefore we have ambiguities in connecting the root to the corner sites, and the middle site to the boundary sites. So look at the upper left corner site e.g., we have the choice between $N, W$. The height is maximal (it could have been 3 as well), so we choose the highest priority, i.e., $W$. For the middle site, we have the minimal height, so we choose the lowest priority amongst our 4 edges, i.e., $N$. So this gives as edge configuration for the corresponding rooted spanning tree

| | | |
|---|---|---|
| W | N | E |
| W | N | E |
| W | S | E |

where we indicated for each site the edge to the site neighboring it that is closer to the root.

Given the preference rule, and the spanning tree one can reconstruct the height configuration. So we have made a bijection between $\mathcal{R}$ and the set of rooted spanning trees, and hence, by the matrix tree theorem we have a first proof of Dhar's formula

$$|\mathcal{R}| = det(\Delta) \tag{3.28}$$

## 3.3 Group structure

In the previous section, we gave a proof of $|\mathcal{R}| = det(\Delta)$ by counting the number of group elements of $G$ in the one-dimensional situation. The basic closure relation

$$\prod_y a_y^{\Delta_{xy}} = e$$

remains unaltered, and derives from the elementary fact that adding $2d$ grains to a site makes it topple for sure, and hence we could as well have added one grain to each neighbor directly. So, as already shown in (3.29), the homeomorphism

$$\Psi : \mathbb{Z}^V \to G : n = (n_x)_{x\in V} \mapsto \prod_x a_x^{(\Delta n)_x} \tag{3.29}$$

satisfies $Ker(\Psi) \supset \Delta\mathbb{Z}^V = \{\Delta n : n \in \mathbb{Z}^V\}$. This gives

**Proposition 3.30.** *Let $\Psi$ be defined by (3.29), then $Ker(\Psi) = \Delta\mathbb{Z}^V$, and hence*

$$G \simeq \mathbb{Z}^V/\Delta\mathbb{Z}^V \tag{3.31}$$

We obtain as an straightforward corollary the following.

**Proposition 3.32.** *For every probability distribution $p(x) > 0, \sum_x p(x) = 1$, the Markov chain with transition operator*

$$Pf(\eta) = \sum_x p(x) f(a_x\eta)$$

*has as a stationary distribution*

$$\mu = \frac{1}{|det(\Delta)|}\sum_{\eta\in\mathcal{R}} \delta_\eta$$

*Moreover, in $L_2(\mu)$ the adjoint transition operator (the transition operator of the time reversed process) is given by*

$$P^*f(\eta) = \sum_x p(x) f(a_x^{-1}\eta)$$

Exactly as we derived in the previous section, we also have

**Proposition 3.33.** *Let $n_\eta^x$ denote the column of toppling numbers needed to stabilize $\eta+\delta_x$. Then we have*

$$\int n_\eta^x(y)\mu(d\eta) = G(x,y) = \Delta_{xy}^{-1} \tag{3.34}$$

By the Markov inequality $G(x,y)$ is also an upper bound for the probability that $y$ has to be toppled upon addition to $\eta$, where $\eta$ is distributed according to $\mu$.

## 3.4 Addition of recurrent configurations

Define for two stable height configurations

$$\eta \oplus \zeta = \mathcal{S}(\eta+\zeta) \tag{3.35}$$

Then we have the following

**Theorem 3.36.** *$\mathcal{R},\oplus$ is a group isomorphic with $G$.*

*Proof.* We can rewrite

$$\eta \oplus \zeta = \mathcal{S}(\eta+\zeta) = \prod_{x\in V} a_x^{\eta_x}(\zeta) = \prod_{x\in V} a_x^{\zeta_x}(\eta) \tag{3.37}$$

Define for some $\eta \in \mathcal{R}$:

$$e := \prod_{x\in V} a_x^{-\eta_x}(\eta) \tag{3.38}$$

This is well-defined because $a_x$ working on $\mathcal{R}$ are invertible. It is easy to see that the set

$$A = \{\xi \in \mathcal{R} : e\oplus\xi = \xi\oplus e = \xi\} \tag{3.39}$$

is closed under the dynamics and non-empty, hence $A=\mathcal{R}$. Every element $\eta\in\mathcal{R}$ has an inverse defined by

$$\ominus\eta = \prod_{x\in V} a_x^{-\eta_x} e \tag{3.40}$$

Hence, $\mathcal{R},\oplus$ is a group. To show that $G$ and $\mathcal{R}$ are isomorphic is left as an (easy) exercise. □

Let us discuss an intersting sconsequence of this theorem. Define $\eta,\xi$ to be equivalent ($\eta\sim\xi$) if there exists $n\in\mathbb{Z}^V$ such that $\eta=\xi-\Delta n$. Then each equivalence class of $\mathbb{Z}^V/\sim$ contains a unique element of $\mathcal{R}$. That an equivalence class contains at least one element of $\mathcal{R}$ is easy. Start from $\eta\in\mathbb{Z}^V$. Addition according to $n=\Delta\bar{1}$ gives an equivalent configuration, and stabilization of this configuration yields another equivalent configuration. Sufficiently many times iterating this special addition and stabilization gives an equivalent recurrent configuration. Suppose now that $\eta\sim\xi$ and both are elements of $\mathcal{R}$. It is easy to see that $\eta\sim\xi$ implies that for all $\zeta\in\mathcal{R}$:

$\eta \oplus \zeta = \xi \oplus \zeta$ (see the closure relation in $G$). Choosing $\zeta = \ominus\eta$ we obtain $e = \xi \oplus (\ominus\eta)$, which gives by adding $\eta$, $\xi \oplus (\ominus\eta) \oplus \eta = \xi = \eta$. Notice that the uniqueness of a representant of every equivalence class does not extend to the stable configurations. E.g., if $\eta$ is stable but not recurrent then upon addition of $\Delta\bar{1}$ and subsequent stabilization, not every site will topple. On the other hand $\eta$ is clearly equivalent with the resulting stable configuration (which is stable and different from $\eta$).

The next proposition shows a curious property of the neutral element $e$ of the group $(\mathcal{R}, \oplus)$.

**Proposition 3.41.** *Upon adding $e$ to a recurrent configuration $\eta$, the number of topplings at each site is independent of $\eta$.*

*Proof.* For a possible unstable configuration $\eta$, denote by $n_\eta$ the column collecting the toppling numbers at each site in order to stabilize $\eta$. By abelianness, we have

$$n_{\xi+\eta+\zeta} = n_{\xi\oplus\eta+\zeta} + n_{\xi+\eta} = n_{\xi+\eta\oplus\zeta} + n_{\eta+\zeta}$$

Choosing $\eta = e$ gives the identity

$$n_{\xi+e} = n_{\zeta+e}$$

which proves the proposition. □

## 3.5 General toppling matrices

The results of the previous subsections apply for a general symmetric toppling matrix, which is defined as follows.

**Definition 3.42.** *A integer valued matrix matrix $\Delta$ is called a toppling matrix if*

1. *For all $x, y \in V$, $\Delta_{xx} \geq 2d$, $\Delta_{xy} \leq 0$ for $x \neq y$*
2. *Symmetry: for all $x, y \in V$ $\Delta_{xy} = \Delta_{yx}$*
3. *Dissipativity: for all $x \in V$ $\sum_y \Delta_{xy} \geq 0$*
4. *Strict dissipativity: $\sum_x \sum_y \Delta_{xy} > 0$*

*A toppling matrix is called irreducible if from every site $x \in V$ there is a path $x_0 = x, \ldots, x_n = y$ where for all $i \in \{1, \ldots, n\}$, $\Delta_{y_{i-1}y_i} < 0$ and $y$ is a dissipative site.*

A particularly important example is the so-called dissipative model with (integer) "mass" $\gamma > 0$, which is defined as follows

$$\begin{aligned} \Delta_{xx} &= 2d + \gamma \\ \Delta_{xy} &= -1 \text{ for } |x - y| = 1, x, y \in V \end{aligned} \tag{3.43}$$

The interpretation of the topplings governed by $\Delta$ is as follows. The maximal stable height of a site is $2d + \gamma$. If an unstable site topples, it looses $2d + \gamma$ grains where (at most) $2d$ are distributed to the neighbors, and looses on top of it $\gamma$ grains. So in that case upon each toppling grains are lost. We will see later that this system is not critical: in the limit of growing volumes, avalanche sizes are exponentially damped. This is related to the fact that the massive Green function $G(x, y) = \Delta^{-1}_{xy}$ decays exponentially in the distance between $x$ and $y$. For $\gamma = 0$ one recovers the original critical model.

Finally, the symmetry requirement can be dropped, at the cost of a much more difficult characterization of recurrent configurations, via the so-called script-algorithm, see [36], [15] for more details. This subject will not be dealt with in this lecture notes. Our main direction is to prove results for the infinite volume system, and there the connection with spanning trees is crucial (and lost in the asymmetric case).

## 3.6 Avalanches and waves

One of the most intriguing properties of the ASM is the fact that power laws turn up in the large volume limit. More precisely, the size and the diameter of avalanche clusters have a power law behavior in the limit of large volumes. Though from a rigorous point of view not very much is known about this, let me explain shortly the claims made by physicists and supported by both numerical experiments as well as by non-rigorous scaling arguments. If we start from a configuration $\eta \in \Omega$, then we have the relation

$$\eta + \delta_i - \Delta n^i_\eta = a_i \eta$$

The avalanche cluster at site $i \in V$ is then defined as

$$\mathcal{C}_V(i, \eta) = \{j \in V : n^i_\eta(j) > 0\}$$

In words, this is the set of sites which have to be toppled at least once upon addition at $i$. The power law behavior of this cluster is expressed by the following conjecture

$$\lim_{V \uparrow \mathbb{Z}^d} \mu_V(|\mathcal{C}_V(i, \eta) > n)| \simeq Cn^{-\delta}$$

where $\mu_V$ denotes the uniform measure on recurrent configurations in the volume $V$ So this conjecture tells us that first of all the limit $V \uparrow \mathbb{Z}^d$ exists and moreover

$$\lim_{n \uparrow \infty} n^\delta \lim_{V \uparrow \mathbb{Z}^d} \mu_V(|\mathcal{C}_V(i, \eta)| > n) = C$$

for some $C > 0$. As in the case of critical percolation, similar cluster characteristics, like the diameter are believed to exhibit power law behavior. From a mathematical point of view even the weaker statement that for some $C, \delta > 0$

$$C/n^\delta \geq \liminf_{V \uparrow \mathbb{Z}^d} \mu_V(|\mathcal{C}_V(i, \eta)| > n)$$

is not proved except in very special cases. It is believed (see [35]) that for $d > 4$ the avalanche exponents are equal to their mean-field value, which is rigorously known to be $\delta = 1/2$.

However, power law decay of certain correlation functions is known rigourously. An example is (see later on, or [12])

$$\lim_{V\uparrow\mathbb{Z}^d} (\mu_V(I(\eta(x) = 1)I(\eta(0) = 1)) - \mu_V(I(\eta(x) = 1))\mu_V(I(\eta(0) = 1))) \simeq C|x|^{-2d} \tag{3.44}$$

The reason for that is that one can explicitly compute all the limits showing up in this expression. This is a kind of "good luck", because the question whether the limit

$$\lim_{V\uparrow\mathbb{Z}^d} \mu_V(I(\eta(0) = 2)$$

*exists* has been open for about fifteen years after the discovery of (3.44). Now we know that the limit exists, but we don't have a clue of its value except in $d = 2$. Similarly, the correlation of the indicator of having height 2 in dimension two is expected to decay as $(\log|x|^2)/|x|^4$, but this is a non-rigorous computation based on conformal field theory see [33].

Avalanches turn out to be rather complicated objects. An interesting finding of Priezzhev was that avalanches can be decomposed in a sequence of simpler events, called waves.

Start from some initial $\eta \in \Omega$ and add to 0. Then proceed as follows. First topple the site where you added (if necessary) and all the other sites except the site where you added (0): there you just stack the grains which are arriving. In that process every site topples at most once. Indeed, the site where you added topples at most once by definition. Its neighbors can not topple more than once, because in order to become unstable for a second time, they have to receive grains from all their neighbors. However, by assumption they do not receive grains from 0, so they are unstable at most once. Continuing this argument, one sees that the neighbors of the neighbors of 0 can topple at most once, etc. This first sequence of topplings is called *the first wave.* The set of sites toppled in this procedure is called *the support of the first wave.*

If at the end of the first wave the site at which the wave began (where sand was added) is still unstable, then we topple that site a second time, and obtain "the second wave". The following lemma shows that the "wave" clusters are simply connected. For a subset $V \subseteq \mathbb{Z}^d$ we say that $V$ is simply connected if the subset $V \subseteq \mathbb{R}^d$ obtained by "filling the squares" in $V$ is simply connected. This means that $V$ does not contain holes, or more formally that every loop can be continuously contracted to a point.

**Lemma 3.45.** *The support of a wave in in a recurrent configuration is simply connected.*

*Proof.* We prove the statement for the first wave. Suppose the support of the first wave contains a maximal (in the sense of inclusion) hole $\varnothing \neq H \subseteq V$. By hypothesis, all the neighbors of $H$ in $V$ which do not belong to $H$ have toppled once. The toppling of the outer boundary gives an addition to the inner boundary equal to $\lambda_H(x)$ at site $x$, where $\lambda_H(x)$ is the number of neighbors of $x$ in $V$, not belonging to $H$. But addition of this to $\eta_H$ leads to one toppling at each site $x \in H$, by recurrence of $\eta_H$. This is a contradiction because we supposed that $H \neq \varnothing$, and $H$ is not contained in the support of the wave. After the first wave the volume $V \setminus \{x\}$ contains a recurrent configuration $\xi_{V\setminus\{x\}}$ *i.e., recurrent in that volume* $V \setminus \{x\}$. Suppose that the second wave contains a hole, then this hole has to be a subset of $V \setminus \{x\}$, and arguing as before one discovers that the subconfiguration $\xi_H$ cannot be recurrent. □

## 3.7 Height one and weakly allowed clusters

In this section we follow [26] to compute

$$\lim_{V\uparrow\mathbb{Z}^2} \mu_V(I(\eta(0) = 1)) = \frac{2}{\pi^2}(1 - \frac{2}{\pi}) \tag{3.46}$$

The original idea comes from [12]. The probability of several other special local events can be computed in the thermodynamic limit with this method, which is called "the Bombay trick".

If a configuration $\eta \in \mathcal{R}$ has height one at the origin, then in the burning algorithm, all lattice sites neighboring the origin will have to be burnt before the origin can be burnt. Let us call $e_1, e_2, -e_2, -e_1$ the neighbors of the origin. Consider the toppling matrix $\Delta'$ constructed from $\Delta$ as follows. $\Delta'_{00} = 1$, $\Delta'_{e_2,e_2} = \Delta'_{-e_1,-e_1} = \Delta'_{-e_2,-e_2} = 3$, $\Delta'_{e_2,e_1} = \Delta'_{e_1,-e_1} = \Delta'_{e_1,-e_2} = 0$. All other $\Delta'_{ij} = \Delta_{ij}$. We can visualize this modification of the toppling matrix as cutting certain edges incident to the origin, and correspondingly putting the matrix element of that edge zero, keeping maximal height ($\Delta'_{ii}$) equal to the number of neighbors in the new lattice.

Suppose now that a configuration with height 1 at 0 can be burnt. Then it is easy to see that the same configuration can be burnt with the new matrix $\Delta'$ and vice versa: every burnable stable configuration in the model with toppling matrix $\Delta'$ corresponds to a burnable configuration with height one in the original model (using the same order of burning). It is important that $\Delta'$ differs from $\Delta$ in a finite number of matrix elements not depending on the volume $V \subseteq \mathbb{Z}^d$. The number of recurrent configurations in the model with toppling matrix $\Delta'$ is equal to $det(\Delta') = det(\Delta(I + GB))$, where $B_{ij} = 0$ except for $B_{00} = -3$, $B_{e_2,e_2} = B_{-e_1,-e_1} = B_{-e_2,-e_2} = -1$, $B_{0,e_2} = B_{e_2,0} = B_{-e_2,0} = B_{0,-e_2} = B_{0,-e_1} = B_{-e_1,0} = 1$, i.e., the non-zero elements form a $4 \times 4$ matrix. Therefore only four columns of the product $GB$ are non-zero and the computation of the determinant $det(I + GB)$ reduces to the computation of the four by four determinant

$$det(I+GB) = det\left(I + \begin{pmatrix} G_{00} & G_{e_2,0} & G_{e_2,0} & G_{e_2,0} \\ G_{e_2,0} & G_{00} & G_{e_2,e_2} & G_{-e_1,0} \\ G_{e_2,0} & G_{e_2,e_2} & G_{00} & G_{e_2,e_2} \\ G_{e_2,0} & G_{-e_1,0} & G_{e_2,e_2} & G_{00} \end{pmatrix} \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\right) \tag{3.47}$$

Where we used the symmetry $G(x,y) = G(y,x)$ of the Green function in replacing $G(-e_2, 0)$ by $G(e_2, 0)$. Since we are in dimension $d = 2$, we should take care now about taking the infinite volume limit because the "bare" Green functions are diverging. However the limit

$$\lim_{V \uparrow \mathbb{Z}^d} (G_V(x,y) - G_V(0,0)) = a(x,y) \tag{3.48}$$

is well defined. Moreover we have the explicit values, see [12], [37]

$$\begin{aligned} a(1,0) = a(0,1) = a(-1,0) = a(0,-1) = -1/4 \\ a(-1,1) = a(1,1) = a(1,-1) = a(-1,-1) = -1/\pi \end{aligned} \tag{3.49}$$

Computing the determinant in (3.47) and taking the infinite volume limit gives

$$\lim_{V \uparrow \mathbb{Z}^2} \mu_V(I(\eta_0 = 1)) = 2(2a(1,1) - 1)(a(1,1))^2 = 2(\pi - 2)/\pi^3 \tag{3.50}$$

We remark here that the modification $\Delta'$ of the matrix $\Delta$ counting the number of recurrent configurations with height one at 0 is not unique. E.g. the matrix $\Delta''_{00} = 0$, $\Delta''_{0e} = 0$, $\Delta''_{ee} = 1$ for $|e| = 1$, and all other $\Delta''_{ij} = \Delta_{ij}$ would also do. However for that modification one has to compute a $6 \times 6$ determinant.

Similarly one can compute the probability that sites 0 and $i$ both have height one. This gives rise to

$$\mu_V(I(\eta_0 = 1)I(\eta_i = 1)) = det(I + G\tilde{B}) \tag{3.51}$$

where $\tilde{B}$ is an $8 \times 8$ matrix having two blocks of the previous $B$ matrix. Define $(G)_{00}$ to be the matrix

$$\begin{pmatrix} G_{00} & G_{e_2,0} & G_{e_2,0} & G_{e_2,0} \\ G_{e_2,0} & G_{00} & G_{e_2,e_2} & G_{-e_1,0} \\ G_{e_2,0} & G_{e_2,e_2} & G_{00} & G_{e_2,e_2} \\ G_{e_2,0} & G_{-e_1,0} & G_{e_2,e_2} & G_{00} \end{pmatrix}$$

used in the computation of the height one probability. Similarly

$$(G)_{0i} = \begin{pmatrix} G_{0i} & G_{e_2,i} & G_{e_2,i} & G_{e_2,i} \\ G_{e_2,i} & G_{0i} & G_{e_2,e_2+i} & G_{-e_1,i} \\ G_{e_2,i} & G_{e_2,e_2+i} & G_{0i} & G_{e_2,e_2+i} \\ G_{e_2,i} & G_{-e_1,i} & G_{e_2,e_2+i} & G_{0i} \end{pmatrix}$$

$$(G)_{i0} = \begin{pmatrix} G_{i0} & G_{i+e_2,0} & G_{i+e_2,0} & G_{i+e_2,0} \\ G_{i+e_2,0} & G_{i0} & G_{i+e_2,e_2} & G_{i-e_1,0} \\ G_{i+e_2,0} & G_{i+e_2,e_2} & G_{i0} & G_{i+e_2,e_2} \\ G_{i+e_2,0} & G_{i-e_1,0} & G_{i+e_2,e_2} & G_{i0} \end{pmatrix}$$

and

$$(G)_{ii} = \begin{pmatrix} G_{ii} & G_{i+e_2,i} & G_{i+e_2,i} & G_{i+e_2,i} \\ G_{i+e_2,i} & G_{ii} & G_{i+e_2,e_2+i} & G_{i-e_1,i} \\ G_{i+e_2,i} & G_{i+e_2,e_2+i} & G_{ii} & G_{i+e_2,e_2+i} \\ G_{i+e_2,i} & G_{i-e_1,i} & G_{i+e_2,e_2+i} & G_{ii} \end{pmatrix}$$

Then the desired probability can be written as

$$\mu_V(I(\eta_0 = 1)I(\eta_i = 1)) = det(I + GB) = det\left(I + \begin{pmatrix} (G)_{00}B & (G)_{0i}B \\ (G)_{i0}B & (G)_{ii}B \end{pmatrix}\right) \tag{3.52}$$

The computation of this determinant is straightforward but a bit tedious without a computer. The main conclusion is that as $|i| \to \infty$,

$$\lim_{V\uparrow\mathbb{Z}^d} \mu_V(I(\eta_0 = 1)I(\eta_i = 1)) - (\mu_V(I(\eta_0 = 1)))^2 \simeq 1/|i|^4 \tag{3.53}$$

where by $a_n \simeq b_n$ we mean that $a_n/b_n \to 1$.

It is easy to see that this behavior arises from the products of matrix elements of $(G)_{0i}$ and $(G)_{i0}$ arising from the development of the determinant in (3.52). Exactly the same computation can be done in higher dimensions $d \geq 3$ (where the Green's function does not diverge in the infinite volume limit) giving rise to the decay

$$\lim_{V\uparrow\mathbb{Z}^d} \mu_V(I(\eta_0 = 1)I(\eta_i = 1)) - (\mu_V(I(\eta_0 = 1)))^2 \simeq 1/|i|^{2d} \tag{3.54}$$

So this is our first explicit computation showing the presence of power law decay of correlations in this model.

Certain higher order correlations can also be computed. The general structure of what can be computed is the probability of those local configurations which can be counted by a finite perturbation of the toppling matrix $\Delta$. A subconfiguration $(W, \eta_W)$ is called *weakly allowed* if diminishing with one unit one of the heights in $\eta_W$ leads to a forbidden subconfiguration. Weakly allowed configurations have the property that in the burning algorithm they will be burnt "in the last stage". Therefore a local modification of the matrix $\Delta$ in the sites of the support $W$ can be found, and the computation of the probability of the occurrence of $\eta_W$ is analogously reduced to the computation of a finite determinant. Here are three examples of weakly allowed subconfigurations.

| | |
|---|---|
| 2 | 1 |

| | | |
|---|---|---|
| 2 | 2 | 1 |

| | | |
|---|---|---|
| | 1 | |
| 2 | 3 | 1 |

Here are some of the probabilities of weakly allowed clusters

$$\lim_{V\uparrow\mathbb{Z}^2} \mu_V(\eta_0 = 1, \eta_{e_1} = 2, \eta_{2e_1} = 2) = \frac{9}{32} - \frac{9}{\pi} + \frac{47}{2\pi^2} - \frac{48}{\pi^3} + \frac{32}{\pi^4}$$

$$\begin{aligned}\lim_{V\uparrow\mathbb{Z}^2} \mu_V(\eta_0 = 1, \eta_{-e_1} = 2, \eta_{-e_1-e_2} = 2) \quad &= \quad -\frac{81}{14} + \frac{525}{\pi} - \frac{1315}{\pi^2} + \frac{60076}{9\pi^3} \\ &- \quad \frac{503104}{27\pi^4} + \frac{257024}{9\pi^5} - \frac{1785856}{81\pi^6} + \frac{524288}{81\pi^7}\end{aligned}$$

More can be found in [26].

# 4 Towards infinite volume: the basic questions

In the previous subsection we showed how to compute probabilities of several subconfigurations in the thermodynamic limit. The computation of

$$\lim_{V\uparrow\mathbb{Z}^2} \mu_V(\eta_0 = k) \tag{4.1}$$

for $k = 2, 3, 4$ is still possible explicitly, but already requires a "tour de force" performed in a seminal paper of Priezzhev [34]. For $d > 2$ the height probabilities $2, \dots, 2d$ cannot be computed explicitly, and probabilities of more complicated local events cannot be computed explicitly either (not even for $\mathbb{Z}^2$).

However the interesting features of this model, like power law decay of correlations, avalanche tail distribution, etc., are all statements about the large volume behavior. A natural question is therefore whether the thermodynamic limit

$$\lim_{V\uparrow\mathbb{Z}^d} \mu_V \tag{4.2}$$

exists as a weak limit (we will define this convergence more accurately later on). Physically speaking this means that enlarging the system more and more leads to converging probabilities for local events, i.e., events like $\eta(x_1) = k_1, \dots \eta(x_n) = k_n$. A priori this is not clear at all. Usually, e.g. in the context of models of equilibrium statistical mechanics, existence of the thermodynamic limit is based on some form of locality. More precisely existence of thermodynamic limits in the context of Gibbs measures is related to "uniform summability" of the "local potential". This "uniform locality" is absent in our model (the burning algorithm is non-local, and application of the addition operator can change the configuration in a large set).

Let us now come to some more formal definitions. For $V \subseteq \mathbb{Z}^d$ a finite set we denote by $\Omega_V = \{1, \dots 2d\}^V$ the set of stable configurations in $V$. The infinite volume stable configurations are collected in the set $\Omega = \{1, \dots, 2d\}^{\mathbb{Z}^d}$. We will always consider $\Omega$ with the product topology. Natural $\sigma$-fields on $\Omega$ are $\mathcal{F}_V = \sigma\{\psi_x : \eta \to \eta_x, x \in V\}$, and the Borel-$\sigma$-field $\mathcal{F} = \sigma\left(\cup_{V\subseteq\mathbb{Z}^d}\mathcal{F}_V\right)$. If we say that $\mu$ is a measure on $\Omega$ we always mean a measure defined on the measurable space $(\Omega, \mathcal{F})$. For a sequence $a_V$ indexed by finite subsets of $\mathbb{Z}^d$ and with values in a metric space $(X, d)$ we say that $a_V \to a$ if for every $\epsilon > 0$ there exists $V_0$ such that for all $V \supset V_0$ finite, $d(a_V, a) < \epsilon$. Remind that $\Omega$ with the product topology is a compact metric space. An example of a metric generating the product topology is

$$d(\eta, \xi) = \sum_{x\in\mathbb{Z}^d} 2^{-|x|} |\eta(x) - \xi(x)| \tag{4.3}$$

where for $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, $|x| = \sum_{i=1}^d |x_i|$.

A function $f : \Omega \to \mathbb{R}$ is said to be local if there exists a finite set $V$ such that $f(\eta_V \xi_{V^c}) = f(\eta_V \zeta_{V^c})$ for all $\eta, \xi, \zeta$, i.e., the value of the function depends only of the heights at a finite number of sites. Of course a local function can be viewed as a function defined on $\Omega_V$ if $V$ is large enough. Local functions are continuous and the set of local functions is uniformly dense in the set of continuous functions (by Stone-Weierstrass theorem). So in words, continuity of a function $f : \Omega \to \mathbb{R}$ means that the value of the function depends only weakly on heights far away (uniformly in these far-away heights).

**Definition 4.4.** *Let $(\mu_V)_{V \subseteq \mathbb{Z}^d}$ be a collection of probability measures on $\Omega_V$ and $\mu$ a probability measure on $\Omega$. We say that $\mu_V$ converges to $\mu$ if for all local functions $f$*

$$\lim_{V \uparrow \mathbb{Z}^d} \mu_V(f) = \mu(f) \tag{4.5}$$

**Remark 4.6.** *If for all local functions the limit in the lhs of (4.5) exists, then by Riesz representation theorem, this limit indeed defines a probability measure on $\Omega$.*

We denote by $\mathcal{R}_V$ the set of recurrent (allowed) configurations in finite volume $V \subseteq \mathbb{Z}^d$. The set $\mathcal{R} \subseteq \Omega$ is the set of allowed configurations in infinite volume, i.e.,

$$\mathcal{R} = \{\eta \in \Omega : \forall V \subseteq \mathbb{Z}^d \text{ finite }, \eta_V \in \mathcal{R}_V\} \tag{4.7}$$

For $\eta \in \Omega$ we define $a_{x,V}$:

$$a_{x,V}\eta = (a_{x,V}\eta_V)\eta_{V^c} \tag{4.8}$$

For a measure $\mu$ concentrating on $\mathcal{R}$ we say that $a_{x,V} \to a_x$ $\mu$-a.s. if there exists a set $\Omega' \subseteq \Omega$ with $\mu(\Omega') = 1$ such that for all $\eta \in \Omega'$, $a_{x,V}\eta$ converges to a configuration $a_x(\eta)$ (in the metric (4.3)) as $V \uparrow \mathbb{Z}^d$. If $a_{x,V} \to a_x$ $\mu$ a.s., then we say that the infinite volume addition operator $a_x$ is *well-defined* $\mu$-almost surely. It is important to realize that we cannot expect $a_{x,V}\eta$ to converge for *all* $\eta \in \mathcal{R}$. For instance consider the maximal configuration (in $d = 2$ e.g.) $\eta_{max} \equiv 4$. Then it is easy to see that $a_{x,V}\eta$ does not converge: taking the limit $V \uparrow \mathbb{Z}^d$ along a sequence of squares or along a sequence of triangles gives a different result. Since we will show later on that we have a natural measure $\mu$ on $\mathcal{R}$, we can hope that "bad" configurations like $\eta_{max}$ are exceptional in the sense of the measure $\mu$.

We now present a list of precise questions regarding infinite volume limits.

1. Do the stationary measures $\mu_V = \frac{1}{|\mathcal{R}_V|}\sum_{\eta \in \mathcal{R}_V} \delta_\eta$ converge to a measure $\mu$ as $V \uparrow \mathbb{Z}^d$, concentrating on $\mathcal{R}$ ? Is the limiting measure $\mu$ translation invariant, ergodic, tail-trivial ?

2. Is $a_x$ well-defined $\mu$-a.s. ? Is $\mu$ invariant under the action of $a_x$ ? Does abelianness still hold (i.e., $a_x a_y = a_y a_x$ on a set of $\mu$-measure one) ?

3. What remains of the group structure of products of $a_x$ in the infinite volume ? E.g., are the $a_x$ invertible (as measurable transformations) ?

4. Is there a natural stationary (continuous time) Markov process $\{\eta_t : t \geq 0\}$ with $\mu$ as an invariant measure ?

5. Has this Markov process good ergodic properties ?

Regarding question 4, a natural candidate is a process generated by Poissonian additions. In words it is described as follows. At each site $x \in \mathbb{Z}^d$ we have a Poisson process $N_t^{\varphi_x}$ with intensity $\varphi_x$, for different sites these processes are independent. Presuppose now that questions 1-2 have a positive answer. Then we can consider the formal product

$$\prod_{x \in \mathbb{Z}^d} a_x^{N_t^{\varphi_x}} \tag{4.9}$$

More precisely we want conditions on the addition rates $\varphi_x$ such that

$$\lim_{V \uparrow \mathbb{Z}^d} \prod_{x \in V} a_x^{N_t^{\varphi_x}}(\eta)$$

exists for $\mu$ almost every $\eta \in \mathcal{R}$ (or preferably even for a bigger class of initial configurations $\eta \in \Omega$). It turns out that a constant addition rate will not be possible (for the non-dissipative model). A sufficient condition is (as we will prove later)

$$\sum_{x \in \mathbb{Z}^d} \varphi_x G(0, x) < \infty \tag{4.10}$$

where $G(0, x)$ is the Green function of the lattice Laplacian on $\mathbb{Z}^d$. This implies that for questions 4-5, we have to restrict to transient graphs (i.e., $d \geq 3$). We remark here that probably the restriction $d \geq 3$ is of a technical nature, and a stationary dynamics probably exists also in $d = 2$, but this is not proved. On the other hand, we believe that the condition (4.10) is necessary and sufficient for the convergence of the formal product (4.9). Results in that direction are presented in the last section of these notes.

## 4.1 General estimates

Suppose that we have solved problem 1 from our list in the previous section, i.e., we know that $\mu_V$ converges to $\mu$. Then we can easily solve problem 2.

**Proposition 4.11.**

*Suppose that $\mu_V$ converges to $\mu$ in the sense of definition 4.4. Suppose that $d \geq 3$. Then $a_x = \lim_{V \uparrow \mathbb{Z}^d} a_{x,V}$ is $\mu$-almost surely well-defined. Moreover there exists a $\mu$-measure one set $\Omega' \subseteq \mathcal{R}$ such that for all $\eta \in \Omega'$, for all $V \subseteq \mathbb{Z}^d$ finite and $n_x \geq 0$, $x \in V$ the products $\prod_{x \in V} a_x^{n_x}(\eta)$ are well-defined.*

*$a_x^{-1} = \lim_{V \uparrow \mathbb{Z}^d} a_{x,V}^{-1}$ is $\mu$-almost surely well-defined. Moreover there exists a $\mu$-measure one set $\Omega' \subseteq \mathcal{R}$ such that for all $\eta \in \Omega'$, for all $V \subseteq \mathbb{Z}^d$ finite and $n_x \geq 0$, $x \in V$ the products $\prod_{x \in V} a_x^{-n_x}(\eta)$ are well-defined.*

*Proof.* Let us prove that $a_x$ is $\mu$-almost surely well-defined. The other statements of the proposition will then follow easily.

Call $N_V(x, y, \eta)$ the number of topplings at $y$ needed to stabilize $\eta + \delta_x$ in the finite volume $V$ (so throwing away the grains falling off the boundary of $V$).

It is easy to see that for $V' \supset V$ and all $y \in \mathbb{Z}^d$, $N_V(x,y,\eta) \leq N_{V'}(x,y\,\eta)$ (this follows from abelianness). Therefore we can define $N(x,y,\eta) = \lim_{V\uparrow\mathbb{Z}^d} N_V(x,y,\eta)$ (which is at this stage possibly $+\infty$). Clearly $N_V(x,\cdot,\eta)$ is only a function of the heights $\eta(z), z \in V$, and thus it is a local function of $\eta$.

Moreover, we have, $\mu_V(N_V(x,y,\eta)) = G_V(x,y)$ (see (2.21), (3.34)). Therefore, using that $\mu_V \to \mu$,

$$
\begin{aligned}
\int d\mu(N(x,y,\eta)) &= \int d\mu(\lim_{V\uparrow\mathbb{Z}^d} N_V(x,y,\eta)) = \lim_V \int d\mu(N_V(x,y,\eta)) \\
&= \lim_V \lim_W \int d\mu_W(N_V(x,y,\eta)) \\
&\leq \lim_W \int d\mu_W(N_W(x,y,\eta)) = \lim_W G_W(x,y) = G(x,y)
\end{aligned} \tag{4.12}
$$

where we used $d \geq 3$, so that $G(x,y) < \infty$ (this works in general for transient graphs).

This proves that $N(x,y,\eta)$ is $\mu$-a.s. well-defined and is an element of $L^1(\mu)$ (so in particular finite, $\mu$-a.s.). Therefore we can define

$$a_x(\eta) = \eta + \delta_x - \Delta N(x,\cdot,\eta) \tag{4.13}$$

Then we have $a_x = \lim_V a_{x,V}$ $\mu$-almost surely, so $a_x$ is well-defined $\mu$-a.s.

Let us now prove the a.s. existence of inverses $a_x^{-1}$. In finite volume we have, for $\eta \in \mathcal{R}_V$

$$a_{x,V}^{-1}\eta = \eta - \delta_x + \Delta n_x^V(\cdot,\eta) \tag{4.14}$$

where now $n_x^V(y,\eta)$ denotes the number of "untopplings" at $y$ in order to make $\eta - \delta_x$ recurrent. Upon an untoppling of a site $y$, the site $y$ *receives* $2d$ grains and all neighbors $z \in V$ of $y$ *lose* one grain. Upon untoppling of a boundary site some grains are *gained*, i.e., the site receives $2d$ grains but only the neigbors in $V$ lose one grain.

The inverse $a_{x,V}^{-1}$ can be obtained as follows. If $\eta - \delta_x$ is recurrent ($\in \mathcal{R}_V$), then it is equal to $a_{x,V}^{-1}\eta$. Otherwise it contains a FSC with support $V_0$. Untopple the sites of $V_0$. If the resulting configuration is recurrent, then it is $a_{x,V}^{-1}\eta$. Otherwise it contains a FSC with support $V_1$, untopple $V_1$, etc. It is clear that in this way one obtains a recurrent configuration $a_{x,V}^{-1}\eta$ such that (4.14) holds. Integrating (4.14) over $\mu_V$ gives

$$\mu_V(n_x^V(y,\eta)) = G_V(x,y) \tag{4.15}$$

Proceeding now in the same way as with the construction of $a_x$, one obtains the construction of $a_x^{-1}$ in infinite volume. The remaining statements of the proposition are obvious. $\square$

The essential point in proving that $a_x$ is well-defined in infinite volume is the fact that the toppling numbers $N(x,y,\eta)$ are well-defined in $\mathbb{Z}^d$, by transience.

For $x \in \mathbb{Z}^d$ and $\eta \in \mathcal{R}$, we define the avalanche initiated at $x$ by

$$Av(x,\eta) = \{y \in \mathbb{Z}^d : N(x,y,\eta) > 0\} \tag{4.16}$$

This is possibly an infinite set, e.g. if $\eta$ is the maximal configuration. In order to proceed with the construction of a stationary process, we need that $\mu$ is invariant under the action of the addition operator $a_x$. This is the content of the following proposition.

**Proposition 4.17.** *Suppose that avalanches are almost surely finite. Then we have that $\mu$ is invariant under the action of $a_x$ and $a_x^{-1}$.*

*Proof.* Let $f$ be a local function. We write, using invariance of $\mu_W$ under the action of $a_{x,W}$:

$$\begin{aligned}
& \int f d\mu - \int a_x f d\mu \\
= & \left(\int (a_x f - a_{x,V} f) d\mu\right) + \left(\int (a_{x,V} f) d\mu - \int (a_{x,V} f) d\mu_W\right) \\
+ & \left(\int (a_{x,V} f) d\mu - \int (a_{x,W} f) d\mu_W\right) + \left(\int f d\mu_W - \int f d\mu\right) \\
:= & A_V + B_{V,W} + C_{V,W} + D_W
\end{aligned} \tag{4.18}$$

For $\epsilon > 0$ given, using that $a_{x,V} \to a_x$, $A$ can be made smaller than $\epsilon/4$ by choosing $V$ large enough. The second and the fourth term $B$ and $D$ can be made smaller than $\epsilon/4$ by choosing $W$ big enough, using $\mu_W \to \mu$, the fact that $f$ is local and the fact that for fixed $V$, $a_{x,V} f$ is also a local function. So the only problematic term left is the third one. If $a_{x,V} \neq a_{x,W}$, then the avalanche is not contained in $V$. Therefore

$$C_{V,W} \leq 2\|f\|_\infty \mu_W (Av(x,\eta) \not\subseteq V)$$

Notice that for fixed $V$, the event $Av(x,\eta) \not\subseteq V$ is a local event (depends only on heights in $V$ together with its exterior boundary). Therefore, we can choose $W$ big enough such that

$$\mu_W (Av(x,\eta) \not\subseteq V) - \mu(Av(x,\eta) \not\subseteq V) < \epsilon/8$$

and by the assumed finiteness of avalanches, we can assume that we have chosen $V$ large enough such that

$$\mu(Av(x,\eta) \not\subseteq V) < \epsilon/8$$

Since $\epsilon > 0$ was arbitrary, and $f$ an arbitrary local function, we conclude that $\mu$ is invariant under the action of $a_x$.

By the finiteness of avalanches, for $\mu$ almost every $\eta \in \mathcal{R}$, there exists $V = V(\eta)$ such that $a_{x,V}(\eta) = a_{x,V(\eta)}(\eta)$ for all $V \supset V(\eta)$. It is then easy to see that $a_x^{-1}(\eta) = a_{x,V}^{-1}(\eta)$ for all $V \supset V(\eta)$. One then easily concludes the invariance of $\mu$ under $a_x^{-1}$. $\square$

## 4.2 Construction of a stationary Markov process

In this subsection we *suppose* that $\mu = \lim_{V \uparrow \mathbb{Z}^d}$ exists, and that $d \geq 3$, so that we have existence of $a_x$ and invariance of $\mu$ under $a_x$. In the next section we will show how to obtain this convergence $\mu_V \to \mu$.

The essential technical tool in constructing a process with $\mu$ as a stationary measure is abelianness. Formally, we want to construct a process with generator

$$Lf(\eta) = \sum_x \varphi_x(a_x f - f) \tag{4.19}$$

where $a_x f(\eta) = f(a_x \eta)$. For the moment don't worry about domains, etc. For a finite volume $V$, the generator

$$L_V f(\eta) = \sum_{x\in V} \varphi_x(a_x f - f) \tag{4.20}$$

defines a bounded operator on the space of bounded measurable functions. This is simply because $a_x$ are well-defined measurable transformations. Moreover it is the generator of a pure jump process, which can explicitly be represented by

$$\eta_t^V = \prod_{x\in V} a_x^{N_t^{\varphi_x}}(\eta) \tag{4.21}$$

where $N_t^{\varphi_x}$ are independent (for different $x \in \mathbb{Z}^d$) Poisson processes with rate $\varphi_x$.

The Markov semigroup of this process is

$$S_V(t) = e^{tL_V} f = \prod_{x\in V} e^{\varphi_x(a_x - I)t} \tag{4.22}$$

Notice that since $\mu$ is invariant under $a_x$, it is invariant under $S_V(t)$ and $S_V(t)$ is a semigroup of contractions on $L^p(\mu)$ for all $1 \le p \le \infty$. We are interested in the behavior of this semigroup as a function of $V$.

**Theorem 4.23.** *Suppose that (4.10) is satisfied. Then, for all $p > 1$, and every local function $f$, $S_V(t)f$ is a $L^p(\mu)$ Cauchy net, and its limit $S(t)f := \lim_{V\uparrow\mathbb{Z}^d} S_V(t)f$ extends to a Markov semigroup on $L^p(\mu)$.*

*Proof.* We use abelianness and the fact that $S_V(t)$ are $L^p(\mu)$ contractions to write, for $V \subseteq W \subseteq \mathbb{Z}^d$

$$\begin{aligned}
\|S_V(t)f - S_W(t)f\|_p &= \|S_V(t)(I - S_{W\setminus V}(t))f\|_p \\
&= \|S_V(t)\int_0^t S_{W\setminus V}(s)(L_{W\setminus V} f)ds\|_p \\
&\le \int_0^t \|S_{W\setminus V}(s)(L_{W\setminus V} f)\|_p ds \\
&\le t\|L_{W\setminus V} f\|_p
\end{aligned} \tag{4.24}$$

Now, suppose that $f$ is a local function with dependence set $D_f$.

$$\begin{aligned}
|L_{W\setminus V} f| &\le \sum_{x\in W\setminus V} \varphi_x|(a_x f - f)| \\
&\le \sum_{x\in W\setminus V} 2\varphi_x\|f\|_\infty I(\exists y \in \overline{D_f} : N(x,y,\eta) > 0)
\end{aligned} \tag{4.25}$$

Where $\overline{D_f}$ is the union of $D_f$ with its external boundary. Hence

$$\begin{aligned}
\|L_{W\setminus V}f\|_p &\leq 2\|f\|_\infty \left\| \sum_{x\in W\setminus V} \varphi_x I(\exists y \in \overline{D_f} : N(x,y,\eta) > 0) \right\|_p \\
&\leq 2\|f\|_\infty \sum_{x\in W\setminus V} \varphi_x \mu(\{\eta : \exists y \in \overline{D_f} : N(x,y,\eta) > 0\}) \\
&\leq 2\|f\|_\infty \sum_{x\in W\setminus V} \varphi_x \sum_{y\in\overline{D_f}} G(x,y) \qquad (4.26)
\end{aligned}$$

This converges to zero as $V, W \uparrow \mathbb{Z}^d$ by assumption (4.10) (remember that $\overline{D_f}$ is a finite set).

Therefore, the limit

$$S(t)f = \lim_{V\uparrow\mathbb{Z}^d} S_V(t)f \qquad (4.27)$$

exists in $L^\infty(\mu)$ for all $f$ local and defines a contraction. Therefore it extends to the whole of $L^\infty(\mu)$ by density of the local functions. To verify the semigroup property, note that

$$\begin{aligned}
\|S(t+s)f - S(t)(S(s)f)\| &= \lim_{W\uparrow\mathbb{Z}^d} \lim_{V\uparrow\mathbb{Z}^d} \|S_V(t+s)f - S_W(t)S_V(s)f\| \\
&= \lim_{W\uparrow\mathbb{Z}^d} \lim_{V\uparrow\mathbb{Z}^d} \|S_V(s)(S_V(t) - S_W(t))f\| \\
&\leq \lim_{W\uparrow\mathbb{Z}^d} \lim_{V\uparrow\mathbb{Z}^d} \|(S_V(t) - S_W(t))f\| = 0 \qquad (4.28)
\end{aligned}$$

It is clear that $S(t)1 = 1$, and $S(t)f \geq 0$ for $f \geq 0$, since for all $V \subseteq \mathbb{Z}^d$ these hold for $S_V(t)$, i.e., $S(t)$ is a Markov semigroup. □

So far, abelianness delivered us a simple proof of the fact that under (4.10) we have a natural candidate semigroup with stationary measure $\mu$. Kolmogorov's theorem gives us the existence of a Markov process with semigroup $S(t)$. The following explicit representation can be used in order to show that this process has a decent (cadlag) version with paths that are right-continuous and have left limits.

**Theorem 4.29.** *Let the addition rate satisfy (4.10). Denote by $\mathbb{P}$ the joint distribution of the independent Poisson processes $N_t^{\varphi_x}$. Then $\mu \times \mathbb{P}$ almost surely, the product*

$$\prod_{x\in V} a_x^{N_t^{\varphi_x}} \eta \qquad (4.30)$$

*converges as $V \uparrow \mathbb{Z}^d$ to a configuration $\eta_t \in \mathcal{R}$. The process $\{\eta_t : t \geq 0\}$ is a version of the process with semigroup $S(t)$ defined in Theorem 4.23, i.e., for all $t \geq 0$, $\eta \in \Omega'$, with $\mu(\Omega') = 1$,*

$$S(t)f(\eta) = \mathbb{E}_\eta f(\eta_t) \qquad (4.31)$$

*Moreover this version concentrates on $D([0,\infty), \mathcal{R})$.*

As a corollary of this theorem, one recovers the generator of the semigroup.

**Proposition 4.32.** *Define*

$$\mathcal{B}_\varphi = \{f \in L^1(\mu) : \sum_{x\in\mathbb{Z}^d} \varphi_x \int |a_x f - f| d\mu < \infty\} \tag{4.33}$$

*For $\varphi$ satisfying (4.10) all local functions are in $\mathcal{B}_\varphi$. For $f \in \mathcal{B}_\varphi$, the expression*

$$Lf = \sum_{x\in\mathbb{Z}^d} (a_x f - f) \tag{4.34}$$

*is well-defined, i.e., the series converges in $L^p(\mu)$ (for all $1 \le p \le \infty$) and moreover,*

$$\lim_{t\downarrow 0} \frac{S(t)f - f}{t} = Lf \tag{4.35}$$

*in $L^p(\mu)$.*

The following theorem shows that we can start the process from a measure stochastically below $\mu$. We remind this notion briefly here, for more details, see e.g. [24] chapter 2. For $\eta, \xi \in \Omega$ we define $\eta \le \xi$ if for all $x \in \mathbb{Z}^d$, $\eta_x \le \xi_x$. Functions $f : \Omega \to \mathbb{R}$ preserving this order are called monotone. Two probability measures $\nu_1, \nu_2$ on $\Omega$ are ordered (notation $\nu_1 \le \nu_2$) if for all $f$ monotone, the expectations are ordered, i.e., $\nu_1(f) \le \nu_2(f)$. This is equivalent with the existence of a coupling $\nu_{12}$ of $\nu_1$ and $\nu_2$ such that

$$\nu\{(\eta, \xi) : \eta \le \xi\} = 1$$

**Theorem 4.36.** *Let $\nu \le \mu$, and $\mathbb{P}$ be as in Theorem 4.29. Then, $\nu \times \mathbb{P}$ almost surely the products $\prod_{x\in V} a_x^{N_t^{\varphi_x}}(\eta)$ converge, as $V \uparrow \mathbb{Z}^d$, to a configuration $\eta_t$. $\{\eta_t, t \ge 0\}$ is a Markov process with initial distribution $\nu$.*

*Proof.* For $V \subseteq \mathbb{Z}^d$ finite and $\eta \in \Omega'$, we define

$$\eta_t^V = \prod_{x\in V} a_x^{N_t^{\varphi_x}}(\eta) \tag{4.37}$$

Then we have the relation

$$\eta_t^V = \eta + N_V^t - \Delta n_t^V(\eta) \tag{4.38}$$

where $N_t^V$ denotes $(N_t^{\varphi_x})_{x\in V}$, and $n_t^V$ collects for all sites in $\mathbb{Z}^d$ the number of topplings caused by addition of $N_t^V$ to $\eta$. Since $\eta \in \Omega'$, these numbers are well-defined. Moreover for $\eta \in \Omega'$, the toppling numbers are non-decreasing in $V$ and converge as $V \uparrow \mathbb{Z}^d$ to their infinite volume counterparts $n_t$ which satisfy

$$\eta_t = \eta + N^t - \Delta n_t(\eta) \tag{4.39}$$

For all $V$ and $\eta \le \xi$ it is clear that $n_t^V(\eta) \le n_t^V(\xi)$. Choose now $\nu_{12}$ to be a coupling of $\nu$ and $\mu$ such that $\nu_{12}(\eta_1 \le \eta_2) = 1$. Clearly, $\eta_1$ belongs to $\Omega'$ with $\nu_{12}$ probability one. Therefore for $\nu_{12}$ almost all $\eta_1$, the limit $\lim_{V\uparrow\mathbb{Z}^d} n_t^V(\eta_1) = (\eta_1)_t$ is well-defined, and the corresponding process $\eta_t$, starting from $\eta = \eta_1$ is then defined via (4.39). □

A trivial example of a possible starting configuration is the minimal configuration $\eta \equiv 1$. Less trivial examples are measures concentrating on minimal recurrent configurations (i.e., minimal elements of $\mathcal{R}$).

**Remark 4.40.** *In general we do not know whether $\nu S(t)$ converges to $\mu$ as $t \to \infty$, unless, as we will see later, $\nu$ is absolutely continuous w.r.t. $\mu$. With respect to that question, it would be nice to find a "successfull" coupling, i.e., a coupling $\mathbb{P}_{\zeta,\xi}$ of two versions of $\eta_t$, starting from $\zeta, \xi$ such that $\lim_{t\to\infty} \mathbb{P}_{\zeta,\xi}(\zeta_V(t) = \xi_V(t)) = 1$ for all finite $V \subseteq \mathbb{Z}^d$. Less ambitious but also interesting would be to obtain this result for $\zeta \in \Omega'$ and $\xi = a_x\zeta$. This would yield that all limiting measures $\lim_{n\to\infty} \nu S(t_n)$ along diverging subsequences $t_n \uparrow \infty$ are $a_x$-invariant. Uniqueness of $a_x$-invariant measures can then possibly be obtained by showing that $\mu$ is a Haar measure on some decent group consisting of products of $a_x$.*

# 5 Infinite volume limits of the dissipative model

As an example of "the dissipative case", we consider the toppling matrix defined $\Delta_{xx} = 2d + \gamma$ with $\gamma \in \mathbb{N}$ and $\Delta_{xy} = -1$ for $xy$ nearest neighbors. Upon toppling of a site $x \in V$ each neighbor in $V$ receives a grain, and $\gamma$ grains are lost. One says that " a mass $\gamma$ is added to the massless (critical) model". Another example is to consider "stripes", i.e., $\mathbb{Z}^d \times \{0, \ldots, k\}$ with $\Delta_{xx} = 2d+1$, $\Delta_{xy} = -1$ for $x, y$ nearest neighbors. In this case only the boundary sites are dissipative but constitute a non-vanishing fraction of the volume of the whole system. In general we define "dissipativity" by a condition on the Green's function.

**Definition 5.1.** *The model with toppling matrix $\Delta$ (indexed by sites in $S$ ) is called dissipative if the inverse $G_{xy} = \Delta^{-1}_{xy}$ exists and satisfies*

$$\sup_{x\in S} \sum_{y\in S} G_{xy} < \infty \tag{5.2}$$

This means that the "random walk associated to $\Delta$" is almost surely killed. E.g. for the case $\Delta_{xx} = 2d + \gamma$, one can interpret this as connecting every site $x \in \mathbb{Z}^d$ with $\gamma$ links to an extra site "$*$" and running a simple symmetric random walk on $\mathbb{Z}^d \cup \{*\}$ killed upon hitting $*$. This random walk has a lifetime $\tau$ with finite exponential moments, i.e., $\mathbb{E}(e^{\epsilon\tau}) < \infty$ for $\epsilon > 0$ small enough, and thus (5.2) is clearly satisfied.

In the "critical case" where $\sum_y G(x, y) = \infty$, condition (4.10) cannot hold for a constant addition rate $\varphi$, i.e., in that case the addition rate $\varphi(x)$ has to depend on $x$ and has to converge to zero sufficiently fast for $x$ far away. In the dissipative case constant addition rate is allowed. In fact, addition $\oplus$ of recurrent configurations (as introduced in finite volume in section 3.4) turns out to be well-defined (in infinite volume) as we will see later on. This gives us a group structure and the group $\mathcal{R}, \oplus$ will turn out to be compact. That is the ideal context for applications of theory of random walks on groups, as is e.g. treated in [20]. In what follows we will always denote by $S$ the infinite set of vertices where the toppling matrix is defined so e.g. $S = \mathbb{Z}^d, S = \mathbb{Z}^d \times \{0, \ldots, k\}$, etc.

In the following lemma we show that certain additions are defined in infinite volume for *every* recurrent configuration. This is different from the non-dissipative case where such additions would only be weel-defined $\mu$-a.s.

**Lemma 5.3.** *Suppose that $n : S \to \mathbb{Z}$ satisfies*

$$B := \sup_{y \in S} \sum_{x \in S} |n_x| G(x, y) < \infty \tag{5.4}$$

*then*

$$\Omega(n) = \{\eta \in \mathcal{R} : \exists m \in \mathbb{N}^S, \exists \xi \in \mathcal{R} : \eta + n = \xi + \Delta m\} = \mathcal{R} \tag{5.5}$$

*Proof.* Clearly, $\Omega(n)$ is a subset of $\mu$ measure one, because the product

$$\prod_x a_x^{n_x}$$

is almost surely well-defined if $n$ satisfies (5.4). This follows from the fact that addition according to $n$ satisfying (5.4) causes $\mu$-a.s. a finite number of topplings at every site $x \in S$.

Therefore $\Omega(n)$ is a dense subset of $\mathcal{R}$. We will prove that $\Omega(n)$ is a closed set. This implies that $\Omega(n) = \mathcal{R}$. Let $\eta_k \to \eta$ be a sequence of elements of $\Omega(n)$. Then there exist $m_k \in \mathbb{N}^S$, $\xi_k \in \mathcal{R}$ such that

$$\eta_k + n = \xi_k + \Delta m_k \tag{5.6}$$

This gives $m_k = G(\eta_k + n - \xi_k)$. Since $G$ is a bounded operator on $l^\infty(S)$ by (5.2), and $\eta_k(x), \xi_k(x)$, $x \in S$ are bounded by a constant, we have by (5.4) that there exists a finite $C > 0$ such that $m_k(x) \in [0, C]$ for all $k, x$. Therefore there exists a subsequence $k_n \in \mathbb{N}$ such that $\xi_{k_n} \to \xi$, $m_{k_n} \to m$ as $n \to \infty$.

Therefore, taking the limit along that subsequence in (5.6) gives

$$\eta + n = \xi + \Delta m \tag{5.7}$$

which shows that $\eta \in \Omega(n)$, because $\xi \in \mathcal{R}$ since $\mathcal{R}$ is closed. □

In finite volume we had the fact that $\xi = \eta + \Delta m$ for some $m \in \mathbb{Z}^V$ together with $\eta, \xi \in \mathcal{R}$ implies $\eta = \xi$. This is not the case anymore in infinite volume. Take e.g. $S = \mathbb{Z} \times \{0, 1\}$ and $\Delta_{xx} = 4$, then the configurations $\eta \equiv 3$ and $\zeta \equiv 4$ are both recurrent and "equivalent" , i.e.,

$$\zeta = \eta + \Delta m \tag{5.8}$$

where $m \equiv 1$. This has as a drawback that we can only define addition of two recurrent configurations modulo this equivalence.

**Definition 5.9.** *For $\eta, \xi \in \mathcal{R}$ we say that $\eta \sim \xi$ if there exists $m \in \mathbb{Z}^V$ such that*

$$\eta = \xi + \Delta m \tag{5.10}$$

**Lemma 5.11.** *The quotient $\mathcal{R}/\sim$ is a compact set.*

*Proof.* It suffices to show that equivalence classes of $\sim$ are closed. Denote for $\eta \in \mathcal{R}$, $[\eta]$ the corresponding equivalence class. Suppose $\xi_n \in [\eta]$ for some $\xi_n \to \xi$. Then for every $n$ there exist $m_n \in \mathbb{Z}^S$ such that

$$\eta = \xi_n + \Delta m_n \tag{5.12}$$

As in the proof of lemma 5.3, $m_n(x)$ is uniformly bounded in $x, n$ and hence there exists a sequence $k_n \uparrow \infty$ of positive integers, such that along $k_n$, $\xi_{k_n} \to \xi$, $m_{k_n} \to m$. Taking the limit in (5.12) gives $\xi = \lim_n \xi_n \in [\eta]$. □

Now for $\eta \in \mathcal{R}$, we have proved in lemma 5.3 that the set $\Omega(\eta)$ of those configurations $\xi$ such that $\xi + \eta$ can be stabilized by a finite number of topplings at each site is the whole of $\mathcal{R}$. Clearly two "stable versions" of $\eta + \xi$ are equivalent in the sense $\sim$. Therefore on $\mathcal{R}/\sim$ we define

$$[\eta] \oplus [\xi] = [\mathcal{S}(\eta + \xi)] \tag{5.13}$$

where $\mathcal{S}(\eta + \xi)$ is any stable version of $\eta + \xi$.

**Lemma 5.14.** *$\mathcal{R}/\sim$ is a compact topological group.*

*Proof.* We leave the proof as an exercise with the following hint. Show that the class $[\eta] \oplus [\xi]$ can be obtained as the set of configurations equivalent with

$$\lim_{V \uparrow S} \eta_V \oplus \xi_V \tag{5.15}$$

along a suitable subsequence, where $\oplus$ is defined as usual in finite volume. □

We then have the following.

**Theorem 5.16.** *Suppose (5.2) is satisfied. Then there exists a unique measure $\mu$ on $\mathcal{R}$ such that $\mu$ is invariant under the action of $a_x$. Moreover, $\mu$ is the "lifting" of the unique Haar measure on $(\mathcal{R}/\sim, \oplus)$. Finally, the finite volume uniform measures $\mu_V$ converge to $\mu$ in the sense of definition 4.4.*

*Proof.* For the complete proof, we refer to [29]. The proof consists of three parts:

1. Show that for all limit points $\mu$ of $\mu_V$, $a_x$ is well-defined $\mu$-a.s. and $\mu$ is invariant under the action of $a_x$. This is easy and based on condition (5.2). Define $\mathcal{I}$ to be the set of limit points of $\mu_V$.

2. Show that for every $\mu \in \mathcal{I}$, there exists a $\mu$-measure one set $\Omega_\mu$ such that for all $\eta \in \Omega_\mu$, the equivalence class $[\eta]$ is a singleton.

3. The proof then essentially follows from the uniqueness of the Haar measure on $(\mathcal{R}/\sim, \oplus)$.

□

## 5.1 Ergodicity of the dissipative infinite volume dynamics

In this section we consider the dissipative case and rate one addition, i.e., the process with generator

$$L = \sum_{x\in\mathbb{Z}^d} (a_x - I) \tag{5.17}$$

We furthermore suppose that "every site is dissipative", i.e., $\Delta_{xx} = 2d + \gamma$ with $\gamma$ a strictly positive integer, and $\Delta_{xy} = -1$ for $x, y$ nearest neighbors. The corresponding Green function then decays exponentially, i.e.,

$$0 \leq G(x,y) = \Delta^{-1}_{xy} \leq Ae^{-B|x-y|} \tag{5.18}$$

We start this process from a measure $\nu$ stochastically dominated by $\mu$; this is well-defined by Theorem 4.36. We then show

**Theorem 5.19.** *There exists $C_2 > 0$ such that for all local functions $f$ there exists $C_f > 0$ with*

$$\left| \int S(t) f d\nu - \int f d\mu \right| \leq C_f e^{-C_2 t} \tag{5.20}$$

*Proof.* We start the proof with a simple lemma.

**Lemma 5.21.** *Let $A$ be a self-adjoint invertible bounded operator on a Hilbert space $H$, then for all $f \in H$ we have the estimate*

$$\|A^{-1} f\|^2 \geq \frac{\|f\|^2}{\|A\|^2} \tag{5.22}$$

*Proof.* Suppose $\|f\| = 1$. Use the spectral theorem and Jensen's inequality to write

$$\begin{aligned} \|A^{-1}f\|^2 &= < A^{-1}f | A^{-1}f > \\ &= \int \frac{1}{\lambda^2} dE_{f,f}(\lambda) \\ &\geq \frac{1}{\int \lambda^2 dE_{f,f}(\lambda)} = \frac{1}{\|Af\|^2} \geq \frac{1}{\|A\|^2} \end{aligned} \tag{5.23}$$

□

We now turn to the proof of the theorem. Fix a local function $f$ with dependence set $D_f$. The idea is to approximate $S(t)$ by finite volume semigroups, and to estimate the speed of convergence as a function of the volume. More precisely, we split

$$\left| \int S(t) f d\nu - \int f d\mu \right| \leq A^V_t(f) + B^V_t(f) + C_V(f) \tag{5.24}$$

with

$$
\begin{aligned}
A_t^V(f) &= \left| \int S(t) f d\nu - \int S_V(t) f d\nu_V \right| && (5.25)\\
B_t^V(f) &= \left| \int S_V(t) f d\nu_V - \int f d\mu_V \right| && (5.26)\\
C_V(f) &= \left| \int f d\mu_V - \int f d\mu \right|. && (5.27)
\end{aligned}
$$

where $\nu_V$ is the restriction of $\nu$ to $V$ and

$$
S_V(t) f(\eta) = \int f \left( \prod_{x \in V} a_{x,V}^{N^{t,x}} \eta \right) d\mathbb{P}
$$

for a collection $\{N^{t,x} : x \in \mathbb{Z}^d\}$ of independent rate one Poisson processes with joint distribution $\mathbb{P}$. By Theorem 5.16,

$$
\lim_{V \uparrow S} C_V(f) = 0. \tag{5.28}
$$

For the first term in the right-hand side of (5.24) we write

$$
A_t^V(f) = \left| \int \int \left( f \left( \prod_{x \in S} a_x^{N^{t,x}} \eta \right) - f \left( \prod_{x \in V} a_{x,V}^{N^{t,x}} \eta \right) \right) d\mathbb{P} d\nu \right| \tag{5.29}
$$

The integrand of the right hand side is zero if no avalanche from $V^c$ has influenced sites of $D_f$ during the interval $[0, t]$, otherwise it is bounded by $2\|f\|_\infty$. More precisely, the difference between the function $f$ evaluated in the two configurations appearing in (5.29) comes from extra additions in $V^c$ which possibly can influence the heights in $D_f$ and boundary topplings where grains are lost (in the second product where we use $a_{x,V}$ versus kept in the first product where we write $a_x$. Therefore, since $N^{t,x}$ are rate one Poisson processes:

$$
A_t^V(f) \le C 2 \|f\|_\infty t \sum_{y \in D_f} \sum_{x \in V^c} G(x, y) \tag{5.30}
$$

The second term in the right hand side of (5.24) is estimated by the relaxation to equilibrium of the finite volume dynamics. The generator $L_V^0$ has the eigenvalues

$$
\sigma(L_V^0) = \left\{ \sum_{x \in V} \left( \exp \left( 2\pi i \sum_{y \in V} G_V(x, y) n_y \right) - 1 \right) : n \in \mathbb{Z}^V / \Delta^V \mathbb{Z}^V \right\} \tag{5.31}
$$

The eigenvalue 0 corresponding to the stationary state corresponds to the choice $n = \overline{0}$. For the speed of relaxation to equilibrium we are interested in the minimum absolute value of the real part of the non-zero eigenvalues. More precisely:

$$
B_t^V(f) \le C_f \exp(-\lambda_V t)
$$

where

$$\begin{aligned}\lambda_V &= \inf\left\{|Re(\lambda)| : \lambda \in \sigma(L_V^0) \setminus \{0\}\right\} \\ &= 2\inf\left\{\sum_{x\in V} \sin^2\left(\pi \sum_{y\in V} G_V(x,y)n_y\right) : n \in \mathbb{Z}^V/\Delta^V\mathbb{Z}^V, n \neq \overline{0}\right\}\end{aligned}$$

by (5.31). Since there is a constant $c$ so that for all real numbers $r$

$$\sin^2(\pi r) \geq c(\min\{|r-k| : k \in \mathbb{Z}\})^2$$

we get

$$\sum_{x\in V} \sin^2\left(\pi((\Delta^V)^{-1}n)_x\right) \geq c\inf\left\{\|(\Delta^V)^{-1}n - k\|^2 : n \in \mathbb{Z}^V/\Delta^V\mathbb{Z}^V, n \neq \overline{0}, k \in \mathbb{Z}^V\right\} \tag{5.32}$$

where $\|\cdot\|$ represents the euclidian norm in $\mathbb{Z}^V$ that we estimate by

$$\|(\Delta^V)^{-1}n - k\|^2 = \|(\Delta^V)^{-1}(n - \Delta^V k)\|^2 \geq \|\Delta^V\|^{-2}\|(n - \Delta^V k)\|^2$$

Taking the infimum of (5.32) we have

$$\lambda_V \geq \|\Delta_V\|^{-2}\kappa_V \tag{5.33}$$

where

$$\kappa_V = \inf\{\|\Delta n - k\|^2 : k \in \mathbb{Z}^V, n \in \mathbb{Z}^V/\Delta\mathbb{Z}^V, n \neq 0\} \tag{5.34}$$

This is simply the smallest non-zero distance between a point in the lattice $\Delta\mathbb{Z}^V$ and a point in the lattice $\mathbb{Z}^V$, that is, $\kappa_V = 1$. For every regular volume we have

$$\|\Delta^V\| \leq \sqrt{2\gamma^2 + 16d^2}$$

This gives

$$B_t^V(f) \leq C_f \exp(-Ct) \tag{5.35}$$

where $C > 0$ is independent of $V$.
The statement of the theorem now follows by combining (5.28), (5.30), (5.35), and choosing $V = V_t$ such that $A_t^{V_t}(f) \vee \mathcal{C}_{V_t}(f) \leq \exp(-Ct)$. □

**Remark 5.36.** *The particular choice of dissipation is not essential for the ergodic theorem, but the rate of convergence will of course depend on the addition rate. So in general we have $\nu S(t) \to \mu$ as $t \to \infty$ (in the weak topology) but the speed of convergence depends on the details of the addition rate, and the decay of the Green function.*

A weaker form of ergodicity is the following

**Theorem 5.37.** *Suppose* (5.2) *is satisfied. For all addition rates $\varphi$ such that (4.10) holds, and for all $f, g \in L_2(\mu)$,*

$$\lim_{t\to\infty} \int S(t)fg\,d\mu = \int f\,d\mu \int g\,d\mu \tag{5.38}$$

*Proof.* Since the generator $L = \sum_x \varphi(x)(a_x - I)$ is a normal operator ($L^* = \sum_x \varphi(x)(a_x^{-1} - I)$ commutes with $L$ on a common core the set of local functions), (5.38) follows from ergodicity, i.e., if $Lf = 0$ then $f = \int f d\mu$ $\mu$-a.s. Suppose $Lf = 0$, with $f \geq 0$ and $\int f d\mu > 0$. Then

$$< Lf|f >= \sum_x \frac{1}{2} \int (a_x f - f)^2 = 0 \tag{5.39}$$

and hence the probability measure

$$d\nu_f = \frac{f d\mu}{\int f d\mu}$$

is invariant under the action of $a_x$, and hence under the group action. It is therefore proportional to the Haar measure, i.e., $f$ is $\mu$-a.s. constant. (Remark that we have hidden here some harmless back and forth liftings from $\mathcal{R}$ to $\mathcal{R}/\sim$). ☐

**Remark 5.40.** *One can show that in the "bulk" dissipative case, $\Delta_{xx} = 2d + \gamma$ with $\gamma$ a strictly positive integer, and $\Delta_{xy} = -1$ for $x, y$ nearest neighbors, there is also exponential decay of spatial correlations. More precisely there exists a constant $\delta > 0$ such that for all $f, g$ local function we have*

$$|\int (f \tau_x g) d\mu - \int f d\mu \int g d\mu| \leq C_{f,g} e^{-\delta |x|} \tag{5.41}$$

*This is proved for "high dissipation" ($\gamma$ large depending on $d$) in [29], using a percolation argument, and for all $\gamma > 0$ in [19], using the correspondence with spanning trees.*

# 6 Back to criticality

The essential simplification of the dissipative case is caused by the summable decay of the Green function. This is not the case anymore if we consider the original critical model. It is clear that one of the consequences of "non-dissipativity" is that we have to give up the idea of a group of addition of recurrent configurations (defined on the whole of $\mathcal{R}$). The reason is that e.g. in $d = 2$ the configuration of all heights equal to eight cannot be stabilized (even the configuration of all four except at the origin height 5 cannot be stabilized). From the work of [23] it follows that the limit of the neutral elements in finite volume in $\mathbb{Z}^2$ is equal to the all-three configuration. But adding this to any recurrent configuration gives an unstable configuration which cannot be stabilized.

Let us come back to the basic questions of section 3.1. We start with the following theorem of [1].

**Theorem 6.1.** *Let $\mu_V$ denote the uniform measure on recurrent configurations $\mathcal{R}_V$ in volume $V \subseteq \mathbb{Z}^d$. Then for all $d \in \mathbb{N}$ as $V \uparrow \mathbb{Z}^d$, $\mu_V \to \mu$, in the sense of (4.4).*

We will give the full proof of this theorem in $1 \leq d \leq 4$ and indicate how the proof goes in $d \geq 5$. For the complete proof we refer to [1]. The basic ingredient is the existence of the weak infinite volume limit $\nu = \lim_{\Lambda \uparrow \mathbb{Z}^d} \nu_\Lambda$ of the uniform measure on rooted spanning trees, and some basic properties of the limiting measure. Let us first explain this convergence. For a set $\Lambda \subseteq \mathbb{Z}^d$ we denote by $E(\Lambda)$ the set of edges inside $\Lambda$, i.e., those edges $e = (xy)$ such that both vertices $x, y \in \Lambda$. The rooted spanning tree measure $\nu_\Lambda$ is the uniform measure on rooted spanning trees $T_\Lambda$ (as explained in section ) viewed as a measure on $\{0,1\}^\Lambda$ (where value 1 is interpreted as presence of the edge), i.e., we do not consider the edges of $T_\Lambda$ going to the root $*$. We say that $\nu_\Lambda \to \nu$, where $\nu$ is a probability measures on $\{0,1\}^{E(\mathbb{Z}^d)}$ if for all finite sets of edges $\{e_1, \ldots, e_k\}$ the numbers $\nu_\Lambda(\xi_{e_1} = 1 = \ldots \xi_{e_k})$ converge to the probability $\nu(\xi_{e_1} = 1, \ldots, \xi_{e_k} = 1)$. The finite volume measure can be viewed as the distribution of a subgraph $(T_\Lambda, E(T_\Lambda))$ of $(\Lambda, E(\Lambda))$, and analogously the infinite volume measure $\nu$ can be viewed as the distribution of a subgraph of $(\mathbb{Z}^d, E(\mathbb{Z}^d))$ which we will denote by $(T, E(T))$.

For an infinite subgraph $(T, E(T)) \subseteq (Z^d, E(\mathbb{Z}^d))$ we call a path in $T$ a set of vertices $\gamma = \{x_1, \ldots, x_k, \ldots\}$ such that for all $i$, $x_i x_{i+1} \in E(T)$. A path is infinite if it contains infinitely many vertices. Two infinite paths $\gamma, \gamma'$ in $T$ are called equivalent if the set $(\gamma \setminus \gamma') \cup (\gamma' \setminus \gamma)$ is finite. An end of $T$ is an equivalence class of infinite paths. The subgraph $T$ is called a tree if it contains no loops and is connected. If it contains no loops (but is not necessarily connected) it is called a forest. The following theorem is proved in [32], see also [4] for the last item. In fact much more is proved in these papers but we summarize in the theorem below the facts we need for the proof of our theorem 6.1.

**Theorem 6.2.** *The uniform measures on rooted spanning trees $\nu_\Lambda$ converge to a measure $\nu$ as $\Lambda \uparrow \mathbb{Z}^d$. The limiting measure $\nu$ is the distribution of the edge set of a subgraph $(T, E(T)) \subseteq (\mathbb{Z}^d, E(\mathbb{Z}^d))$ which has the following properties*

1. *For $1 \leq d \leq 4$, $T$ is $\nu$-almost surely a tree with a single end.*
2. *For $d \geq 5$, $T$ is $\nu$-almost surely a forest with infinitely many components which all have a single end.*

**Definition 6.3.** *Let a probability measure $\nu$ be given on $A^S$ with $S$ an infinite set, and $A$ a finite set (think of $S = \mathbb{Z}^d$ or $S = E(\mathbb{Z}^d)$, $A = \{1, \ldots, 2d\}$, $A = \{0,1\}$). A function $f : A^S \to \mathbb{R}$ is called $\nu$ almost local if for all $\eta \in A^S$ there exists $V(\eta) \subseteq S$ such that $\nu(\{\eta : |V(\eta)| < \infty\}) = 1$ and such that for all $\eta, \xi \in A^S$,*

$$f(\eta) = f(\eta_{V(\eta)} \xi_{S \setminus V(\eta)}) \tag{6.4}$$

*In words, the value of $f$ depends only on the coordinates of $\eta$ in $V(\eta)$.*

This notion of "almost" local will show up naturally if we want to reconstruct the height variables from a realization of the uniform spanning forest (USF) on $\mathbb{Z}^d, d \leq 4$. The height will depend on a finite portion of the tree with probability one, but the size of this portion is not uniformly bounded (the height variable is not a local function). We can now sketch the proof of theorem 6.1.

*Proof.* Suppose we have a rooted spanning tree $T_\Lambda$ in the finite volume $\Lambda$. We can view the tree as a realization of the burning algorithm. The corresponding recurrent height configuration can be reconstructed if we know the burning times $t_x$ for all sites. The burning time $t_x$ is exactly equal to the length of the unique (non-back-tracing) path from $x$ to the root $*$ (this path is unique if we do not allow back-tracing, i.e. each edge in the path is traversed at most once). If for a given site $x$ we know the order of the numbers $t_x, t_y$ for all neighbors $y$ of $x$, then we can already determine the height. E.g. if $t_x < \min_{y\sim x} t_y$ this means that the site is burnt only after all its neighbors, which implies that its height is one. Similarly, given the preference rule we can reconstruct the height from the ordering of the numbers $t_x, t_{y_{y\sim x}}$. Suppose now a realization of the uniform rooted spanning tree is given as well as a finite number of sites $A = \{x_1, \ldots, x_k\}$. Call $\bar{A}$ the set of sites obtained from $A$ by adding all the neighbors of $A$. We suppose that $\Lambda$ is big enough such that $\bar{A} \subseteq \Lambda$. Call $\eta_x(\Lambda)(T_\Lambda)$ the height at site $x$ corresponding to the rooted spanning tree $T_\Lambda$. For an infinite tree $T$, we can reconstruct the height $\eta_x(T)$ if the tree has one end. This goes as follows. If the tree has one end, then all paths starting from $x \in A$ or a neighbor $y \sim x$ and going to infinity (think of infinity as being the root) coincide from some point $a(x)$ on, i.e., $a(x)$ is a common ancestor of the set $\bar{A}$. Consider the subtree $T_\Lambda(A)$ of all descendants of $a(x)$. This is a finite tree, and the height $h_x$ is reconstructed from the order of the distances from $x$ and his neighbors to $a(x)$. The heights $\eta_x, x \in A$ are completely determined by the tree $T_\Lambda(A)$. Notice that as long as $\Lambda$ is finite, it is possible that $a(x)$ coincides with the root $*$, but the probability of that event under $\nu_\Lambda$ will become small as $\Lambda$ increases. Finally, denote by $T_\Lambda(A, \eta)$ the edge configuration of the tree $T_\Lambda(A)$ corresponding to height configuration $\eta_A$ on $A$. With the notation we just introduced we can now write, for fixed finite $V_0 \subseteq \mathbb{Z}^d$

$$
\begin{aligned}
\mu_\Lambda(\eta_A) &= \nu_\Lambda(T_\Lambda(A) = T_\Lambda(A,\eta)) \\
&= \nu_\Lambda\Big(T_\Lambda(A) = T_\Lambda(A,\eta), T_\Lambda(A) \subseteq V_0\Big) + \nu_\Lambda\Big(T_\Lambda(A) = T_\Lambda(A,\eta), T_\Lambda(A) \not\subseteq V_0\Big)
\end{aligned}
$$

Since the indicator $T_\Lambda(A) \subseteq V_0$ is local, we conclude that

$$
\limsup_{\Lambda\uparrow\mathbb{Z}^d} \mu_\Lambda(\eta_A) \le \nu\Big(T_{\mathbb{Z}^d}(A) = T_{\mathbb{Z}^d}(A,\eta), T_{\mathbb{Z}^d}(A) \subseteq V_0\Big) + \nu(T_{\mathbb{Z}^d}(A) \not\subseteq V_0) \tag{6.5}
$$

and similarly

$$
\liminf_{\Lambda\uparrow\mathbb{Z}^d} \mu_\Lambda(\eta_A) \ge \nu\Big(T_{\mathbb{Z}^d}(A) = T_{\mathbb{Z}^d}(A,\eta), T_{\mathbb{Z}^d}(A) \subseteq V_0\Big) - \nu(T_{\mathbb{Z}^d}(A) \not\subseteq V_0) \tag{6.6}
$$

Since $T_{\mathbb{Z}^d}(A)$ is $\nu$-almost surely a finite set, we obtain from combining (6.5) and (6.6) and letting $V_0 \uparrow \mathbb{Z}^d$:

$$
\lim_{\Lambda\uparrow\mathbb{Z}^d} \mu_\Lambda(\eta_A) = \nu(T_{\mathbb{Z}^d}(A) = T_{\mathbb{Z}^d}(A,\eta)) \tag{6.7}
$$

which shows the convergence of $\mu_\Lambda$ to a unique probability measure $\mu$.

For the case $d \ge 5$, things become more complicated, because the height at a site $x$ cannot be recovered from the edge configuration of the spanning forest in infinite volume. This is because if $x$ and his neighbors belong to different infinite trees, we cannot directly order the lengths of the paths going to infinity. The idea now is that if $x$ and his neighbors belong to $r$ components, then in each component one can look for a common ancestor of $x$ and the neighbors in that component. The infinite paths going from the $r$ common ancestors will now have a completely random order. More precisely, in bigger and bigger volumes, the lengths $l_1, \ldots, l_r$ of the paths from the $r$ common ancestors to the root will satisfy

$$\nu_\Lambda(l_{\sigma(1)} < l_{\sigma(2)}, \ldots < l_{\sigma(r)}) = \frac{1}{r!} + o(|\Lambda|)$$

for every permutation $\sigma : \{1, \ldots, r\} \to \{1, \ldots r\}$. One can imagine this by realizing that the paths going to the root behave like (loop-erased) random walkers, so the hitting times of the boundary of a large set will have a completely random order. This implies that the height cannot be reconstructed from the edge configuration, but can be reconstructed if we add extra randomness, corresponding to a random ordering of the components of the infinite volume spanning forest. This goes as follows. Given a realization of the infinite volume spanning forest, we associate to each component $\mathcal{T}_i$ a uniform random variable $U_i$ on the interval $[0, 1]$. For different components these $U_i$'s are independent. The height at site $x$ can now be reconstructed from the edge configuration of the spanning forest and these random variables $U_i$. Within each component we can order those sites from the set consisting of $x$ and his neighbors according to the lengths of the paths going to the common ancestor in the given component. This leads say to the order $x^i_1 < x^i_2 < \ldots < x^i_{k_i}$ within the component $\mathcal{T}_i$ The uniform variables determine an order of the components, or equivalently a permutation $\sigma : \{1, \ldots, r\} \to \{1, \ldots r\}$. The final order of $x$ and his neighbors is then given by

$$x^{\sigma(1)}_1 < x^{\sigma(1)}_2 < \ldots < x^{\sigma(1)}_{k_{\sigma(1)}} < x^{\sigma(2)}_1 < x^{\sigma(2)}_2 < \ldots < x^{\sigma(2)}_{k_{\sigma(2)}} < \ldots < x^{\sigma(r)}_1 < x^{\sigma(r)}_2 < \ldots < x^{\sigma(r)}_{k_{\sigma(r)}}$$

and this order determines the height. We finish here the discussion of the case $d \geq 5$; the complete proof can be found in [1] $\square$

## 6.1 Wilson's algorithm and two-component spanning trees

In order to obtain more insight in the nature of avalanches, we already mentioned the idea of decomposing an avalanche into a sequence of waves. We will now obtain a spanning tree representation of the first wave, and derive from that its a.s. finiteness in $d > 4$. The first wave starts upon addition at a site in a recurrent configuration. For simplicity suppose that the site where a grain is added is the origin. Consider a finite volume $V \subseteq \mathbb{Z}^d$ containing 0, and a recurrent configuration $\eta_V \in \mathcal{R}_V$. Its restriction $\eta_{V\setminus\{0\}}$ is recurrent in $V \setminus \{0\}$. Therefore in the burning algorithm applied to $\eta_{V\setminus\{0\}}$, all sites will be burnt. We now split this burning procudure in two phases. In phase one at time $T = 0$, we burn neigbors of 0 (if possible), and continue at time $T = 1$ with neigbors of the sites burnt at time $T = 0$, and so on until no sites can be burnt which are neighbors of the previously burnt sites. The set of all sites which are burnt in the first phase is the support of the first wave. The second phase of the burning algorithm then starts from the boundary of $V$ and burns all the sites not contained in the first wave. The spanning tree respresentation of both phases gives a graph with two components: the first (resp. second) component corresponding to the spanning tree representation of the first (resp. second) phase of the burning algorithm applied to $\eta_{V\setminus\{0\}}$. We call this two-component graph *the two-component spanning tree representation of the wave.*

Under the uniform measure $\mu_V$ on recurrent configurations in $V$, this two-component spanning tree is not uniformly distributed, but we will show that it has a distribution with bounded density w.r.t. the uniform two-component spanning tree (bounded uniformly in the volume $V$). Let us first describe the uniform measure on two-component spanning trees in finite volume $V$. Consider $V_0 := V\setminus\{0\}$ and the multigraph $(V_0^*, E^*)$ obtained by adding the additional site $*$ ("root") and putting $\alpha_V(x)$ edges between sites of the inner boundary and $*$ where

$$\alpha_{V_0}(x) = 2d - \{y : y \text{ is neighbor of } \ x, y \notin V_0\}$$

A rooted spanning tree is a connected subgraph of $(V^*, E^*)$ containing all sites and no loops. One can split such a spanning tree $\mathcal{T}$ in paths going off to $*$ via a neigbor of the origin (notation $\mathcal{T}_0$), and paths going off to $*$ via the boundary of $V$. With a slight abuse of words, we call $\mathcal{T}_0$ "the component containing the origin". We will be interested in the distribution of $\mathcal{T}_0$ under the uniform measure on rooted spanning trees of $V_0$, as $V \uparrow \mathbb{Z}^d$. Notice that the recurrent configurations $\eta_{V_0} \in \mathcal{R}_{V_0}$ are in one-to-one correspondence with the rooted spanning trees of $V_0$, and hence the distribution of the tree $\mathcal{T}$ under the uniform measure $\mu_{V_0}$ on $\mathcal{R}_{V_0}$ is uniform. We will now first show that the distribution of $\mathcal{T}_0$ under $\mu_V$ is absolutely continuous w.r.t. the distribution of $\mathcal{T}_0$ under $\mu_{V_0}$, with a uniformly (in $V$) bounded density.

**Lemma 6.8.** *There is a constant $C(d) > 0$ such that for all $d \geq 3$*

$$\sup_{V\subseteq\mathbb{Z}^d} \frac{|\mathcal{R}_{V\setminus\{0\}}|}{|\mathcal{R}_V|} \leq C(d) \tag{6.9}$$

*Proof.* By Dhar's formula,

$$|\mathcal{R}_{V\setminus\{0\}}| = \det(\Delta_{V\setminus\{0\}}) = \det(\Delta'_V)$$

where $\Delta'_V$ denotes the matrix indexed by sites $y \in V$ and defined by $(\Delta'_V)_{yz} = (\Delta_{V\setminus\{0\}})_{yz}$ for $y, z \in V \setminus \{0\}$, and $(\Delta'_V)_{0z} = (\Delta'_V)_{z0} = \delta_{0z}$. Clearly,

$$\Delta_V + P = \Delta'_V$$

where $P$ is a matrix which has only non-zero entries $P_{yz}$ for $y, z \in N = \{u : |u| \leq 1\}$. Moreover, $\max_{y,z\in V} P_{yz} \leq 2d - 1$. Hence

$$\frac{|\mathcal{R}_{V\setminus\{0\}}|}{|\mathcal{R}_V|} = \frac{\det(\Delta_V + P)}{\det(\Delta_V)} = \det(I + G_V P),$$

where $G_V = (\Delta_V)^{-1}$. Here $(G_V P)_{yz} = 0$ unless $z \in N$. Therefore

$$\det(I + G_V P) = \det(I + G_V P)_{u\in N, v\in N} \tag{6.10}$$

By transience of the simple random walk in $d \geq 3$, we have $\sup_V \sup_{y,z} G_V(y,z) \leq G(0,0) < \infty$, and therefore the determinant of the finite matrix $(I + G_V P)_{u\in N, v\in N}$ in (6.10) is bounded by a constant depending on $d$. $\square$

From this lemma we derive the following.

**Proposition 6.11.** *For all events $A$ depending on heights in $V_0$,*

$$\frac{\mu_V(A)}{\mu_{V_0}(A)} \leq C \tag{6.12}$$

*Proof.* We have, for all $\sigma \in \Omega_V$

$$\mu_V(\eta_{V_0} = \sigma_{V_0}) = \frac{1}{|\mathcal{R}_V|}|\{k \in \{1, \ldots, 2d\} : k_0(\sigma)_{V_0} \in \mathcal{R}_V\}| \leq 2d\frac{|\mathcal{R}_{V_0}|}{|\mathcal{R}_V|} \leq 2dC(d) \leq C \tag{6.13}$$

where $C(d)$ is the constant of lemma 6.8. The statement of the proposition now follows from the elementary inequality

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i} \leq \max_{i=1}^n \frac{x_i}{y_i}$$

for non-negative numbers $x_i, y_i$, with $\sum_{i=1}^n x_i > 0, \sum_{i=1}^n y_i > 0$. $\square$

The distribution of $\mathcal{T}_0$ under $\mu_V$ is that of the first wave. After the first wave the restriction of the initial configuration $\eta \in \mathcal{R}_V$ on $V_0$ is given by

$$S_V^1(\eta) = \left(\prod_{j\sim 0} a_{j,V_0}\right)(\eta_{V_0})$$

Similarly if there exists a $k$-th wave, then its result on the restriction $\eta_{V_0}$ is given by

$$S_V^k(\eta) = \left(\prod_{j\sim 0} a_{j,V_0}\right)^k(\eta_{V_0})$$

Denote by $\Xi_V^k$ he $k$-th wave in volume $V$.

We can then write, for a fixed $W \subseteq \mathbb{Z}^d$ containing the origin,

$$\begin{aligned}
\mu_V\left(\Xi_V^k \not\subseteq W\right) &\leq \mu_V(\mathcal{T}_0^V(\eta_V) \not\subseteq W) \\
&\leq C\mu_{V_0}(\mathcal{T}_V^0(S_V^{k-1}(\eta_{V_0})) \not\subseteq W) \\
&= C\mu_{V_0}(\mathcal{T}_V^0(\eta_{V_0}) \not\subseteq W)
\end{aligned} \tag{6.14}$$

where in the second line we used proposition 6.11 ad in the third line, we used that $\mu_{V_0}$ is invariant under the addition operators $a_{x,V_0}, x \in V_0$.

Therefore if we can prove that under the uniform measure on two-component spanning trees, "the component of the origin" stays almost surely finite, then we obtain that all waves are finite with probability one (w.r.t. the infinite volume limit of $\mu_V$). Let us denote by $\nu_V^0$ this uniform measure on two-component spanning trees of $V$. We then have the following

**Theorem 6.15.** *1. For all $d \geq 1$ the limit $\lim_{V \uparrow \mathbb{Z}^d} \nu_V^0 := \nu^0$ exists*

*2. Assume $d > 4$. The "component of the origin" $\mathcal{T}_0$ is $\nu^0$-almost surely finite.*

*3. Assume $d > 4$. Under the infinite volume limit $\mu$ of the uniform measures on recurrent configurations, all waves are finite with probability one.*

We will now proceed by Wilson's algorithm to generate a sample of $\nu^0$. Then we will give a sketch of proof of item 2 of theorem 6.15. Item 3 follows by previous considerations from item 2. Item 1 is a standard result of Pemantle.

To define Wilson's algorithm directly in infinite volume, we assume $d \geq 3$. Enumerate the sites of $\mathbb{Z}^d \setminus \{0\} := \{x_0, x_1, \ldots, x_n, \ldots\}$. Pick the first point and start simple random walk killed upon hitting 0. This gives a (possibly infinite) path $\gamma_1$. Let $LR(\gamma_1)$ denote its loop-erasure (which is well-defined since the random walk is transient). Pick a new point $x$ not yet visited by $LR(\gamma_1)$. Start simple random walk killed upon hitting $\{0\} \cup LR(\gamma_1)$. This gives a second path $\gamma_2$, which we again loop-erase to generate $LR(\gamma_2)$. Continue this procedure until every point $x \in \mathbb{Z}^d \setminus \{0\}$ has been visited (to be precise this of course requires some limiting procedure which we will not detail here). The graph which is created by this algorithm, restricted to $\mathbb{Z}^d \setminus \{0\}$ (but keeping track of the exiting bonds when hitting 0) has distribution $\nu^0$. The component of the origin corresponds to those paths "killed by hitting the origin or by a path that has been killed by hitting the origin or...".

We can now give a sketch of the proof of item 2 of theorem 6.15.

*Proof.* The idea is to make a coupling between $\nu^0$ and $\nu$, where $\nu$ is the uniform spanning forest (USF) measure on $\mathbb{Z}^d$. In a spanning forest $T$ (distributed according to $\nu$) we make the convention to direct the edges towards infinity. If two sites $x, y$ are joined by an directed edge, we say that $x$ is a descendant of $y$, or $y$ is an ancestor of $x$. Put $V_N := [-N, N]^d \cap \mathbb{Z}^d$.

Define the event

$$G(M, N) = \{desc(V_M; T) \subseteq V_n\} \tag{6.16}$$

where $desc(V)$ denotes the set of descendants of $V$. Another way to describe the event $G(M, N)$ is to say that there exists a connected set $V_M \subseteq D \subseteq V_N$ such that is no directed edge of from $Zd \setminus D$ to $D$ (there is of course a directed edge from $D$ to $\mathbb{Z}^d \setminus D$).

In [4], theorem 10.1, it is proved that each component of the USF has one end. This means that for two sites $x, y$ in the same component, the paths going to infinity coincide at some moment (more formally two infinite paths containing $x$ and $y$

have infinitely many vertices in common). In particular this implies that the set of descendant of a given site is always finite. This implies that for all $M \geq 1$

$$\lim_{N\to\infty} \nu(G(M,N)) = 1 \tag{6.17}$$

The idea of the proof is then the following: first generate that part of the tree containing $D$, then, in the two-component spanning tree, the component of the origin will be inside $D$ provided no walker used in generating the part that contains $D$ hits the origin on a loop (because then in the measure $\nu^0$, this walker will be killed). If $d > 4$ , then

$$\sum_x G(0,x)^2 < \infty \tag{6.18}$$

which implies that in the simple random walk, with probability one, there exists $M_0$ such that loops containing the origin and a point $x$ with $|x| > M_0$ do not occur. Therefore by choosing $M$ large enough, with probability arbitrary close to one, the walks used to generate the part of the tree with distribution $\nu$ containing $D$ do not contain a loop (which is afterwards erased) containing the origin. In that case, the component of the origin will be enclosed by $D$ and hence be finite. □

## 6.2 A second proof of finiteness of waves

For a finite graph $G$, we write $T_G$ for the uniform spanning tree on the graph $G$, and write $\mathbb{P}$ to denote its law. Let $0 \in \Lambda \subseteq \mathbb{Z}^d$, $d \geq 1$. Let $\nu_\Lambda$ denote the wired spanning tree measure in volume $\Lambda$. We can think of $\nu_\Lambda$ in the following way. We add a vertex $\delta$ to $\Lambda$, which is joined to each boundary vertex of $\Lambda$ by the appropriate number of edges. Denote the graph obtained this way by $\tilde{\Lambda}$. Then $\nu_\Lambda$ is the marginal law of $T_{\tilde{\Lambda}}$ on the "ordinary" edges, that is on edges not touching $\delta$. Introduce another graph $\hat{\Lambda}$ by adding an edge $e$ to $\tilde{\Lambda}$, between $\delta$ and 0. Then the two-component spanning tree measure $\nu_\Lambda^{(0)}$ is the marginal law of $T_{\hat{\Lambda}}$ conditioned on $\{e \in T_{\hat{\Lambda}}\}$ on the ordinary edges. Finally, $\nu_\Lambda$ is also the marginal law of $T_{\hat{\Lambda}}$ conditioned on $\{e \notin T_{\hat{\Lambda}}\}$.

**Lemma 6.19.** *Let $d \geq 1$. For any finite $0 \in \Lambda \subseteq \mathbb{Z}^d$, $\nu_\Lambda$ stochastically dominates $\nu_\Lambda^{(0)}$.*

*Proof.* It is shown in [4], Theorem 4.1, that for any finite graph $G$, $e$ an edge of $G$ and an increasing event $A$ that ignores the edge $e$, one has

$$\mathbb{P}[T_G \in A \,|\, e \in T_G] \leq \mathbb{P}[T_G \in A]. \tag{6.20}$$

This in turn implies that

$$\mathbb{P}[T_G \in A \,|\, e \notin T_G] \geq \mathbb{P}[T_G \in A]. \tag{6.21}$$

Now let $A$ be an increasing event on ordinary edges of $\Lambda$. Then

$$\nu_\Lambda^{(0)}(A) = \mathbb{P}(T_{\hat{\Lambda}} \in A \,|\, e \in T_{\hat{\Lambda}}) \leq \mathbb{P}(T_{\hat{\Lambda}} \in A) \leq \mathbb{P}(T_{\hat{\Lambda}} \in A \,|\, e \notin T_{\hat{\Lambda}}) = \nu_\Lambda(A). \tag{6.22}$$

□

Taking the weak limits as $\Lambda \to \mathbb{Z}^d$, we obtain that for any increasing event $A$ that only depends on finitely many edges of $\mathbb{Z}^d$, we have $\nu^{(0)}[A] \leq \nu[A]$. By Strassen's theorem, there is a monotone coupling between $\nu^{(0)}$ and $\nu$. Let $\nu^*$ denote the coupling measure on the space $\Omega \times \Omega$. Let $\mathcal{T}_0$ denote the component of 0 under $\nu^{(0)}$, and let $\mathcal{T}_1$ denote the union of the other components. (Actually, $\mathcal{T}_1$ is a single component, but we do not need this.)

**Proposition 6.23.** *For $d = 3$ or $4$,*

$$\nu^{(0)}[|\mathcal{T}_0| = \infty] = 0. \tag{6.24}$$

*Proof.* By transience of the simple random walk, it is not hard to see that when $d \geq 3$, $\mathcal{T}_1$ contains an infinite path $\pi_1$ $\nu^{(0)}$-a.s. On the event $\{|\mathcal{T}_0| = \infty\}$, there is an infinite path $\pi_0$ inside $\mathcal{T}_0$, that is of course disjoint from $\pi_1$. By the coupling, we get two disjoint infinite paths in the uniform spanning forest. When $d \leq 4$, this is almost surely impossible, because there is only one component, and that has one end. $\square$

Here is a proof that works for all $d \geq 3$.

**Proposition 6.25.** *For all $d \geq 3$,*

$$\nu^{(0)}[|\mathcal{T}_0| = \infty] = 0. \tag{6.26}$$

*Proof.* Assume that $\nu^{(0)}(|\mathcal{T}_0| = \infty) = c_1 > 0$, and we try to reach a contradiction. We consider the construction of the configuration under $\nu^{(0)}$ via Wilson's algorithm. Suppose that the first random walk, call it $S^{(1)}$, starts from $x \neq 0$. Then we have

$$\nu^{(0)}(x \not\leftrightarrow 0) = \mathbb{P}(S^{(1)} \text{ does not hit } 0) = 1 - \frac{G(x,0)}{G(0,0)} \to 1 \qquad \text{as } |x| \to \infty. \tag{6.27}$$

In particular, there exists an $x \in \mathbb{Z}^d$, such that

$$\nu^{(0)}(|\mathcal{T}_0| = \infty,\, x \not\leftrightarrow 0) \geq c_1/2. \tag{6.28}$$

Fix such an $x$. Let $B(x,n)$ denote the box of radius $n$ centred at $x$. Fix $n_0$ such that $0 \in B(x, n_0)$, and 0 is not a boundary point of $B(x, n_0)$. By inclusion of events, (6.28) implies

$$\nu^{(0)}(0 \leftrightarrow \partial B(x,n),\, x \not\leftrightarrow 0) \geq c_1/2 \tag{6.29}$$

for all $n \geq n_0$. For fixed $n \geq n_0$, let $y_1 = x$, and let $y_2, \dots, y_K$ be an enumeration of the sites of $\partial B(x,n)$. We use Wilson algorithm with this enumeration of sites. Let $S^{(i)}$ and $\tau^{(i)}$ denote the $i$-th random walk and the corresponding hitting time determined by the algorithm. We use these random walks to analyze the configuration under both $\nu^{(0)}$ and $\nu$.

The event on the left hand side of (6.29) can be recast as

$$\{\tau^{(1)} = \infty,\, \exists\, 2 \leq j \leq K \text{ such that } \tau^{(j)} < \infty,\, S^{(j)}_{\tau^{(j)}} = 0\}, \tag{6.30}$$

and hence this event has probability at least $c_1/2$. On the above event, there is a first index $2 \leq N \leq K$, such that the walk $S^{(N)}$ hits $B(x, n_0)$ at some random time $\sigma$, where $\sigma < \tau^{(N)}$. Therefore this latter event, call it $A$, also has probability at least $c_1/2$. Let $p = p(x, n_0)$ denote the minimum over $z \in \partial B(x, n_0)$ of the probability that a random walk started at $z$ hits $x$ before 0 without exiting $B(x, n_0)$. Clearly, $p > 0$.

Let $B$ denote the subevent of $A$ on which after time $\sigma$, the walk $S^{(N)}$ hits the loop-erasure of $S^{(1)}$ before hitting 0 (and without exiting $B(x, n_0)$). We have $\mathbb{P}(B \mid A) \geq p$. Now we regard the random walks as generating $\nu$. By the definition of $N$, on the event $A \cap B$, the hitting times $\tau^{(1)}, \ldots, \tau^{(N)}$, have the same values as in the construction for $\nu^{(0)}$. Moreover, on $A \cap B$, the tree containing $x$ has two disjoint paths from $\partial B(x, n_0)$ to $\partial B(x, n)$: one is part of the infinite path generated by $S^{(1)}$, the other part of the path generated by $S^{(N)}$. Therefore, the probability of the existence of two such paths is at least $p(c_1/2)$, for all $n \geq n_0$. However, this probability should go to zero, because under $\nu$, each tree has one-end almost surely. This is a contradicion, proving the Proposition. $\square$

# 7 Stabilizability and "the organization of self-organized criticality"

In this section we introduce the notion of "stabilizability", and show that in some sense the stationary measure of the ASM defines a transition point between stabilizable and non-stabilizable measures. It is based upon material of [30] and [14].

We denote by $\mathcal{H}$ the set of infinite height configurations, i.e., $\mathcal{H} = \{1, 2, 3, \ldots\}^{\mathbb{Z}^d}$. $\mathcal{P}_t = \mathcal{P}_t(\mathcal{H})$ denotes the set of translation invariant probability measure on the Borel-$\sigma$-field of $\mathcal{H}$. As before, $\Omega$ denotes the set of stable configurations.

For $\eta \in \mathcal{H}$ $V \subseteq \mathbb{Z}^d$, we define $N_V^\eta$ to be the column collecting the toppling numbers at each site $x \in V$, when we stabilize $\eta_V$ in the finite volume $V$, i.e., grains are lost when boundary sites of $V$ topple. We then have the relation

$$\eta_V - \Delta_V N_V^\eta = \mathcal{S}_V(\eta) \tag{7.1}$$

where $\mathcal{S}_V$ denotes stabilization in $V$, i.e., $\mathcal{S}_V(\eta) = (\mathcal{S}_V(\eta_V))\eta_{V^c}$, and $(\Delta_V)_{xy} = \Delta_{xy} 1\!\!1_{x,y \in V}$.

Remember that for fixed $\eta \in \mathcal{H}$ and $x \in \mathbb{Z}^d$, $N_V^\eta(x)$ is non-decreasing in $V$.

**Definition 7.2.** *We call $\eta \in \mathcal{H}$* stabilizable *if for all $x \in \mathbb{Z}^d$, $\lim_{V \uparrow \mathbb{Z}^d} N_V^\eta(x) < \infty$. We denote by $\mathbb{S}$ the set of all stabilizable configurations. Notice that since $N_V^\eta$ are (Borel) measurable functions of $\eta$, $\mathbb{S}$ is a (Borel) measurable set.*

*A probability measure $\nu$ on $\mathcal{H}$ is called stabilizable if $\nu(\mathbb{S}) = 1$.*

It is clear that $\mathbb{S}$ is a translation invariant set, therefore if $\nu$ is an ergodic probability measure (under spatial translations), then $\nu(\mathbb{S}) \in \{0, 1\}$.

The set of stabilizable configurations clearly satisfies the following monotonicity property: if $\eta \in \mathbb{S}, \zeta \leq \eta$ then $\zeta \in \mathbb{S}$.

Therefore the following "critical density" is well-defined:

**Definition 7.3.** *The critical density for stabilizability is defined as*

$$\begin{aligned}\rho_c &= \sup\{\rho : \exists \mu \in \mathcal{P}_t : \int \eta(0) d\mu = \rho, \mu(\mathbb{S}) = 1\} \\ &= \inf\{\rho : \exists \mu \in \mathcal{P}_t : \int \eta(0) d\mu = \rho, \mu(\mathbb{S}) = 0\}\end{aligned} \tag{7.4}$$

**Definition 7.5.** *A configuration $\eta \in \mathcal{H}$ is* weakly stabilizable *if there exists $N^\eta$ : $\mathbb{Z}^d \to \mathbb{N}$, and $\xi \in \Omega$ such that*

$$\eta - \Delta N^\eta = \xi \tag{7.6}$$

**Remark 7.7.** *It is clear that stabilizability implies weak stabilizability, but the converse is not clear because it is not obvious that a given stabilizing toppling vector $n \in \mathbb{N}^{\mathbb{Z}^d}$ can be realized as a (possibly infinite) sequence of legal topplings.*

Let us denote by $\rho_{\mathbb{Z}^d}$ the expected height of the stationary measure of the ASM, in the thermodynamic limit.

We have the following theorem

**Theorem 7.8.** *1. For all $d \geq 1$, $\rho_c = d + 1$. In $d = 1$, $\rho_c = 2 = \rho_{\mathbb{Z}^d}$, while for $d = 2$, $\rho_c < \rho_{\mathbb{Z}^d}$.*

*2. If $\nu$ is a translation invariant ergodic probability measure on $\mathcal{H}$ with $\nu(\eta_0) = \rho > 2d$, then $\nu$ is not weakly stabilizable.*

*Proof.* We start with item 2 Suppose that there exist $m \in \mathbb{N}^{\mathbb{Z}^d}$ such that

$$\eta - \Delta m = \xi \tag{7.9}$$

with $\xi$ stable and $\eta$ a sample from $\nu$. Let $X_n$ be the position of simple random walk (starting at the origin at time 0) at time $n$. From (7.9) it follows that

$$m(X_n) - m(0) - \frac{1}{2d}\sum_{k=1}^{n-1}(\xi(X_k) - \eta(X_k)) \tag{7.10}$$

is a mean-zero martingale. Therefore taking expectations w.r.t. the random walk

$$\frac{1}{n}\left(\mathbb{E}_0(m(X_n)) - m(0))\right) = \frac{1}{n}\mathbb{E}_0\left(\frac{1}{2d}\sum_{k=1}^{n-1}(\xi(X_k) - \eta(X_k))\right) \tag{7.11}$$

By stability of $\xi$

$$\frac{1}{n}\left(\sum_{k=1}^{n-1}(\xi(X_k) - \eta(X_k))\right) \leq 2d - \frac{1}{n}\sum_{k=1}^{n-1}(\eta(X_k)) \tag{7.12}$$

Therefore, using dominated convergence and $\nu(\eta(0)) = 2d + 2d\delta$, for some $\delta > 0$

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty} \frac{1}{n} \left(\mathbb{E}_0(m(X_n)) - m(0)\right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \mathbb{E}_0 \left( \frac{1}{2d} \sum_{k=1}^{n-1} (\xi(X_k) - \eta(X_k)) \right) \\
&\leq 1 - \mathbb{E}_0 \left( \frac{1}{2d} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \eta(X_k) \right) < -\delta
\end{aligned} \tag{7.13}$$

where in the last step we used te ergodicity of the “scenery process” $\{\tau_{X_k}\eta : k \in \mathbb{N}\}$, which follows from the ergodicity of $\nu$ (under spatial translations). Cleary, (7.13) is a contradiction.

Nex, we prove item 1. Suppose that $\nu$ is not stabilizable. Since for every $x \in \mathbb{Z}^d$, $N^\eta_V(x)$ goes to infinity as $V \uparrow \mathbb{Z}^d$, it follows that the restriction of $\eta$ to some fixed volume $V_0$ must be recurrent in $V_0$ (i.e., an element of $\mathcal{R}_{V_0}$). By definition of $\mathcal{R}$, this implies that $\nu(\mathcal{R}) = 1$. We can then use the following lemma of which item one comes from [10], and item 2 follows straighforwardly from item 1.

**Lemma 7.14.** *1. Suppose that $\xi \in \mathcal{R}_V$ is minimally recurrent, i.e., diminishing the height by one at any site $x \in V$ creates a forbidden subconfiguration. Then the total number of grains $\sum_{x\in V} \xi(x)$ equals the number of sites + the number of edges in $V$.*

*2. Suppose that $\nu$ is a translation invariant probability measure concentrating on minimally recurrent configurations, i.e., such that $\nu$-a.s. every restriction $\eta_{V_0}$ is minimally recurrent in $V_0$. Then $\int \eta_0 \, d\nu = d + 1$*

This implies that $\nu$ has a minimal expected height equal to $d + 1$. We then show that there exist non stabilizable measures with density $d + 1 + \delta$ for every $\delta > 0$. We give the proof for $d = 2$, the generalization to arbitrary $d$ is obvious.

Let $\nu$ be a stabilizable measure concentrating on $\mathcal{R}$ such that $\nu(\eta_0) = d + 1$. Let $\nu_p$ denote the Bernoulli measure on $\{0, 1\}^{\mathbb{Z}}$ with $\nu_p(\omega_0 = 1) = p$. Let $\omega, \omega'$ be independently drawn from $\nu_p$. Consider the random field $X(x, y) = \omega(x) + \omega'(y)$. Let $\nu \oplus \nu_p$ denote the distribution of the pointwise addition of $\eta$ and $X$, where $\eta$ is drawn from $\nu$ and $X$ is independent of $\eta$ as defined before. In $X$ we almost surely have infinitely many rectangles $R_1 \ldots R_n, \ldots$ surrounding the origin where at least “two grains are added” on the corner sites and one grain on the other boundary sites. Therefore, since $\nu(\mathcal{R}) = 1$, upon stabilization in a set $V \supset R_n$, we have at least $n$ topplings at the origin. Hence, $\nu \oplus \nu_p$ is not stabilizable, and by choosing $p, q$ small enough, the expected height can be chosen less than or equal to $d + 1 + \delta$. The inequality $\rho_c = 3 < \rho_{\mathbb{Z}^2}$ follows from [34]. $\square$

**Remark 7.15.** *1. If in the definition of the critical density for stabilizability one restricts to product measures (in stead of general translation invariant measures), then it is believed that $\rho_c$ equals the expected height in the stationary* *measure of the ASM, in the thermodynamic limit. In that sense $\rho_c$ would be a "real critical point" (transition point between stabilizable and non-stabilizable).*

2. *The proof of item two of theorem 7.8 gives the result that if*
$$\eta - \Delta m = \xi \tag{7.16}$$
*then the (height) density of $\eta$ cannot be larger that the density of $\xi$. I.e., one cannot by topplings "sweep away" a density of grains. However, the equality* (7.16) *does not imply that the densities of $\eta$ and $\xi$ are equal. Indeed, consider in $d = 2$, $m(x, y) = x^2 + y^2$, then we have*
$$\overline{2} - \Delta m = \overline{6}$$
*i.e., mass can be "obtained from infinity".*

3. *In $d = 1$, $\rho_c = 2$ is also the expected height of the stationary measure of the ASM in the thermodynamic limit. For $\rho = 2$ one can have both stabilizability and non-stabilizability, e.g.,* 313131313131313... *and its shift* 131313131313131313... *are not stabilizable, whereas* 2222... *is stable and hence trivially stabilizable.*

The following notion of metastability formalizes the idea that there are trivially stabilizable measures such as the Dirac measure concentrating on the maximal configuration, which however have the property of "being at the brink of non-stabilizability". For $\mu$ and $\nu$ two probability measure on $\mathcal{H}$, we denote by $\mu \oplus \nu$ the distribution of the pointwise addition of two independent configurations drawn from $\nu$, resp. $\mu$.

**Definition 7.17.** *A probability measure $\nu$ on $\mathcal{H}$ is called metastable if it is stabilizable but $\nu \oplus \delta_0$ is not stabilizable with non-zero probability, .i.e, $\nu \oplus \delta_0(\mathbb{S}) < 1$.*

The following theorem gives a sufficient condition for metastability. We only give the idea of the proof, its formailization is straightforward.

**Theorem 7.18.** *Suppose that $\nu$ is a stationary and ergodic probability measure on $\Omega$, concentrating on the set of recurrent configurations $\mathcal{R}$. Define $I_\eta(x) = \mathbb{1}_{\eta(x)=2d}$ and call $\tilde{\nu}$ the distribution of $I_\eta$. Suppose that $\tilde{\nu}$ dominates a bernoulli measure $\mathbb{P}_p$ with $p$ sufficiently close to one such that the 1's percolate and the zeros do not percolate. Then $\nu$ is metastable.*

**Idea of proof.** Suppose that we have a "sea" of height $2d$ and "islands" of other heights, and such that the configuration is recurrent. Suppose the origin belongs to the sea, and we add a grain at the origin. The first wave must be a simply connected subset of $\mathbb{Z}^d$ because the configuration is recurrent. It is clear that the "sea" of height $2d$ is part of the wave, and therefore every site is contained in the wave (because if an island is not contained then the wave would not be simply connected). So in the first wave every site topples exactly once, but this implies that the resulting configuration is exactly the same. Hence we have infinitely many waves.

The following theorem shows that the thermodynamic limit of the stationary measures of the ASM is in some sense "maximally stabilizable".

**Theorem 7.19.** *Let $d > 4$. Let $\mu$ denote the thermodynamic limit of the stationary measures of the ASM. Let $\nu$ be ergodic under translations such that $\nu(\eta_0) > 0$. Then $\mu \oplus \nu$ is almost surely not stabilizable.*

*Proof.* We have to prove that $\mu \oplus \nu$ is not stabilizable for any $\nu$ stationary and ergodic such that $\nu(\eta(0)) > 0$. A configuration drawn from $\mu \oplus \nu$ is of the form $\eta + \alpha$, where $\eta$ is distributed according to $\mu$ and $\alpha$ independently according to $\nu$.

Suppose $\eta + \alpha$ can be stabilized, then we can write

$$\eta_V + \alpha_V - \Delta_V m^1_V = \xi^1_V \tag{7.20}$$

with $m^1_V \uparrow m^1_{\mathbb{Z}^d}$ as $V \uparrow \mathbb{Z}^d$. We define $m^{2,V} \in \mathbb{N}^{\mathbb{Z}^d}$ by

$$\eta + \alpha^0_V - \Delta m^{2,V} = \xi^{2,V} \tag{7.21}$$

where $\alpha^0_V : \mathbb{Z}^d \to \mathbb{N}$ is defined $\alpha^0_V(x) = \alpha(x) 1\!\!1_{x \in V}$. In words this means that we add according to $\alpha$ only in the finite volume $V$ but we *stabilize in infinite volume.* The fact that $m^{2,V}$ is finite follows from the fact that the addition operators $a_x$ and finite products of these are well-defined in infinite volume on $\mu$ almost every configuration. Since for $W \supset V$

$$\alpha^0_V \le \alpha^0_W \tag{7.22}$$

and $m^1_V$ does not diverge, it is clear that $m^{2,V}$ is well-defined, by approximating the equation (7.21) in growing volumes. Moreover, for $\Lambda \subseteq \mathbb{Z}^d$ fixed, it is also clear that $(m^{2,V})_\Lambda$ and $(m^1_V)_\Lambda$ will coincide for $V \supset V_0$ large enough. Otherwise, the stabilization of $\eta_V + \alpha_V$ would require additional topplings in $\Lambda$ for infinitely many $V$'s, which clearly contradicts that $m^1_V$ converges (and hence remains bounded). But then we have that for $V$ large enough, $(\xi^1_V)_\Lambda$ and $\xi^{2,V}_\Lambda$ coincide. For any $V$, the distribution of $\xi^{2,V}$ is $\mu$, because $\mu$ is stationary under the infinite volume addition operators. Therefore, we conclude that the limit $\lim_V \xi^1_V = \lim_V \xi^2_V$ is distributed according to $\mu$. Hence, passing to the limit $V \uparrow \mathbb{Z}^d$ in (7.21) we obtain

$$\eta + \alpha - \Delta m = \xi \tag{7.23}$$

where $\eta$ and $\xi$ have the same distribution $\mu$, and where $m \in \mathbb{N}^{\mathbb{Z}^d}$. Let $\{X_n, n \in \mathbb{N}\}$ be simple random walk starting at the origin. Then for any function $f : \mathbb{Z}^d \to \mathbb{R}$,

$$M_n = f(X_n) - f(X_0) - \frac{1}{2d} \sum_{i=1}^{n-1} (-\Delta) f(X_i) \tag{7.24}$$

is a mean zero martingale. Applying this identity with $f(x) = m_x$, using (7.23) gives that

$$M_n = m(X_n) - m(X_0) + \sum_{i=1}^{n-1} (\eta(X_i) + \alpha(X_i) - \xi(X_i)) \tag{7.25}$$

is a mean zero martingale (w.r.t. the filtration $\mathcal{F}_n = \sigma(X_r : 0 \le n)$, so $\eta$ and $\xi$ are *fixed* here). This gives, upon taking expectation over the random walk,

$$\frac{1}{n} \left( \mathbb{E}_0(m(X_n)) - m(0) \right) = \frac{1}{n} \mathbb{E}_0 \left( \sum_{i=1}^{n-1} (\xi(X_i) - \eta(X_i) - \alpha(X_i)) \right) \tag{7.26}$$

Using now that $m(0) < \infty$ by assumption, the ergodicity of $\mu$ and $\nu$, and the fact that both $\xi$ and $\eta$ have distribution $\mu$, we obtain from (7.26) upon taking the limit $n \to \infty$ that

$$0 \leq \lim_{n\to\infty} \frac{1}{n} \left(\mathbb{E}_0(m(X_n)) - m(0)\right) = \lim_{n\to\infty} \mathbb{E}_0 \left( \frac{1}{n} \sum_{i=1}^{n-1} (\xi(X_i) - \eta(X_i) - \alpha(X_i)) \right) = -\alpha \tag{7.27}$$

which is a contradiction. □

**Open question**
Are there probability measures $\nu \in \mathcal{P}_t$ such that $\nu(\eta_0) > \rho_{\mathbb{Z}^d}$ and $\nu$ is not metastable ? This question is closely related to the question whether the density is the only relevant parameter in separating metastable measures from stabilizable measures.

# 8 Acknowledgement

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# 9 Recommended literature

The following papers provide excellent introductory material.

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