

(Γ, s) sandpile graph

(See the end for a brief introduction
to rotor routers.)

The **support** of a configuration c is $\text{Supp}(c) := \{v \in \tilde{V} : c_v \neq 0\}$.

Def. A configuration b on Γ is a **burning configuration** if

(1) $b = 0 \bmod \tilde{L}$ Note: this means $b \geq 0$ componentwise.

(2) $\forall v \in \tilde{V}, \exists$ directed path from w to v for some $w \in \text{Supp}(b)$.

If b is a burning configuration, we call $\sigma_b := \hat{\Delta}^{-1} b$ its **script** or **firing vector**.

Thm. Let b be burning configuration with burning script σ_b ,
and let e denote the identity of $S(\Gamma, s)$.

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$$1) (kb)^{\circ} = e \text{ for } k \gg 0.$$

$$2) \text{ A configuration } c \text{ is recurrent iff } (c+b)^{\circ} = c.$$

3) A configuration c is recurrent iff the firing vector
for $(c+b) \rightsquigarrow (c+b)^{\circ}$ is σ_b .

$$4) \sigma_b \geq \vec{1} \text{ (componentwise).}$$

PF/ 1) By choosing k large enough, then selectively firing unstable vertices, property (2) of the definition of a burning configurations says $kb \rightsquigarrow c + c_{\max}$ for some configuration c . Hence,

$(kb)^{\circ}$ is recurrent since it can be obtained by adding sand to c_{\max} and stabilizing. However, $b = 0 \pmod{\vec{1}}$ and each

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equivalence class mod \tilde{L} has a unique recurrent element. The recurrent element equal to 0 mod \tilde{L} is the identity, e .
Hence, $b = e$.

2) (\Rightarrow) If c is recurrent, then $(c+b)^\circ$ is recurrent, but also $c = (c+b)^\circ \text{ mod } \tilde{L}$. Arguing as just above, $c = (c+b)^\circ$.

(\Leftarrow) Suppose $c = (c+b)^\circ$. It follows that $c = (c+kb)^\circ$ for all $k \geq 1$. Use part (1) to get k large enough so that

$(kb)^\circ = e$. Then $c = (c+kb)^\circ = (c+e)^\circ$. Since e is recurrent, so is c .

3) Let ϕ be the firing vector for $c+b \rightarrow (c+b)^\circ$. Then c is recurrent $\Leftrightarrow (c+b)^\circ = c \Leftrightarrow c+b - \tilde{\Delta} \phi = c \Leftrightarrow b = \tilde{\Delta} \phi \Leftrightarrow \phi = \tilde{\Delta}^{-1} b = \sigma_b$.

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4) Since c_{\max} is recurrent the firing vector for

$c_{\max} + b \rightarrow (c_{\max} + b)^0 = c_{\max}$ is σ_b . Given any vertex $w \in \tilde{V}$,

take $v \in \text{Supp}(b)$ such that there exists a directed path

$v = v_1, v_2, \dots, v_m = w$. Then v_1, \dots, v_m is a valid firing sequence for $c_{\max} + b$. Recall that the firing vector of a stabilization is independent of any particular firing sequence. Thus, each vertex fires at least once in the stabilization $c_{\max} + b \rightarrow (c_{\max} + b)^0$.

So $\sigma_b \geq 1$. \square

Example For an undirected graph, $b = \tilde{\Delta}^{\top}$ is always a burning configuration. It can be thought of as resulting from firing the sink vertex.

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Conjecture Let $\Gamma = \Gamma_0$ be any undirected graph. Let Γ_i be obtained by gluing the cycle graph with n vertices, C_n , to Γ along an edge. For $i \geq 1$, let Γ_i be obtained by gluing C_n to Γ_{i-1} along an edge of the cycle glued to Γ_{i-2} to form Γ_{i-1} . Then

$$|S(\Gamma_i)| = n |S(\Gamma_{i-1})| - |S(\Gamma_{i-2})|$$

for $i \geq 2$.

Idea for proof: Count trees. ?

Rotor Rotors

Let (Γ, s) be a sandpile graph. For each $v \in \tilde{V}$, cyclically order the outedges from v . To play the **rotor-rotter** game select an outedge to be “active”, and place sand on the vertex. Then, ① choose a non-sink vertex and its active edge; make that edge inactive and make the next outedge at v (in the cyclic order at v) active; ③ and shift one grain of sand along the active edge. Thus, the number grains of sand at v decreases by 1 and some vertex adjacent to v gains a grain of sand.

Bijection between recurrents and spanning trees directed into v .

Fix a spanning tree into v . Choose the edges of the tree as the active edges for the rotor rotter game on Γ .

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Now place a recurrent element on the graph and play the rotor-rotter game until all the sand is directed into the sink (it turns out this must happen eventually). When the game stops like this, the set of active edges will form a spanning tree directed into s . This sets up a bijection between currents and spanning trees. The original tree corresponds to the identity configuration. See the next page for an example.



Rotor-Router

