

Thm. (Specr) There exists a unique burning configuration b with script $\sigma_b = \tilde{\Delta}^{-1} b$ having the following property: if $\sigma_{b'}$ is the burning script for a burning configuration b' , then $\sigma_{b'} \geq \sigma_b$ (componentwise).

For this b ,

- 1) $\forall v \in \tilde{V}$, we have $b_v < \text{outdeg}(v)$ unless v is a source ($\text{indeg}(v) = 0$), in which case $b_v = \text{outdeg}(v)$.
- 2) $\sigma_b \geq \vec{1}$ with equality iff T has no selfish vertices, i.e., no vertex $v \in \tilde{V}$ with $\text{indeg}(v) > \text{outdeg}(v)$.

Pf/ Create σ_b with a greedy algorithm: define $\sigma_0 = \vec{1}$ and $b^0 = \tilde{\Delta} \sigma_0$. For $i \geq 0$, if $b^i \geq 0$, stop, letting $b = b^i$ and $\sigma_b = \sigma_i$. Otherwise, choose a vertex w s.t. $b_w^i < 0$ and define

$$\sigma_{i+1} = \sigma_i + W, \quad b^{i+1} = b^i + \Delta w = \tilde{\Delta} \sigma_{i+1}.$$

(2)

With w as above, $\forall v \in \tilde{V}$

$$\star \quad b_v^{i+1} = \begin{cases} b_w^i + \text{outdeg}(w) - \text{wt}(w,w) < \text{outdeg}(w) - \text{wt}(w,w) & \text{if } v = w \\ b_v^i - \text{wt}(w,v) \leq b_v^i & \text{if } v \neq w. \end{cases}$$

Since $b_v^0 = \text{outdeg}(v) - \text{indeg}(v)$ it follows ^{for all i} from \star that $b_v^i \leq \text{outdeg}(v)$ with equality iff $\text{indeg}(v) = 0$. Thus, (1) follows once we prove our process halts. Part (2) is immediate, then, by construction.

We now show \dagger process halts. Suppose not. By \star we know $b_v^i \leq \text{outdeg}(v) \quad \forall i$. Let

$$W = \left\{ v \in \tilde{V} : \sigma_i(v) \rightarrow \infty \text{ as } i \rightarrow \infty \right\}.$$

We'll get a contradiction by showing $\sum_{w \in W} b_w^i \rightarrow \infty$ as $i \rightarrow \infty$.

③

Order the vertices so that $\tilde{\Delta}$ becomes a matrix. Let

$\tilde{\Delta}_W$ be the submatrix of $\tilde{\Delta}$ formed by the rows of $\tilde{\Delta}$ corresponding to vertices in W :

$$\tilde{\Delta} = \begin{array}{c} W \\ W^c \end{array} \left[\begin{array}{c|c} W & W^c \\ \hline & \tilde{\Delta}_W \\ \hline & * \end{array} \right]$$

Let c_v denote the sum of the entries of $\tilde{\Delta}_W$ in the column corresponding to $v \in \tilde{V}$. Then

$$\sum_{w \in W} b_w^i = \sum_{v \in \tilde{V}} \sigma_{i,v} c_v = \sum_{w \in W} \sigma_{i,w} c_w + \sum_{v \notin W} \sigma_{i,v} c_v.$$

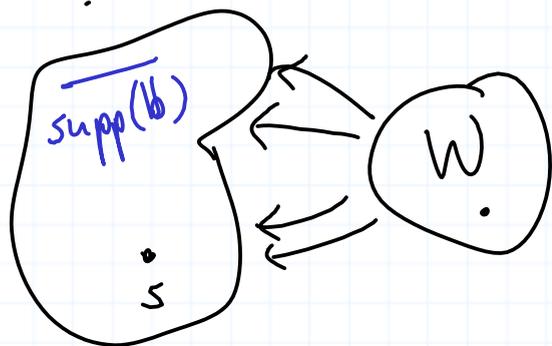
For $v \notin W$, we have that $\sigma_{i,v}$ becomes constant as $i \rightarrow \infty$.

For all $w \in W$, we have $c_w \geq 0$ (but not necessarily for $w \in W^c$). ④

Since every vertex has a path to the sink, \exists some edge connecting a vertex $w' \in W$ to a vertex outside W (possibly the sink). For this w' , we have $c_{w'} > 0$. Hence, $\prod_{i, w'} c_{w'} \rightarrow \infty$ as $i \rightarrow \infty$ and thus, $\sum_{w \in W} b_w^i \rightarrow \infty$, too.

Thus, the process halts, producing a configuration b satisfying (1) + (2).

Let $\overline{\text{supp}(b)}$ denote the collection of $v \in V$ for which there is a path from some element of $\text{supp}(b)$. To show that b is a burning configuration, it remains to show that $\overline{\text{supp}(b)} = V$. Let $W = \overline{\text{supp}(b)}^c$.



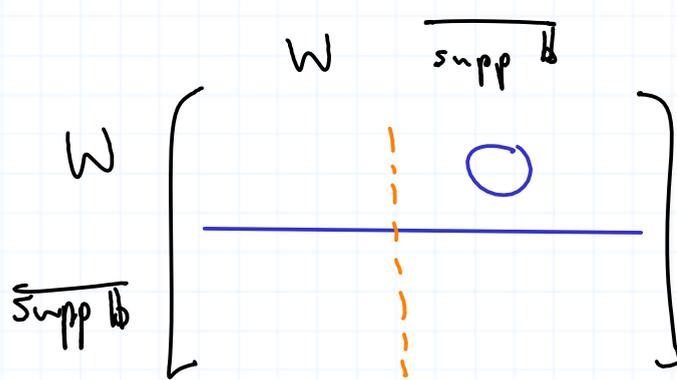
Define $\tilde{\Delta}_W$ and c_v as above.

We have

$$\begin{aligned} \star \star \quad 0 &= \sum_{w \in W} b_w = \sum_{v \in \tilde{V}} \sigma_{b,v} c_v \\ &= \sum_{w \in W} \sigma_{b,w} c_w. \end{aligned}$$

We have $c_v \geq 0 \quad \forall v \in \tilde{V}$ and since there is a path from every vertex to the sink, if $W \neq \emptyset$, \exists some $w \in W$ s.t. $c_w > 0$, contradiction $\star \star$. Hence, $W = \emptyset$.

It remains to show that $\sigma_b \leq \sigma_{b'}$ \forall burning configurations b' .
 Let b' be a burning configuration with script σ' . We would like to show $\sigma_i \leq \sigma'_i \quad \forall i$, whence $\sigma_b \leq \sigma'$. From last



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time, we know $\sigma' \geq \vec{1} = \sigma_0$. Suppose $\exists i$ s.t. $\sigma_i \leq \sigma'$
 but $\sigma_{i+1} \neq \sigma'$. Then $\exists v \in \tilde{V}$ s.t. $b_v^i < 0$ and $\sigma_{i+1} = \sigma_i + v$.
 Thus, $\sigma_{i,w} \leq \sigma'_w$ for $w \neq v$ and $\sigma_{i,v} = \sigma'_v$. Then, since
 $\tilde{\Delta}_{vw} \leq 0 \quad \forall w \neq v$,

$$\begin{aligned} b_v^i &= \sum_w \sigma'_w \tilde{\Delta}_{vw} \\ &= \sigma'_v \tilde{\Delta}_{vv} + \sum_{w \neq v} \sigma'_w \tilde{\Delta}_{vw} \\ &= \sigma_{i,v} \tilde{\Delta}_{vv} + \sum_{w \neq v} \sigma'_w \tilde{\Delta}_{vw} \\ &\leq \sigma_{i,v} \tilde{\Delta}_{vv} + \sum_{w \neq v} \sigma_{i,w} \tilde{\Delta}_{vw} \\ &= b_v^i < 0, \end{aligned}$$

contradicting the fact that $b' \geq 0$. \square

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