

ker Δ

Let (Γ, s) be a sandpile graph with Laplacian $\Delta: \mathbb{Z}V \rightarrow \mathbb{Z}V$.

For each $v \in V$, let τ_v denote the sum of the weights of all spanning trees rooted into v . Let $\gamma = \gcd(\tau_v)$ and define

$$\tilde{\tau}_v = \tau_v / \gamma.$$

Thm. * $\ker \Delta = \sum_{v \in V} \tilde{\tau}_v v$.

* This thm. appears in Jeff Crase's
Read thesis.

Pf/ Order the vertices: $V = \{v_1, \dots, v_{n+1}\}$ with $v_{n+1} = s$. Let $\Delta^{(ij)}$ be the matrix formed by removing the i^{th} row and j^{th} column of Δ , and let $C_{ij} = (-1)^{i+j} \det \Delta^{(ij)}$ be the ij -th cofactor of Δ .

It would be better to make this a separate lemma.

(2)

We first show that since the sum of the rows of Δ is the zero vector, if we fix j , then C_{ij} is independent of i .

Let u_1, \dots, u_{n+1} be the rows of Δ after removing the j^{th} column of A . Let e_1, \dots, e_n be the standard basis vectors for \mathbb{Z}^n .

We then have

$$u_2 \wedge \cdots \wedge u_{n+1} = (\det \Delta^{(ij)}) e_1 \wedge \cdots \wedge e_n.$$

Pick any i with $2 \leq i \leq n+1$. Then, since $\sum_{k=1}^{n+1} u_k = 0$,

$$\begin{aligned} (\det \Delta^{(ij)}) e_1 \wedge \cdots \wedge e_n &= u_1 \wedge \cdots \wedge \overset{\text{unit}}{\cancel{u_i}} \wedge \cdots \wedge u_{n+1} \\ &= (-\sum_{k=2}^{n+1} u_k) \wedge \cdots \wedge \overset{\text{unit}}{\cancel{u_i}} \wedge \cdots \wedge u_{n+1} \\ &= -\sum_{k=2}^{n+1} (u_k \wedge \cdots \wedge \overset{\text{unit}}{\cancel{u_i}} \wedge \cdots \wedge u_{n+1}) \\ &= -u_i \wedge u_2 \wedge \cdots \wedge \overset{\text{unit}}{\cancel{u_i}} \wedge \cdots \wedge u_{n+1} \end{aligned}$$

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$$\begin{aligned}
 &= (-1)^{i+1} u_2 \wedge \cdots \wedge u_i \wedge \cdots \wedge u_{n+1} \\
 &= (-1)^{i+1} (\det \Delta^{(ij)}) e_1 \wedge \cdots \wedge e_n.
 \end{aligned}$$

Thus,

$$C_{ij} = (-1)^{i+j} \det \Delta^{(ij)} = (-1)^{i+j} [(-1)^{i+1} \det \Delta^{(ij)}] = (-1)^{i+j} \det \Delta^{(ij)} = C_{ij}.$$

Next, fix i and calculate $\det \Delta$ by expanding along the i^{th} row:

$$\begin{aligned}
 0 = \det \Delta &= \sum_{j=1}^{n+1} \Delta_{ij} C_{ij} \\
 &= \sum_{j=1}^{n+1} \Delta_{ij} C_{jj} \quad (\text{by independence of } C_{ij} \text{ on } i) \\
 &= \sum_{j=1}^{n+1} \Delta_{ij} \tau_{v_j} \quad (\text{matrix-tree theorem})
 \end{aligned}$$

Thus, $(\tau_{v_1}, \dots, \tau_{v_{n+1}}) \in \ker \Delta$.

Since Γ has at least one directed spanning tree (into s),

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the matrix-tree theorem says Δ has at least one non-zero $n \times n$ minor; hence, the rank of Δ is at least n . Since the rows of Δ add up to $\vec{0}$, the rank is at most n . Thus, $\ker \Delta$ (over \mathbb{Z}) consists of all integer multiples of a single vector. Since $(\tau_{v_1}, \dots, \tau_{v_{n+1}}) \in \ker \Delta$, it follows that the single vector generating $\ker \Delta$ must be $(\tilde{\tau}_{v_1}, \dots, \tilde{\tau}_{v_{n+1}})$. \square

Examples

1) If Γ is undirected, $\ker \Delta = \vec{1}$

$$\Delta = \begin{bmatrix} p & -q \\ -p & q \end{bmatrix}$$



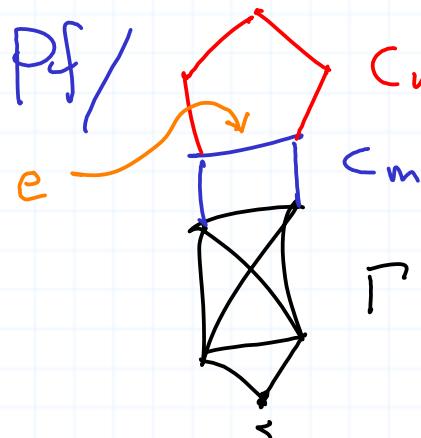
2) If $\Gamma = P \uparrow \downarrow Q$ with p, q relatively prime, $\ker \Delta = (q, p)$.

Proof of conjecture

Thm. (Tianyan Xu) Let (Γ, s) be a sandpile graph. Let

Γ' be the graph obtained from Γ by attaching an m -cycle graph C_m to Γ along an edge, and let Γ'' be obtained from Γ' by attaching an n -cycle graph C_n to Γ' along any edge of C_m not shared with Γ . Then

$$|S(\Gamma'', s)| = n |S(\Gamma', s)| - |S(\Gamma, s)|.$$



Let e be the edge along which C_n is attached to C_m . There are two types of spanning trees in Γ'' :

- ① Those that are spanning trees of Γ' after the removal of the non- e edges of C_n and

② those that are not.

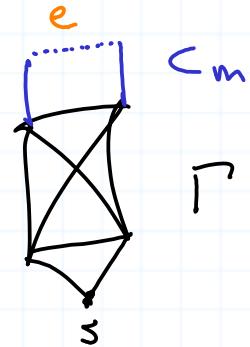
⑥

Spanning trees of T'' of type ① are obtained from spanning trees of T' by add all but one non-e edge of C_n . (Adding e would create a cycle.). Thus, there are $(n-1) |S(T'; s)|$ trees of type ①.

Spanning trees of T'' of type ② require all of the non-e edges of C_n (and do not contain e). To get a spanning tree of type ②, start with a spanning tree for T' containing e, then remove e and add the non-e edges of C_n . All trees of type ② arise this way. So we need to count the spanning trees of T' that contain e. We do this by counting all those that do not contain e and subtracting from the total, $|S(T'; s)|$.

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Spanning trees of Γ' does not contain e come from spanning trees of Γ to which the non-e edges of C_m have been added. Thus, the total number is $|S(\Gamma, s)|$.



Adding up the spanning trees of types ① and ② gives the total:

$$\begin{aligned}
 |S(\Gamma'', s)| &= (n-1) |S(\Gamma', s)| + [S(\Gamma', s) - S(\Gamma, s)] \\
 &= n |S(\Gamma', s)| - S(\Gamma, s). \quad \square
 \end{aligned}$$