

$K$  complete digraph on  $n+1$  vertices, edges weighted by indeterminates  $\{x_{ij}\}$ .

$L$  = generalized Laplacian;  $L^{(k)}$  =  $k^{\text{th}}$  reduced Laplacian

$T_k$  = spanning trees rooted into  $v_k$ .

Thm. (Matrix-tree) For each  $k$

$$\det L^{(k)} = \prod_{\tau \in T_k} \text{wt}(\tau).$$

Pf / See handout.  $\square$

Cor. Let  $(\Gamma, s)$  be a sandpile graph with reduced Laplacian  $\tilde{\Delta}$ .

(a)  $|S(\Gamma, s)| = \det \tilde{\Delta}$

(b) Order the vertices of  $\Gamma$  to identify configurations with  $\mathbb{N}^n$

and  $\tilde{\Delta}$  becomes a matrix. Let  $U, V$  be square integer matrices,  $(2)$   
 invertible over the integers such that  $U \tilde{\Delta} V = D$ , where  
 $D = \text{diag}(d_1, \dots, d_n)$  is the Smith normal form of  $\tilde{\Delta}$ .  
 Then

$$S(\Gamma, s) \longrightarrow \frac{\mathbb{Z}}{d_1 \mathbb{Z}} \times \dots \times \frac{\mathbb{Z}}{d_n \mathbb{Z}}$$

$$c \longmapsto U_c$$

*Pf/* We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathbb{Z}^n & \xrightarrow{\tilde{\Delta}} & \mathbb{Z}^n & \longrightarrow & S(\Gamma, s) & \longrightarrow & 0 \\ \downarrow V^{-1} & & \downarrow U & & \downarrow & & \\ \mathbb{Z}^n & \xrightarrow{D} & \mathbb{Z}^n & \longrightarrow & \frac{\mathbb{Z}}{d_1 \mathbb{Z}} \times \dots \times \frac{\mathbb{Z}}{d_n \mathbb{Z}} & \longrightarrow & 0 \end{array}$$

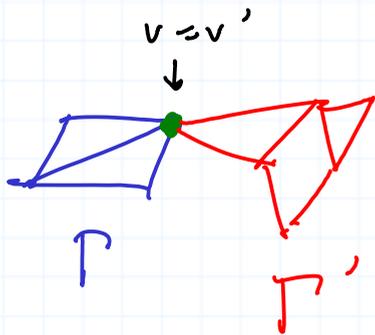
with  $V^{-1}$  and  $U$  isomorphisms.  $\square$

Cor. (Cayley's thm.) The number of (undirected) trees on  $n$  labeled vertices is  $n^{(n-2)}$  (3)

Pf/ HW.  $\square$

### Conjectures from homework

1. Let  $\Gamma$  and  $\Gamma'$  be undirected graphs, and let  $v, v'$  be vertices of  $\Gamma, \Gamma'$ , respectively. Let  $J = J(\Gamma, v; \Gamma', v')$  be the graph formed by gluing  $v$  to  $v'$



Then  $S(J) = S(\Gamma) \times S(\Gamma')$ . Note: we will see that

the sandpile group for a (connected) undirected graph does not depend on the choice of sink.

(4)

Pf/ Since the sandpile groups involve are independent of the choice of sink, take  $v$  and  $v'$  to be the sinks for  $\Gamma$ ,  $\Gamma'$ , respectively, and take to corresponding single vertex  $v=v'$  to be the sink of  $J$ . Then the reduced Laplacian of  $J$  is

$$\tilde{\Delta}_J = \begin{bmatrix} \tilde{\Delta}_\Gamma & 0 \\ 0 & \tilde{\Delta}_{\Gamma'} \end{bmatrix}$$

and the result follows.  $\square$

2. A connected undirected graph has trivial sandpile group iff the graph is a tree.

Pf/ ( $\Leftarrow$ ) Apply the previous result.

( $\Rightarrow$ ) Matrix-tree theorem.  $\square$

⑤